Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 03-j

Z Transform and Discrete-Time LTI Systems -1

Contents

Reference:

Signals and Systems 2nd Edition – Oppenheim, Willsky
Schaum's Outline Series: Signals and Systems

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1. The inverse Z Transform

Inversion of the *z*-transform to find the sequence $x[n]$ from its *z*-transform $X(z)$ is called the inverse *z*-transform, symbolically denoted as

$$x[n] = \mathcal{Z}^{-1}\{X(z)\} \tag{4.27}$$

A. Inversion Formula:

As in the case of the Laplace transform, there is a formal expression for the inverse *z*-transform in terms of an integration in the *z*-plane; that is,

$$x[n] = \frac{1}{2\pi j}\oint_C X(z) z^{n-1}\, dz \tag{4.28}$$

where C is a counterclockwise contour of integration enclosing the origin. Formal evaluation of Eq. (4.28) requires an understanding of complex variable theory.

B. Use of Tables of *z*-Transform Pairs:

In the second method for the inversion of $X(z)$, we attempt to express $X(z)$ as a sum

$$X(z) = X_1(z) + \cdots + X_n(z) \tag{4.29}$$

where $X_1(z), \ldots, X_n(z)$ are functions with known inverse transforms $x_1[n], \ldots, x_n[n]$. From the linearity property (4.17) it follows that

$$x[n] = x_1[n] + \cdots + x_n[n] \tag{4.30}$$

C. Power Series Expansion:

The defining expression for the *z*-transform [Eq. (4.3)] is a power series where the sequence values $x[n]$ are the coefficients of z^{-n}. Thus, if $X(z)$ is given as a power series in the form

$$\begin{aligned} X[z] &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ &= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \end{aligned} \tag{4.31}$$

we can determine any particular value of the sequence by finding the coefficient of the appropriate power of z^{-1}. This approach may not provide a closed-form solution but is very useful for a finite-length sequence where $X(z)$ may have no simpler form than a polynomial in z^{-1} (see Prob. 4.15). For rational *z*-transforms, a power series expansion can be obtained by long division as illustrated in Probs. 4.16 and 4.17.

D. Partial-Fraction Expansion:

As in the case of the inverse Laplace transform, the partial-fraction expansion method provides the most generally useful inverse z-transform, especially when $X(z)$ is a rational function of z. Let

$$X(z) = \frac{N(z)}{D(z)} = k\frac{(z-z_1)\cdots(z-z_m)}{(z-p_1)\cdots(z-p_n)} \tag{4.32}$$

Assuming $n \geq m$ and all poles p_k are simple, then

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \cdots + \frac{c_n}{z-p_n} = \frac{c_0}{z} + \sum_{k=1}^{n}\frac{c_k}{z-p_k} \tag{4.33}$$

where

$$c_0 = X(z)\Big|_{z=0} \qquad c_k = (z-p_k)\frac{X(z)}{z}\Big|_{z=p_k} \tag{4.34}$$

Hence, we obtain

$$X(z) = c_0 + c_1\frac{z}{z-p_1} + \cdots + c_n\frac{z}{z-p_n} = c_0 + \sum_{k=1}^{n} c_k\frac{z}{z-p_k} \tag{4.35}$$

Inferring the ROC for each term in Eq. (4.35) from the overall ROC of $X(z)$ and using Table 4-1, we can then invert each term, producing thereby the overall inverse z-transform (see Probs. 4.19 to 4.23).

If $m > n$ in Eq. (4.32), then a polynomial of z must be added to the right-hand side of Eq. (4.35), the order of which is $(m - n)$. Thus for $m > n$, the complete partial-fraction expansion would have the form

$$X(z) = \sum_{q=0}^{m-n} b_q z^q + \sum_{k=1}^{n} c_k\frac{z}{z-p_k} \tag{4.36}$$

If $X(z)$ has multiple-order poles, say, p_i is the multiple pole with multiplicity r, then the expansion of $X(z)/z$ will consist of terms of the form

$$\frac{\lambda_1}{z-p_i} + \frac{\lambda_2}{(z-p_i)^2} + \cdots + \frac{\lambda_r}{(z-p_i)^r} \tag{4.37}$$

where

$$\lambda_{r-k} = \frac{1}{k!}\frac{d^k}{dz^k}\left[(z-p_i)^r\frac{X(z)}{z}\right]\Bigg|_{z=p_i} \tag{4.38}$$

2. The system functions of discrete time LTI systems

A. The System Function:

In Sec. 2.6 we showed that the output $y[n]$ of a discrete-time LTI system equals the convolution of the input $x[n]$ with the impulse response $h[n]$; that is [Eq. (2.35)],

$$y[n] = x[n] * h[n] \tag{4.39}$$

Applying the convolution property (4.26) of the z-transform, we obtain

$$Y(z) = X(z)H(z) \tag{4.40}$$

where $Y(z)$, $X(z)$, and $H(z)$ are the z-transforms of $y[n]$, $x[n]$, and $h[n]$, respectively. Equation (4.40) can be expressed as

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.41}$$

The z-transform $H(z)$ of $h[n]$ is referred to as the *system function* (or the *transfer function*) of the system. By Eq. (4.41) the system function $H(z)$ can also be defined as the ratio of the z-transforms of the output $y[n]$ and the input $x[n]$. The system function $H(z)$ completely characterizes the system. Fig. 4-3 illustrates the relationship of Eqs. (4.39) and (4.40).

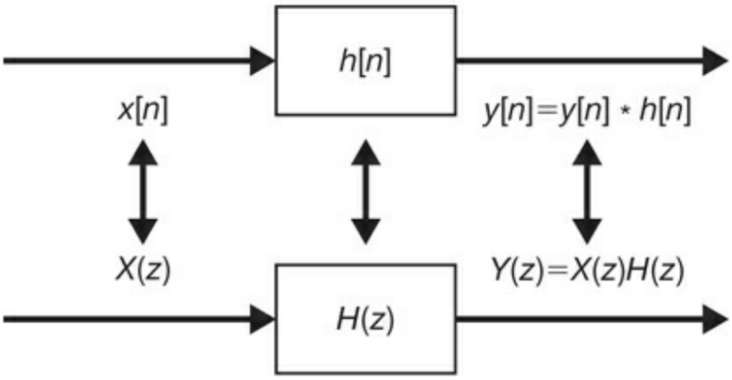


Fig. 4-3 Impulse response and system function.

B. Characterization of Discrete-Time LTI Systems:

1. Causality:

For a causal discrete-time LTI system, we have [Eq. (2.44)]

$$h[n] = 0 \qquad n < 0$$

since $h[n]$ is a right-sided signal, the corresponding requirement on $H(z)$ is that the ROC of $H(z)$ must be of the form

$$|z| > r_{\max}$$

That is, the ROC is the exterior of a circle containing all of the poles of $H(z)$ in the z-plane. Similarly, if the system is anticausal, that is,

$$h[n] = 0 \qquad n \geq 0$$

then $h[n]$ is left-sided and the ROC of $H(z)$ must be of the form

$$|z| < r_{\min}$$

That is, the ROC is the interior of a circle containing no poles of $H(z)$ in the z-plane.

2. Stability:

In Sec. 2.7 we stated that a discrete-time LTI system is BIBO stable if and only if [Eq. (2.49)]

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

The corresponding requirement on $H(z)$ is that the ROC of $H(z)$ contains the unit circle (that is, $|z| = 1$). (See Prob. 4.30.)

3. Causal and Stable Systems:

If the system is both causal and stable, then all of the poles of $H(z)$ must lie

inside the unit circle of the z-plane because the ROC is of the form $|z| > r_{\max}$, and since the unit circle is included in the ROC, we must have $r_{\max} < 1$.

C. System Function for LTI Systems Described by Linear Constant-Coefficient Difference Equations:

In Sec. 2.9 we considered a discrete-time LTI system for which input $x[n]$ and output $y[n]$ satisfy the general linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{4.42}$$

Applying the z-transform and using the time-shift property (4.18) and the linearity property (4.17) of the z-transform, we obtain

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

or

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k} \tag{4.43}$$

Thus,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{4.44}$$

D. Systems Interconnection:

For two LTI systems (with $h_1[n]$ and $h_2[n]$, respectively) in cascade, the

overall impulse response $h[n]$ is given by

$$h[n] = h_1[n] * h_2[n] \tag{4.45}$$

Thus, the corresponding system functions are related by the product

$$H(z) = H_1(z) H_2(z) \qquad R \supset R_1 \cap R_2 \tag{4.46}$$

Similarly, the impulse response of a parallel combination of two LTI systems is given by

$$h[n] = h_1[n] + h_2[n] \tag{4.47}$$

and

$$H(z) = H_1(z) + H_2(z) \qquad R \supset R_1 \cap R_2 \tag{4.48}$$

3. Block Diagram Representation for Causal LTI Systems

The system function algebra for analyzing discrete-time block diagrams such as series, parallel, and feedback interconnections is exactly the same as that for the corresponding continuous-time systems in Section 9.8.1. For example, the system function for the cascade of two discrete-time LTI systems is the product of the system functions for the individual systems in the cascade. Also, consider the feedback interconnection of two systems, as shown in Figure 10.17. It is relatively involved to determine the difference equation or impulse response for the overall system working directly in the time domain. However, with the systems and sequences expressed in terms of their z-transforms, the analysis involves only algebraic equations. The specific equations for the interconnection of Figure 10.17 exactly parallel eqs. (9.159)–(9.163), with the final result that the overall system function for the feedback system of Figure 10.17 is

$$\frac{Y(z)}{X(z)} = H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}. \tag{10.115}$$

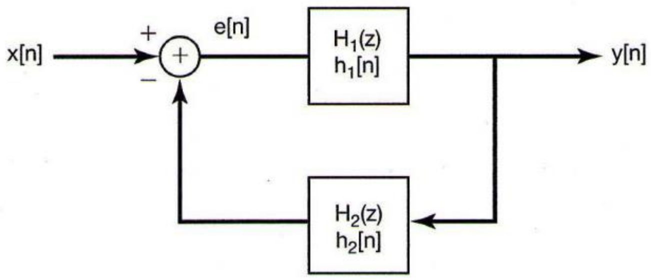


Figure 10.17 Feedback interconnection of two systems.

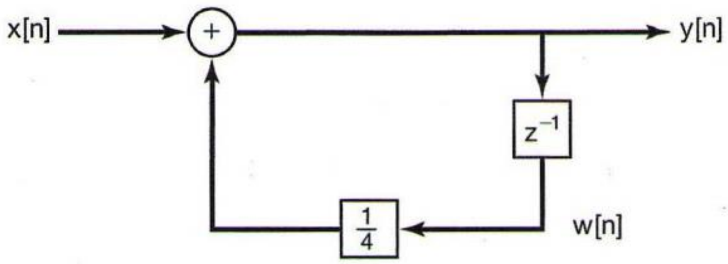
Example 10.28

Consider the causal LTI system with system function

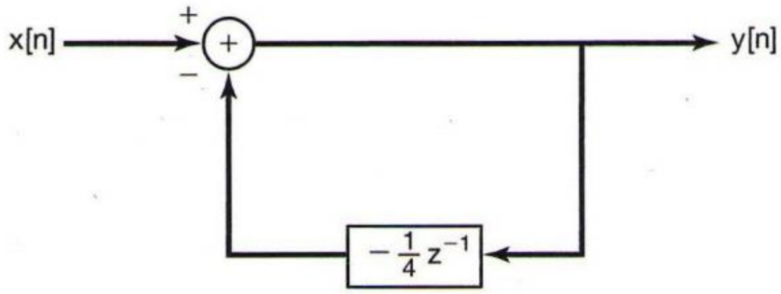
$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}. \tag{10.116}$$

Using the results in Section 10.7.3, we find that this system can also be described by the difference equation

$$y[n] - \frac{1}{4}y[n-1] = x[n],$$



(a)



(b)

Figure 10.18 (a) Block diagram representations of the causal LTI system in Example 10.28; (b) equivalent block diagram representation.

<u>Example 2</u>

Suppose we now consider the causal LTI system with system function

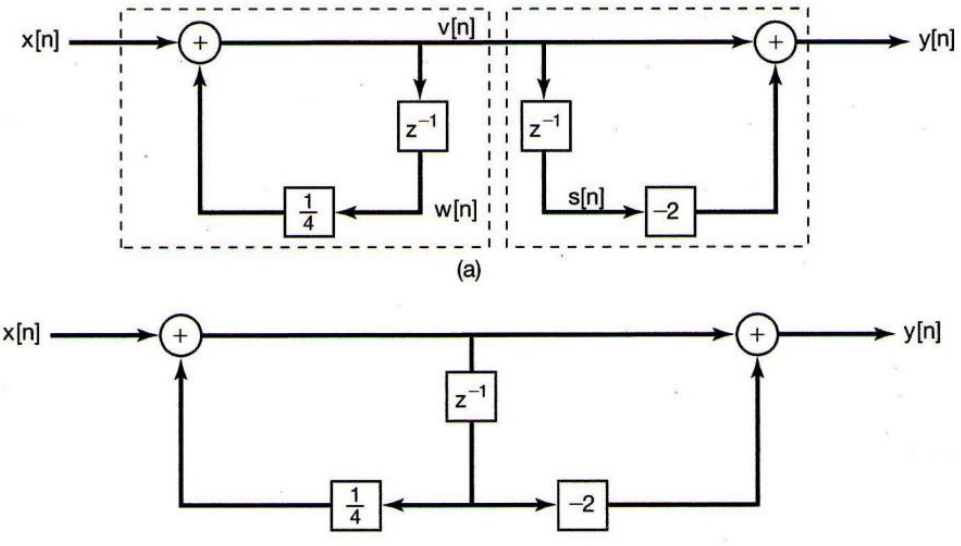
$$H(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}} = \left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right)(1 - 2z^{-1}). \qquad (10.117)$$

Let

$$y[n] = v[n] - 2v[n-1].$$

And

$$w[n] = s[n] = v[n-1].$$



(b)

Figure 10.19 (a) Block-diagram representations for the system in Example 10.29; (b) equivalent block-diagram representation using only one unit delay element.

Example 3

Next, consider the second-order system function

$$H(z) = \frac{1}{(1 + \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} = \frac{1}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}, \qquad (10.118)$$

which is also described by the difference equation

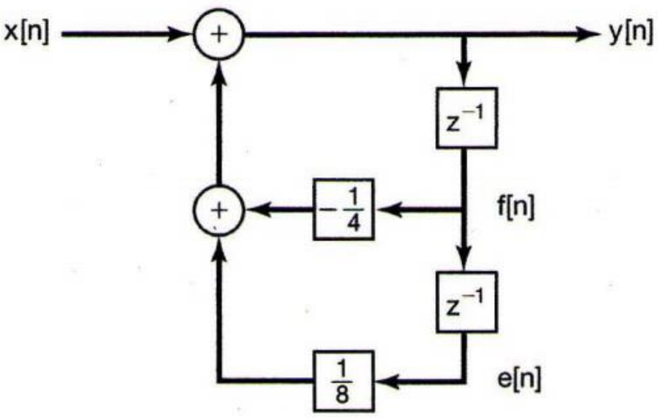
$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]. \qquad (10.119)$$

Using the same ideas as in Example 10.28, we obtain the block-diagram representation for this system shown in Figure 10.20(a). Specifically, since the two system function blocks in this figure with system function z^{-1} are unit delays, we have

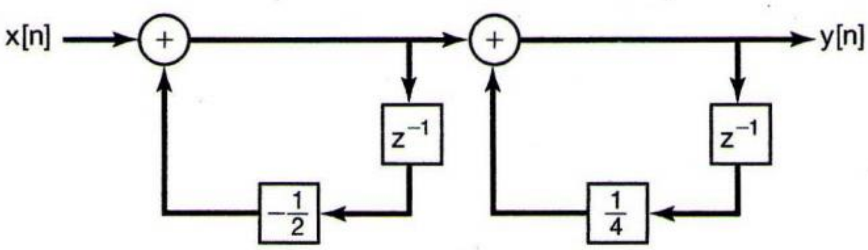
$$f[n] = y[n-1],$$
$$e[n] = f[n-1] = y[n-2],$$

so that eq. (10.119) can be rewritten as

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n],$$



(a)



(b)

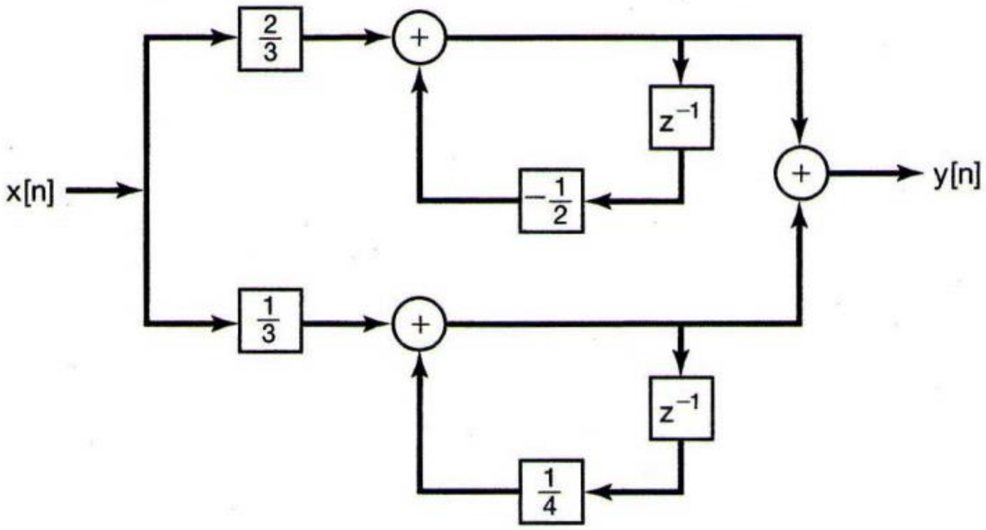


Figure 10.20 Block-diagram representations for the system in Example 10.30: (a) direct form; (b) cascade form; (c) parallel form.

$$y[n] = -\frac{1}{4}f[n] + \frac{1}{8}e[n] + x[n],$$

which is exactly what the figure represents.

The block diagram in Figure 10.20(a) is commonly referred to as a *direct-form* representation, since the coefficients appearing in the diagram can be determined by inspection from the coefficients appearing in the difference equation or, equivalently, the system function. Alternatively, as in continuous time, we can obtain both *cascade-form* and *parallel-form* block diagrams with the aid of a bit of system function algebra. Specifically, we can rewrite eq. (10.118) as

$$H(z) = \left(\frac{1}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right), \tag{10.120}$$

which suggests the cascade-form representation depicted in Figure 10.20(b) in which the system is represented as the cascade of two systems corresponding to the two factors in eq. (10.120).

Also, by performing a partial-fraction expansion, we obtain

$$H(z) = \frac{\frac{2}{3}}{1+\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1-\frac{1}{4}z^{-1}},$$

which leads to the parallel-form representation depicted in Figure 10.20(c).

4. The Unilateral Z Transform

The *unilateral* (or *one-sided*) *z*-transform $X_I(z)$ of a sequence $x[n]$ is defined as [Eq. (4.5)]

$$X_I(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \tag{4.49}$$

and differs from the bilateral transform in that the summation is carried over only $n \geq 0$. Thus, the unilateral *z*-transform of $x[n]$ can be thought of as the bilateral transform of $x[n]u[n]$. Since $x[n]u[n]$ is a right-sided sequence, the ROC of $X_I(z)$ is always outside a circle in the *z*-plane.

Basic Properties

Most of the properties of the unilateral *z*-transform are the same as for the bilateral *z*-transform. The unilateral *z*-transform is useful for calculating the response of a causal system to a causal input when the system is described by a linear constant-coefficient difference equation with nonzero initial conditions. The basic property of the unilateral *z*-transform that is useful in

this application is the following time-shifting property which is different from that of the bilateral transform.

Time-Shifting Property:

If $x[n] \leftrightarrow X_I(z)$, then for $m \geq 0$,

$$x[n-m] \leftrightarrow z^{-m} X_I(z) + z^{-m+1} x[-1] + z^{-m+2} x[-2] + \cdots + x[-m] \tag{4.50}$$

$$x[n+m] \leftrightarrow z^{m} X_I(z) - z^{m} x[0] - z^{m-1} x[1] - \cdots - z x[m-1] \tag{4.51}$$

The proofs of Eqs. (4.50) and (4.51) are given in Prob. 4.36.

D. System Function:

Similar to the case of the continuous-time LTI system, with the unilateral *z*-transform, the system function $H(z) = Y(z)/X(z)$ is defined under the condition that the system is relaxed; that is, all initial conditions are zero.

S3. Glossary – English/Chinese Translation

English	Chinese
inverse z transform	逆 z 变换
discrete time LTI system	离散时间 LTI 系统
block diagram	方框图
causal system	因果系统
unilateral z transform	单边 z 变换
power series expansion	电源系列扩展
partial fraction expansion	部分分数展开
impulse response	脉冲响应
linear constant coefficient -	线性常数系数
difference equation	差分方程
system interconnection	系统互联互通

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