Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 03-k

Discrete Fourier Transform -1

Contents

Reference:

Signals and Systems 2nd Edition – Oppenheim, Willsky

Schaum's Outline Series: Signals and Systems

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1. The Discrete Fourier Series

A. Periodic Sequences:

In Chap. 1 we defined a discrete-time signal (or sequence) $x[n]$ to be periodic if there is a positive integer N for which

$$x[n+N] = x[n] \qquad \text{all } n \tag{6.1}$$

The fundamental period N_0 of $x[n]$ is the smallest positive integer N for which Eq. (6.1) is satisfied.

As we saw in Sec. 1.4, the complex exponential sequence

$$x[n] = e^{j(2\pi/N_0)n} = e^{j\Omega_0 n} \tag{6.2}$$

where $\Omega_0 = 2\pi/N_0$, is a periodic sequence with fundamental period N_0. As we discussed in Sec. 1.4C, one very important distinction between the discrete-time and the continuous-time complex exponential is that the signals $e^{j\omega_0 t}$ are distinct for distinct values of ω_0, but the sequences $e^{j\Omega_0 n}$, which differ in frequency by a multiple of 2π, are identical. That is,

$$e^{j(\Omega_0 + 2\pi k)n} = e^{j\Omega_0 n}\, e^{j2\pi kn} = e^{j\Omega_0 n} \tag{6.3}$$

Let

$$\Psi_k[n] = e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0} \qquad k = 0, \pm 1, \pm 2, \ldots \tag{6.4}$$

Then by Eq. (6.3) we have

$$\Psi_0[n] = \Psi_{N_0}[n] \qquad \Psi_1[n] = \Psi_{N_0+1}[n] \qquad \ldots \qquad \Psi_k[n] = \Psi_{N_0+k}[n] \qquad \ldots \tag{6.5}$$

and more generally,

$$\Psi_k[n] = \Psi_{k+mN_0}[n] \qquad m = \text{integer} \tag{6.6}$$

Thus, the sequences $\Psi_k[n]$ are distinct only over a range of N_0 successive values of k.

B. Discrete Fourier Series Representation:

The discrete Fourier series representation of a periodic sequence $x[n]$ with fundamental period N_0 is given by

$$x[n] = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0} \tag{6.7}$$

where c_k are the Fourier coefficients and are given by (Prob. 6.2)

$$c_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-jk\Omega_0 n} \tag{6.8}$$

Because of Eq. (6.5) [or Eq. (6.6)], Eqs. (6.7) and (6.8) can be rewritten as

$$x[n] = \sum_{k=\langle N_0 \rangle} c_k e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0} \tag{6.9}$$

$$c_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n] e^{-jk\Omega_0 n} \tag{6.10}$$

where $\Sigma_k =_{<N0>}$ denotes that the summation is on k as k varies over a range of N_0 successive integers. Setting $k = 0$ in Eq. (6.10), we have

$$c_0 = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n] \tag{6.11}$$

which indicates that c_0 equals the average value of $x[n]$ over a period.

The Fourier coefficients c_k are often referred to as the *spectral coefficients* of $x[n]$.

C. Convergence of Discrete Fourier Series:

Since the discrete Fourier series is a finite series, in contrast to the continuous-time case, there are no convergence issues with discrete Fourier series.

D. Properties of Discrete Fourier Series:

1. Periodicity of Fourier Coefficients:

From Eqs. (6.5) and (6.7) [or (6.9)], we see that

$$c_{k+N_0} = c_k \tag{6.12}$$

which indicates that the Fourier series coefficients c_k are periodic with fundamental period N_0.

2. Duality:

From Eq. (6.12) we see that the Fourier coefficients c_k form a periodic sequence with fundamental period N_0. Thus, writing c_k as $c[k]$, Eq. (6.10) can be rewritten as

$$c[k] = \sum_{n=\langle N_0 \rangle} \frac{1}{N_0} x[n] \, e^{-jk\Omega_0 n} \tag{6.13}$$

Let $n = -m$ in Eq. (6.13). Then

$$c[k] = \sum_{m=\langle N_0 \rangle} \frac{1}{N_0} x[-m] e^{jk\Omega_0 m}$$

Letting $k = n$ and $m = k$ in the above expression, we get

$$c[n] = \sum_{k=\langle N_0 \rangle} \frac{1}{N_0} x[-k] e^{jk\Omega_0 n} \tag{6.14}$$

Comparing Eq. (6.14) with Eq. (6.9), we see that $(1/N_0)x[-k]$ are the Fourier coefficients of $c[n]$. If we adopt the notation

$$x[n] \xleftrightarrow{\text{DFS}} c_k = c[k] \tag{6.15}$$

to denote the discrete Fourier series pair, then by Eq. (6.14) we have

$$c[n] \xleftrightarrow{\text{DFS}} \frac{1}{N_0} x[-k] \tag{6.16}$$

Equation (6.16) is known as the *duality* property of the discrete Fourier series.

3. Other Properties:

When $x[n]$ is real, then from Eq. (6.8) or [Eq. (6.10)] and Eq. (6.12) it follows that

$$c_{-k} = c_{N_0 - k} = c_k^* \tag{6.17}$$

where * denotes the complex conjugate.

Even and Odd Sequences:

When $x[n]$ is real, let

$$x[n] = x_e[n] + x_0[n]$$

where $x_e[n]$ and $x_0[n]$, are the even and odd components of $x[n]$, respectively. Let

$$x[n] \overset{\text{DFS}}{\longleftrightarrow} c_k$$

Then

$$x_e[n] \overset{\text{DFS}}{\longleftrightarrow} \text{Re}[c_k] \tag{6.18a}$$

$$x_o[n] \overset{\text{DFS}}{\longleftrightarrow} j\,\text{Im}[c_k] \tag{6.18b}$$

Thus, we see that if $x[n]$ is real and even, then its Fourier coefficients are real, while if $x[n]$ is real and odd, its Fourier coefficients are imaginary.

E. Parseval's Theorem:

If $x[n]$ is represented by the discrete Fourier series in Eq. (6.9), then it can be shown that (Prob. 6.10)

$$\frac{1}{N_0} \sum_{n=\langle N_0 \rangle} |x[n]|^2 = \sum_{k=\langle N_0 \rangle} |c_k|^2 \tag{6.19}$$

Equation (6.19) is called *Parseval's identity* (or *Parseval's theorem*) for the discrete Fourier series.

6.3 The Fourier Transform

A. From Discrete Fourier Series to Fourier Transform:

Let $x[n]$ be a nonperiodic sequence of finite duration. That is, for some positive integer N_1,

$$x[n] = 0 \qquad |n| > N_1$$

Such a sequence is shown in Fig. 6-1(*a*). Let $x_{N_0}[n]$ be a periodic sequence formed by repeating $x[n]$ with fundamental period N_0 as shown in Fig. 6-1(*b*). If we let $N_0 \to \infty$, we have

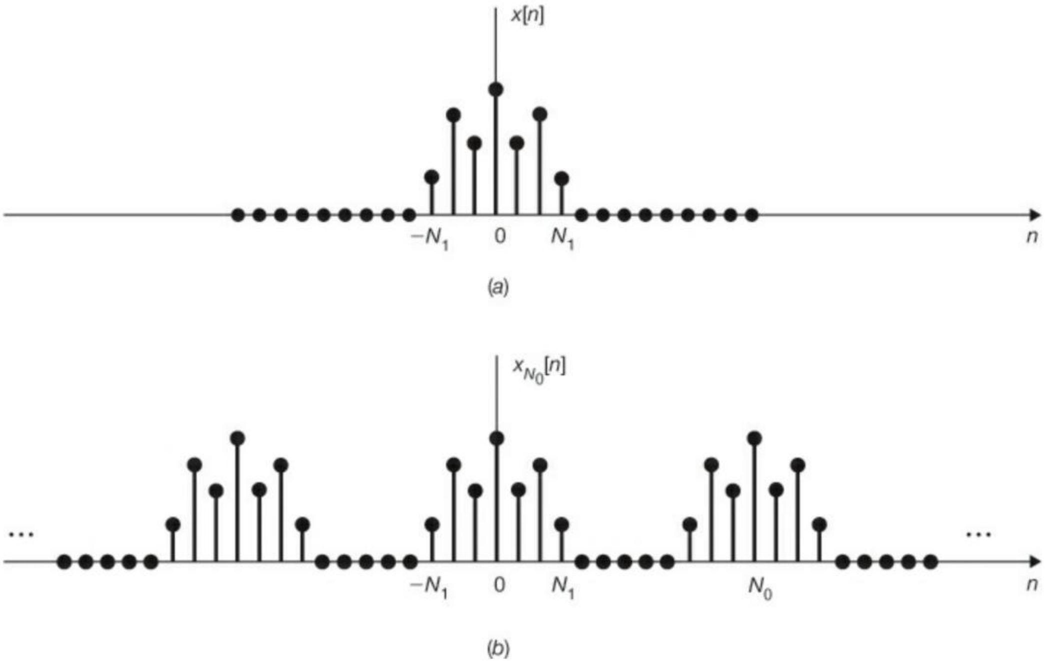

Fig. 6-1 (*a*) Nonperiodic finite sequence $x[n]$; (*b*) periodic sequence formed by periodic extension of $x[n]$.

$$\lim_{N_0 \to \infty} x_{N_0}[n] = x[n] \tag{6.20}$$

The discrete Fourier series of $x_{N_0}[n]$ is given by

$$x_{N_0}[n] = \sum_{k=\langle N_0 \rangle} c_k e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0} \tag{6.21}$$

where

$$c_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x_{N_0}[n]\, e^{-jk\Omega_0 n} \tag{6.22a}$$

Since $x_{N_0}[n] = x[n]$ for $|n| \le N_1$ and also since $x[n] = 0$ outside this interval, Eq. (6.22a) can be rewritten as

$$c_k = \frac{1}{N_0} \sum_{n=-N_1}^{N_1} x[n]\, e^{-jk\Omega_0 n} = \frac{1}{N_0} \sum_{n=-\infty}^{\infty} x[n]\, e^{-jk\Omega_0 n} \tag{6.22b}$$

Let us define $X(\Omega)$ as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \tag{6.23}$$

Then, from Eq. (6.22b) the Fourier coefficients c_k can be expressed as

$$c_k = \frac{1}{N_0} X(k\Omega_0) \tag{6.24}$$

Substituting Eq. (6.24) into Eq. (6.21), we have

$$x_{N_0}[n] = \sum_{k=\langle N_0 \rangle} \frac{1}{N_0} X(k\Omega_0)\, e^{jk\Omega_0 n}$$

or

$$x_{N_0}[n] = \frac{1}{2\pi} \sum_{k=\langle N_0 \rangle} X(k\Omega_0)\, e^{jk\Omega_0 n}\Omega_0 \tag{6.25}$$

From Eq. (6.23), $X(\Omega)$ is periodic with period 2π and so is $e^{j\Omega n}$. Thus, the product $X(\Omega)\, e^{j\Omega n}$ will also be periodic with period 2π. As shown in Fig. 6-2, each term in the summation in Eq. (6.25) represents the area of a rectangle of height $X(k\Omega_0)e^{jk\Omega 0n}$ and width Ω_0. As $N_0 \to \infty$, $\Omega_0 = 2\pi/N_0$ becomes infinitesimal ($\Omega_0 \to 0$) and Eq. (6.25) passes to an integral. Furthermore, since the summation in Eq. (6.25) is over N_0 consecutive intervals of width $\Omega_0 = 2\pi/N_0$, the total interval of integration will always have a width 2π. Thus, as $N_0 \to \infty$ and in view of Eq. (6.20), Eq. (6.25) becomes

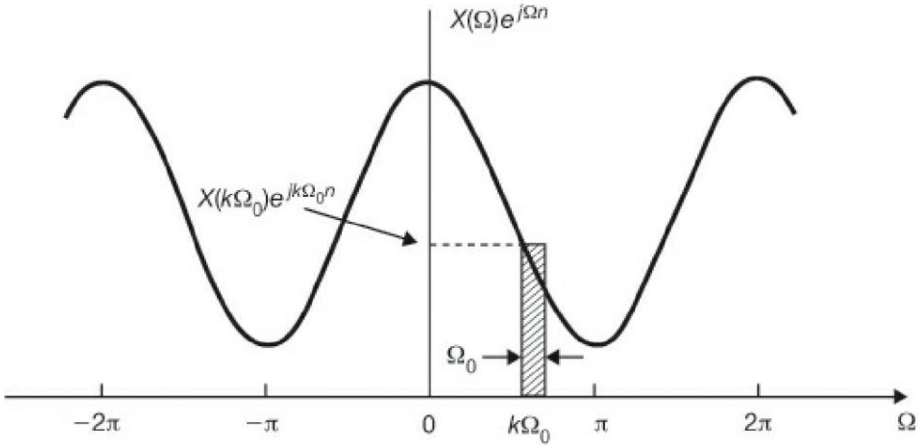

Fig. 6-2 Graphical interpretation of Eq. (6.25).

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega \tag{6.26}$$

Since $X(\Omega)e^{j\Omega n}$ is periodic with period 2π, the interval of integration in Eq. (6.26) can be taken as any interval of length 2π.

B. Fourier Transform Pair:

The function $X(\Omega)$ defined by Eq. (6.23) is called the *Fourier transform* of $x[n]$, and Eq. (6.26) defines the *inverse Fourier transform* of $X(\Omega)$. Symbolically they are denoted by

$$X(\Omega) = \mathscr{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \tag{6.27}$$

$$x[n] = \mathscr{F}^{-1}\{X(\Omega)\} = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega \tag{6.28}$$

and we say that $x[n]$ and $X(\Omega)$ form a Fourier transform pair denoted by

$$x[n] \leftrightarrow X(\Omega) \tag{6.29}$$

Equations (6.27) and (6.28) are the discrete-time counterparts of Eqs. (5.31) and (5.32).

C. Fourier Spectra:

The Fourier transform $X(\Omega)$ of $x[n]$ is, in general, complex and can be expressed as

$$X(\Omega) = |X(\Omega)|\, e^{j\phi(\Omega)} \tag{6.30}$$

As in continuous time, the Fourier transform $X(\Omega)$ of a nonperiodic sequence $x[n]$ is the frequency-domain specification of $x[n]$ and is referred to as the *spectrum* (or *Fourier spectrum*) of $x[n]$. The quantity $|X(\Omega)|$ is called the *magnitude spectrum* of $x[n]$, and $\phi(\Omega)$ is called the *phase spectrum* of $x[n]$. Furthermore, if $x[n]$ is real, the amplitude spectrum $|X(\Omega)|$ is an even function and the phase spectrum $\phi(\Omega)$ is an odd function of Ω.

D. Convergence of $X(\Omega)$:

Just as in the case of continuous time, the sufficient condition for the convergence of $X(\Omega)$ is that $x[n]$ is absolutely summable, that is,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \tag{6.31}$$

E. Connection between the Fourier Transform and the *z*-Transform:

Equation (6.27) defines the Fourier transform of $x[n]$ as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \tag{6.32}$$

The z-transform of $x[n]$, as defined in Eq. (4.3), is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{6.33}$$

Comparing Eqs. (6.32) and (6.33), we see that if the ROC of $X(z)$ contains the unit circle, then the Fourier transform $X(\Omega)$ of $x[n]$ equals $X(z)$ evaluated on the unit circle, that is,

$$X(\Omega) = X(z)\big|_{z=e^{j\Omega}} \tag{6.34}$$

Note that since the summation in Eq. (6.33) is denoted by $X(z)$, then the summation in Eq. (6.32) may be denoted as $X(e^{j}\Omega)$. Thus, in the remainder of

this book, both $X(\Omega)$ and $X(e^{j\Omega})$ mean the same thing whenever we connect the Fourier transform with the z-transform. Because the Fourier transform is the z-transform with $z = e^{j}\Omega$, it should not be assumed automatically that the Fourier transform of a sequence $x[n]$ is the z-transform with z replaced by $e^{j\Omega}$. If $x[n]$ is absolutely summable, that is, if $x[n]$ satisfies condition (6.31), the Fourier transform of $x[n]$ can be obtained from the z-transform of $x[n]$ with $z = e^{j\Omega}$ since the ROC of $X(z)$ will contain the unit circle; that is, $|e^{j\Omega}| = 1$. This is not generally true of sequences which are not absolutely summable. The following examples illustrate the above statements.

EXAMPLE 6.1 Consider the unit impulse sequence $\delta[n]$.

From Eq. (4.14) the z-transform of $\delta[n]$ is

$$\mathcal{Z}\{\delta[n]\} = 1 \qquad \text{all } z \tag{6.35}$$

By definitions (6.27) and (1.45), the Fourier transform of $\delta[n]$ is

$$\mathcal{F}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = 1 \tag{6.36}$$

Thus, the z-transform and the Fourier transform of $\delta[n]$ are the same. Note that $\delta[n]$ is absolutely summable and that the ROC of the z-transform of $\delta[n]$ contains the unit circle.

EXAMPLE 6.2 Consider the causal exponential sequence

$$x[n] = a^n u[n] \qquad a \text{ real}$$

From Eq. (4.9) the z-transform of $x[n]$ is given by

$$X(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

Thus, $X(e^{j\Omega})$ exists for $|a| < 1$ because the ROC of $X(z)$ then contains the unit circle. That is,

$$X(e^{j\Omega}) = \frac{1}{1 - ae^{-j\Omega}} \qquad |a| < 1 \tag{6.37}$$

Next, by definition (6.27) and Eq. (1.91) the Fourier transform of $x[n]$ is

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{\infty} (ae^{-j\Omega})^n \\ &= \frac{1}{1 - ae^{-j\Omega}} \qquad |ae^{-j\Omega}| = |a| < 1 \end{aligned} \tag{6.38}$$

Thus, comparing Eqs. (6.37) and (6.38), we have

$$X(\Omega) = X(z)|_{z = e^{j\Omega}}$$

Note that $x[n]$ is absolutely summable.

EXAMPLE 6.3 Consider the unit step sequence $u[n]$.
From Eq. (4.16) the z-transform of $u[n]$ is

$$\mathfrak{Z}\{u[n]\} = \frac{1}{1 - z^{-1}} \qquad |z| > 1 \tag{6.39}$$

The Fourier transform of $u[n]$ cannot be obtained from its z-transform because the ROC of the z-transform of $u[n]$ does not include the unit circle. Note that the unit step sequence $u[n]$ is not absolutely summable. The Fourier transform of $u[n]$ is given by (Prob. 6.28)

$$\mathscr{F}\{u[n]\} = \pi\delta(\Omega) + \frac{1}{1 - e^{-j\Omega}} \qquad |\Omega| \le \pi \tag{6.40}$$

6.4 Properties of the Fourier Transform

Basic properties of the Fourier transform are presented in the following. There are many similarities to and several differences from the continuous-time case. Many of these properties are also similar to those of the *z*-transform when the ROC of $X(z)$ includes the unit circle.

A. Periodicity:

$$X(\Omega + 2\pi) = X(\Omega) \tag{6.41}$$

As a consequence of Eq. (6.41), in the discrete-time case we have to consider values of Ω (radians) only over the range $0 \le \Omega \le 2\pi$ or $-\pi \le \Omega \le \pi$, while in the continuous-time case we have to consider values of ω (radians/second) over the entire range $-\infty < \omega < \infty$.

B. Linearity:

$$a_1x_1[n] + a_2x_2[n] \leftrightarrow a_1X_1(\Omega) + a_2X_2(\Omega) \tag{6.42}$$

C. Time Shifting:

$$x[n-n_0] \leftrightarrow e^{-j\Omega n_0} X(\Omega) \tag{6.43}$$

D. Frequency Shifting:

$$e^{j\Omega_0 n}x[n] \leftrightarrow X(\Omega-\Omega_0) \tag{6.44}$$

E. Conjugation:

$$x^*[n] \leftrightarrow X^*(-\Omega) \tag{6.45}$$

where * denotes the complex conjugate.

F. Time Reversal:

$$x[-n] \leftrightarrow X(-\Omega) \tag{6.46}$$

G. Time Scaling:

In Sec. 5.4D the scaling property of a continuous-time Fourier transform is expressed as [Eq. (5.52)]

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right) \tag{6.47}$$

However, in the discrete-time case, $x[an]$ is not a sequence if a is not an integer. On the other hand, if a is an integer, say $a = 2$, then $x[2n]$ consists of only the even samples of $x[n]$. Thus, time scaling in discrete time takes on a

form somewhat different from Eq. (6.47).

Let m be a positive integer and define the sequence

$$x_{(m)}[n] = \begin{cases} x[n/m] = x[k] & \text{if } n = km, k = \text{integer} \\ 0 & \text{if } n \neq km \end{cases} \tag{6.48}$$

Then we have

$$x_{(m)}[n] \leftrightarrow X(m\Omega) \tag{6.49}$$

Equation (6.49) is the discrete-time counterpart of Eq. (6.47). It states again the inverse relationship between time and frequency. That is, as the signal spreads in time ($m > 1$), its Fourier transform is compressed (Prob. 6.22). Note that $X(m\Omega)$ is periodic with period $2\pi/m$ since $X(\Omega)$ is periodic with period 2π.

H. Duality:

In Sec. 5.4F the duality property of a continuous-time Fourier transform is expressed as [Eq. (5.54)]

$$X(t) \leftrightarrow 2\pi\, x(-\omega) \tag{6.50}$$

There is no discrete-time counterpart of this property. However, there is a duality between the discrete-time Fourier transform and the continuous-time Fourier series. Let

$$x[n] \leftrightarrow X(\Omega)$$

From Eqs. (6.27) and (6.41)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \tag{6.51}$$

$$X(\Omega + 2\pi) = X(\Omega) \tag{6.52}$$

Since Ω is a continuous variable, letting $\Omega = t$ and $n = -k$ in Eq. (6.51), we have

$$X(t) = \sum_{k=-\infty}^{\infty} x[-k]\, e^{jkt} \qquad (6.53)$$

Since $X(t)$ is periodic with period $T_0 = 2\pi$ and the fundamental frequency $\omega_0 = 2\pi/T_0 = 1$, Eq. (6.53) indicates that the Fourier series coefficients of $X(t)$ will be $x[-k]$. This duality relationship is denoted by

$$X(t) \xleftrightarrow{\text{FS}} c_k = x[-k] \qquad (6.54)$$

where FS denotes the Fourier series and c_k are its Fourier coefficients.

I. Differentiation in Frequency:

$$nx[n] \leftrightarrow j\frac{dX(\Omega)}{d\Omega} \qquad (6.55)$$

J. Differencing:

$$x[n] - x[n-1] \leftrightarrow (1 - e^{-j\Omega})X(\Omega) \qquad (6.56)$$

The sequence $x[n] - x[n-1]$ is called the *first difference* sequence. Equation (6.56) is easily obtained from the linearity property (6.42) and the time-shifting property (6.43).

K. Accumulation:

$$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \pi X(0)\,\delta(\Omega) + \frac{1}{1-e^{-j\Omega}}X(\Omega) \qquad |\Omega| \le \pi \qquad (6.57)$$

Note that accumulation is the discrete-time counterpart of integration. The impulse term on the right-hand side of Eq. (6.57) reflects the dc or average value that can result from the accumulation.

L. Convolution:

$$x_1[n] * x_2[n] \leftrightarrow X_1(\Omega)\, X_2(\Omega) \qquad (6.58)$$

As in the case of the *z*-transform, this convolution property plays an important role in the study of discrete-time LTI systems.

M. Multiplication:

$$x_1[n]\, x_2[n] \leftrightarrow \frac{1}{2\pi} X_1(\Omega) \otimes X_2(\Omega) \qquad (6.59)$$

where $\otimes$ denotes the periodic convolution defined by [Eq. (2.70)]

$$X_1(\Omega) \otimes X_2(\Omega) = \int_{2\pi} X_1(\theta)\, X_2(\Omega - \theta)\, d\theta \qquad (6.60)$$

N. Additional Properties:

If $x[n]$ is real, let

$$x[n] = x_e[n] + x_0[n]$$

where $x_e[n]$ and $x_0[n]$ are the even and odd components of $x[n]$, respectively. Let

$$x[n] \leftrightarrow X(\Omega) = A(\Omega) + jB(\Omega) = |X(\Omega)|e^{j\theta(\Omega)} \tag{6.61}$$

Then

$$X(-\Omega) = X^*(\Omega) \tag{6.62}$$

$$x_e[n] \leftrightarrow \mathrm{Re}\{X(\Omega)\} = A(\Omega) \tag{6.63a}$$

$$x_o[n] \leftrightarrow j\mathrm{Im}\{X(\Omega)\} = jB(\Omega) \tag{6.63b}$$

Equation (6.62) is the necessary and sufficient condition for $x[n]$ to be real. From Eqs. (6.62) and (6.61) we have

$$A(-\Omega) = A(\Omega) \qquad B(-\Omega) = -B(\Omega) \tag{6.64a}$$

$$|X(-\Omega)| = |X(\Omega)| \qquad \theta(-\Omega) = -\theta(\Omega) \tag{6.64b}$$

From Eqs. (6.63a), (6.63b), and (6.64a) we see that if $x[n]$ is real and even, then $X(\Omega)$ is real and even, while if $x[n]$ is real and odd, $X(\Omega)$ is imaginary and odd.

O. Parseval's Relations:

$$\sum_{n=-\infty}^{\infty} x_1[n]\, x_2[n] = \frac{1}{2\pi}\int_{2\pi} X_1(\Omega)\, X_2(-\Omega)\, d\Omega \tag{6.65}$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2\, d\Omega \tag{6.66}$$

Equation (6.66) is known as *Parseval's identity* (or *Parseval's theorem*) for the discrete-time Fourier transform.

5. Glossary – English/Chinese Translation

English	Chinese
Discrete Fourier Transform	离散傅里叶变换
Discrete Fourier Series	离散傅里叶级数
Fourier Coefficients	傅里叶系数
periodic sequence	周期序列
successive values	连续值
spectra coefficients	光谱系数
duality	二重性
Parseval's Theorem	帕斯瓦尔定理
Fourier Transform Pair	傅里叶变换对
Fourier Spectra	傅里叶光谱

----- END -----

Your Notes: