1-03-j Tutorial

Question 1

4.15. Find the inverse z-transform of

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})(1 + 2z^{-1}) \qquad 0 < |z| < \infty \tag{4.79}$$

Question 2

4.17. Find the inverse z-transform of the following $X(z)$:

$$(a)\ X(z) = \log\left(\frac{1}{1 - az^{-1}}\right), |z| > |a|$$

$$(b)\ X(z) = \log\left(\frac{1}{1 - a^{-1}z}\right), |z| < |a|$$

Question 3

4.18. Using the power series expansion technique, find the inverse z-transform of the following $X(z)$:

$$(a)\ X(z) = \frac{z}{2z^2 - 3z + 1} \qquad |z| < \frac{1}{2}$$

$$(b)\ X(z) = \frac{z}{2z^2 - 3z + 1} \qquad |z| > 1$$

Question 4

4.20. Find the inverse z-transform of

$$X(z) = \frac{z}{z(z-1)(z-2)^2} \qquad |z| > 2$$

Solution

Q 1

$$X(z)=z^2\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})(1+2z^{-1}) \qquad 0<|z|<\infty \tag{4.79}$$

Multiplying out the factors of Eq. (4.79), we can express $X(z)$ as

$$X(z)=z^2+\frac{1}{2}z-\frac{5}{2}+z^{-1}$$

Then, by definition (4.3),

$$X(z)=x[-2]z^2+x[-1]z+x[0]+x[1]z^{-1}$$

and we get

$$X[n]=\left\{\ldots,0,1,\frac{1}{2},-\underset{\uparrow}{\frac{5}{2}},1,0,\ldots\right\}$$

Q2

(a)

(*a*) The power series expansion for $\log(1-r)$ is given by

$$\log(1-r)=-\sum_{n=1}^{\infty}\frac{1}{n}r^n \qquad |r|<1 \tag{4.80}$$

Now

$$X(z)=\log\left(\frac{1}{1-az^{-1}}\right)=-\log(1-az^{-1}) \qquad |z|>|a|$$

Since the ROC is $|z|>|a|$, that is, $|az^{-1}|<1$, by Eq. (4.80), $X(z)$ has the power series expansion

$$X(z)=\sum_{n=1}^{\infty}\frac{1}{n}(az^{-1})^n=\sum_{n=1}^{\infty}\frac{1}{n}a^n z^{-n}$$

from which we can indentify $x[n]$ as

$$x[n] = \begin{cases} (1/n)a^n & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

or

$$x[n] = \frac{1}{n}a^n u[n-1] \tag{4.81}$$

(b)

(*b*)

$$X(z) = \log\left(\frac{1}{1-a^{-1}z}\right) = -\log(1-a^{-1}z) \qquad |z| < |a|$$

Since the ROC is $|z| < |a|$, that is, $|a^{-1}z| < 1$, by Eq. (4.80), $X(z)$ has the power series expansion

$$X(z) = \sum_{n=1}^{\infty} \frac{1}{n}(a^{-1}z)^n = \sum_{n=-1}^{-\infty} -\frac{1}{n}(a^{-1}z)^{-n} = \sum_{n=-1}^{-\infty} -\frac{1}{n}a^n z^{-n}$$

from which we can identify $x[n]$ as

$$x[n] = \begin{cases} 0 & n \geq 0 \\ -(1/n)a^n & n \leq -1 \end{cases}$$

or

$$x[n] = -\frac{1}{n}a^n u[-n-1] \tag{4.82}$$

Q3

(a)

(*a*) Since the ROC is $|z| < \frac{1}{2}$, $x[n]$ is a left-sided sequence. Thus, we must divide to obtain a series in power of *z*. Carrying out the long division, we obtain

$$
\begin{array}{r|l}
 & z + 3z^2 + 7z^3 + 15z^4 + \cdots \\
\hline
1 - 3z + 2z^2 & z \\
 & \underline{z - 3z^2 + 2z^3} \\
 & \quad 3z^2 - 2z^3 \\
 & \quad \underline{3z^2 - 9z^3 + 6z^4} \\
 & \qquad 7z^3 - 6z^4 \\
 & \qquad \underline{7z^3 - 21z^4 + 14z^5} \\
 & \qquad\quad 15z^4 \cdots
\end{array}
$$

Thus,

$$X(z) = \cdots + 15z^4 + 7z^3 + 3z^2 + z$$

and so by definition (4.3) we obtain

$$x[n] = \{\ldots, 15, 7, 3, 1, \underset{\uparrow}{0}\}$$

(*b*) Since the ROC is $|z| > 1$, $x[n]$ is a right-sided sequence. Thus, we must divide so as to obtain a series in power of z^{-1} as follows:

$$
\begin{array}{r|l}
 & \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \cdots \\
\hline
2z^2 - 3z + 1 & z \\
 & \underline{z - \frac{3}{2} - \frac{1}{2}z^{-1}} \\
 & \quad \frac{3}{2} - \frac{1}{2}z^{-1} \\
 & \quad \underline{\frac{3}{2} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2}} \\
 & \qquad \frac{7}{4}z^{-1} - \frac{3}{4}z^{-2} \\
 & \qquad \vdots
\end{array}
$$

Thus,

$$X(z) = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \cdots$$

and so by definition (4.3) we obtain

$$x[n] = \left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\right\}$$

Q4

$$X(z) = \frac{z}{z(z-1)(z-2)^2} \qquad |z| > 2$$

Using partial-fraction expansion, we have

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z-2)^2} = \frac{c_1}{z-1} + \frac{\lambda_1}{z-2} + \frac{\lambda_2}{(z-2)^2} \tag{4.83}$$

where

$$c_1 = \left.\frac{1}{(z-2)^2}\right|_{z=1} = 1 \qquad \lambda_2 = \left.\frac{1}{z-1}\right|_{z=2} = 1$$

Substituting these values into Eq. (4.83), we have

$$\frac{1}{(z-1)(z-2)^2} = \frac{1}{z-1} + \frac{\lambda_1}{z-2} + \frac{1}{(z-2)^2}$$

Setting $z = 0$ in the above expression, we have

$$-\frac{1}{4} = -1 - \frac{\lambda_1}{2} + \frac{1}{4} \rightarrow \lambda_1 = -1$$

Thus,

$$X(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{(z-2)^2} \qquad |z| > 2$$

Since the ROC is $|z| > 2$, $x[n]$ is a right-sided sequence, and from Table 4-1 we get

$$x[n] = (1 - 2^n + n2^{n-1})u[n]$$