1-03-i Tutorial

Question 1

4.1. Find the *z*-transform of

(a) $x[n] = -a^n u[-n-1]$

Question 2

4.3. A finite sequence $x[n]$ is defined as

$$x[n] = \{5, 3, \underset{\uparrow}{-2}, 0, 4, -3\}$$

Find $X(z)$ and its ROC.

Question 3

4.6. Find the *z*-transform $X(z)$ and sketch the pole-zero plot with the ROC for each of the following sequences:

(a) $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$

Question 4

4.7. Let

$$x[n] = a^{|n|} \qquad a > 0 \tag{4.66}$$

(a) Sketch $x[n]$ for $a < 1$ and $a > 1$.

(b) Find $X(z)$ and sketch the zero-pole plot and the ROC for $a < 1$ and $a > 1$.

Solution

Q 1

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \qquad |\alpha|<1 \tag{1.91}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{4.3}$$

(*a*) From Eq. (4.3)

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} (a^{-1}z)^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n \end{aligned}$$

By Eq. (1.91)

$$\sum_{n=0}^{\infty} (a^{-1}z)^n = \frac{1}{1-a^{-1}z} \qquad \text{if} \left|a^{-1}z\right| < 1 \text{ or } |z| < |a|$$

Thus,

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a} = \frac{1}{1-az^{-1}} \qquad |z|<|a| \tag{4.52}$$

Q2

From Eq. (4.3) and given *x*[*n*] we have

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-2}^{3} x[n] z^{-n} \\ &= x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} \\ &= 5z^2 + 3z - 2 + 4z^{-2} - 3z^{-3} \end{aligned}$$

For *z* not equal to zero or infinity, each term in *X*(*z*) will be finite and consequently *X*(*z*) will converge. Note that *X*(*z*) includes both positive powers of *z* and negative powers of *z*. Thus, from the result of Prob. 4.2 we conclude that the ROC of *X*(*z*) is $0 < |z|\ \infty$.

Q3

(a) From Table 4-1

$$\left(\frac{1}{2}\right)^n u[n] \leftrightarrow \frac{z}{z-\frac{1}{2}} \qquad |z| > \frac{1}{2} \tag{4.58}$$

$$\left(\frac{1}{3}\right)^n u[n] \leftrightarrow \frac{z}{z-\frac{1}{3}} \qquad |z| > \frac{1}{3} \tag{4.59}$$

We see that the ROCs in Eqs. (4.58) and (4.59) overlap, and thus,

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{3}} = \frac{2z\left(z-\frac{5}{12}\right)}{\left(s-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} \qquad |z| > \frac{1}{2} \tag{4.60}$$

From Eq. (4.60) we see that $X(z)$ has two zeros at $z = 0$ and $z = \frac{5}{12}$ and two poles at $z = \frac{1}{2}$ and $z = \frac{1}{3}$ and that the ROC is $|z| > \frac{1}{2}$, as sketched in Fig. 4-5(a).

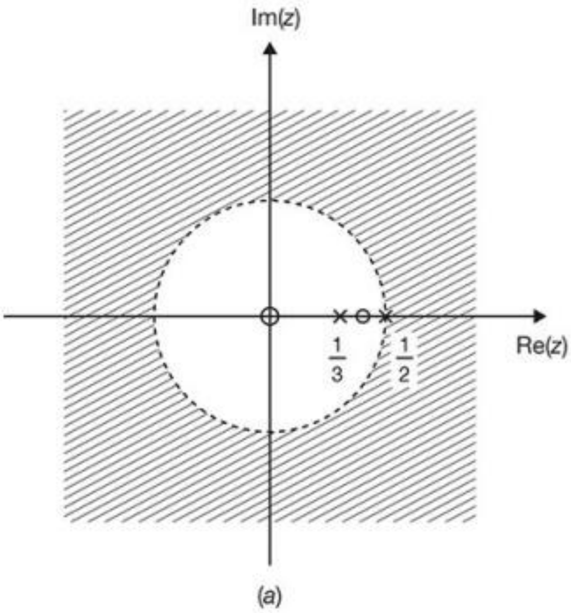

Fig. 4-5

Q4

(a) The sequence $x[n]$ is sketched in Figs. 4-6(a) and (b) for both $a < 1$ and $a > 1$.

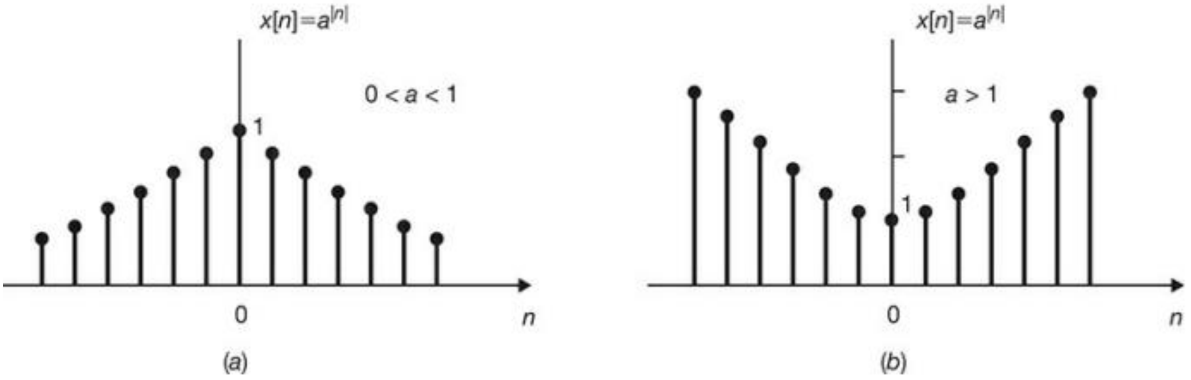

Fig. 4-6

(b) Since $x[n]$ is a two-sided sequence, we can express it as

$$x[n] = a^n u[n] + a^{-n} u[-n-1] \tag{4.67}$$

From Table 4-1

$$a^n u[n] \leftrightarrow \frac{z}{z-a} \qquad |z| > a \tag{4.68}$$

$$a^{-n} u[-n-1] \leftrightarrow -\frac{z}{z-1/a} \qquad |z| < \frac{1}{a} \tag{4.69}$$

If $a < 1$, we see that the ROCs in Eqs. (4.68) and (4.69) overlap, and thus,

$$X(z) = \frac{z}{z-a} - \frac{z}{z-1/a} = \frac{a^2-1}{a} \frac{z}{(z-a)(z-1/a)} \qquad a < |z| < \frac{1}{a} \tag{4.70}$$

From Eq. (4.70) we see that $X(z)$ has one zero at the origin and two poles at $z = a$ and $z = 1/a$ and that the ROC is $a < |z| < 1/a$, as sketched in Fig. 4-7. If $a > 1$, we see that the ROCs in Eqs. (4.68) and (4.69) do not overlap and that there is no common ROC, and thus $x[n]$ will not have $X(z)$.

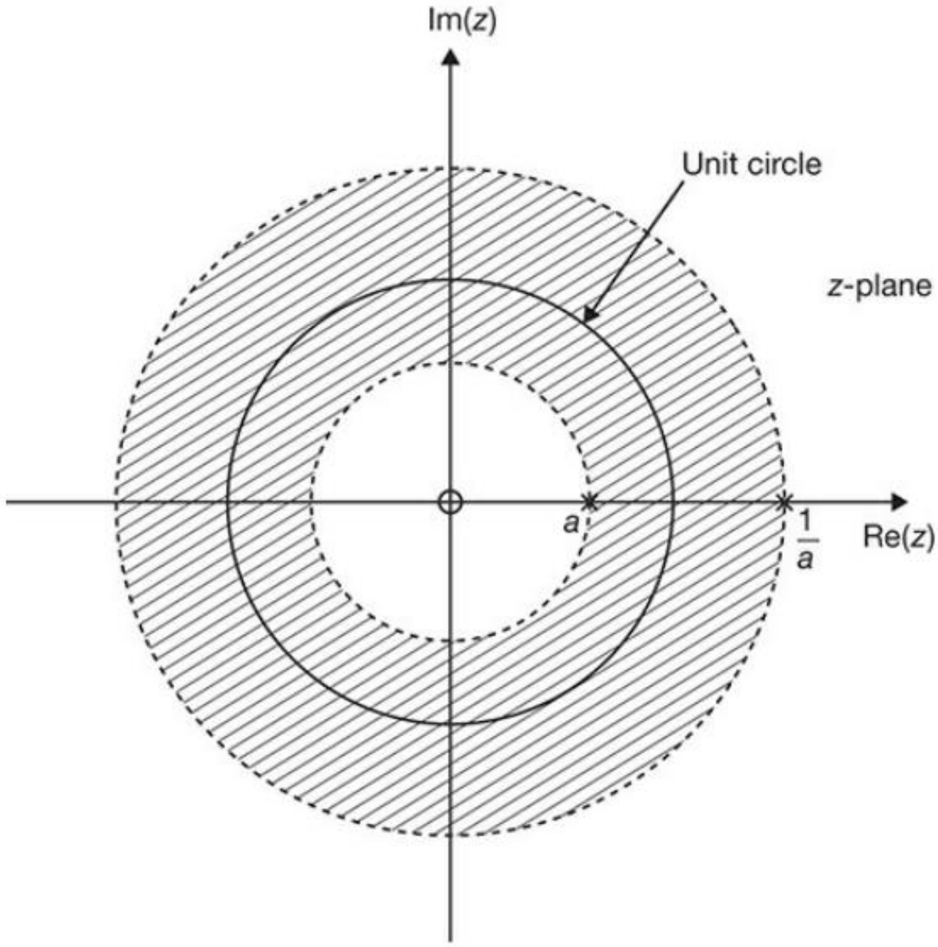

Fig. 4-7