Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 03-i

Z Transform and Discrete-Time LTI Systems -1

Contents

1. Using difference equation to describe a system
2. The Z Transform
3. Z Transform of some common sequences
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Reference:

Signals and Systems 2nd Edition – Oppenheim, Willsky
Schaum's Outline Series: Signals and Systems

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1. Using difference equation to describe a system

- Analogue description: y(t) = F(x(t)). For example: y(t)= sin(x(t))
- For discrete control, we no longer have a continuous time domain function. Instead we have a series of numbers: 0, 1, 2, 3, 5, 10,……….
- Let us consider a set of real numbers with index value k, where k = (0, 1, 2, 3…).

For a transfer function with input u(k), and output y(k):

$$u(k) = f[y(k), y(k-1), \ldots, y(k-m), u(k-1), u(k-2), \ldots, u(k-n)] \qquad (2.2.1)$$

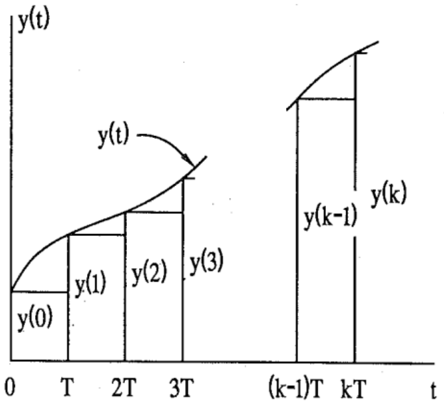
Of course there are an infinite number of ways the $n + m + 1$ values on the right side can be combined to form $u(k)$, but for the majority of this book we shall be interested only in the case where the right side involves a linear combination of the measurements and past controls, or

$$u(k) = b_{n-1}u(k-1) + \cdots + b_0u(k-n) + a_my(k) + a_{m-1}y(k-1) + \cdots + a_0y(k-m) \qquad (2.2.2)$$

k:	present
k-1:	previous sample
a_i, b_j:	weighing factors

For example, if we want to integrate the function of curve below by rectangular approximation:

$$x(k) = x(k-1) + y(k-1)T$$



1. The Z Transform

In Sec. 2.8 we saw that for a discrete-time LTI system with impulse response $h[n]$, the output $y[n]$ of the system to the complex exponential input of the form z^n is

$$y[n] = \mathbf{T}\{z^n\} = H(z)z^n \tag{4.1}$$

where

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \tag{4.2}$$

A. Definition:

The function $H(z)$ in Eq. (4.2) is referred to as the z-transform of $h[n]$. For a general discrete-time signal $x[n]$, the z-transform $X(z)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{4.3}$$

The variable z is generally complex-valued and is expressed in polar form as

$$z = re^{j\Omega} \tag{4.4}$$

where r is the magnitude of z and Ω is the angle of z. The z-transform defined in Eq. (4.3) is often called the *bilateral* (or *two-sided*) z-transform in contrast to the *unilateral* (or *one-sided*) z-transform, which is defined as

$$X_I(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \tag{4.5}$$

The $x[n]$ and $X(z)$ are said to form a z-transform pair denoted as

$$x[n] \leftrightarrow X(z) \tag{4.7}$$

B. The Region of Convergence:

EXAMPLE 4.1 Consider the sequence

$$x[n] = a^n u[n] \qquad a \text{ real} \tag{4.8}$$

Then by Eq. (4.3) the z-transform of $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

For the convergence of $X(z)$ we require that

$$\sum_{n=0}^{\infty} \left| az^{-1} \right|^n < \infty$$

Thus, the ROC is the range of values of z for which $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$. Then

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} \qquad |z| > |a| \tag{4.9}$$

Alternatively, by multiplying the numerator and denominator of Eq. (4.9) by z, we may write $X(z)$ as

$$X(z) = \frac{z}{z - a} \qquad |z| > |a| \tag{4.10}$$

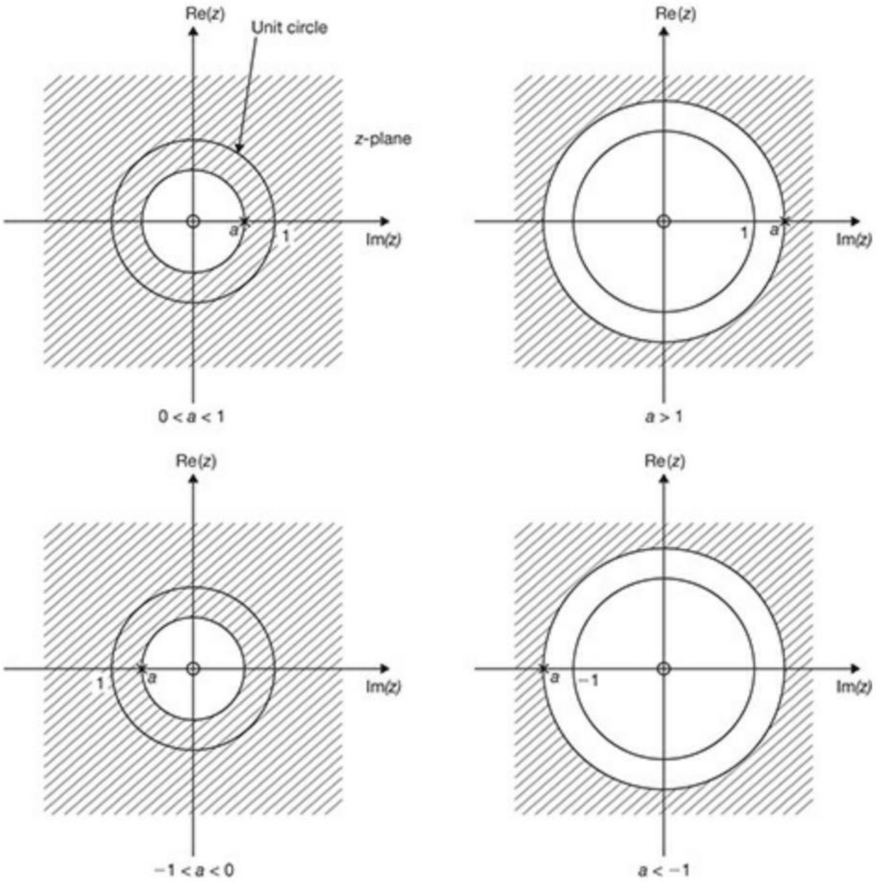


Fig. 4-1 ROC of the form $|z| > |a|$.

EXAMPLE 4.2 Consider the sequence

$$x[n] = -a^n u[-n-1] \tag{4.11}$$

Its z-transform $X(z)$ is given by (Prob. 4.1)

$$X(z) = \frac{1}{1-az^{-1}} \qquad |z| < |a| \tag{4.12}$$

Again, as before, $X(z)$ may be written as

$$X(z) = \frac{z}{z-a} \qquad |z| < |a| \tag{4.13}$$

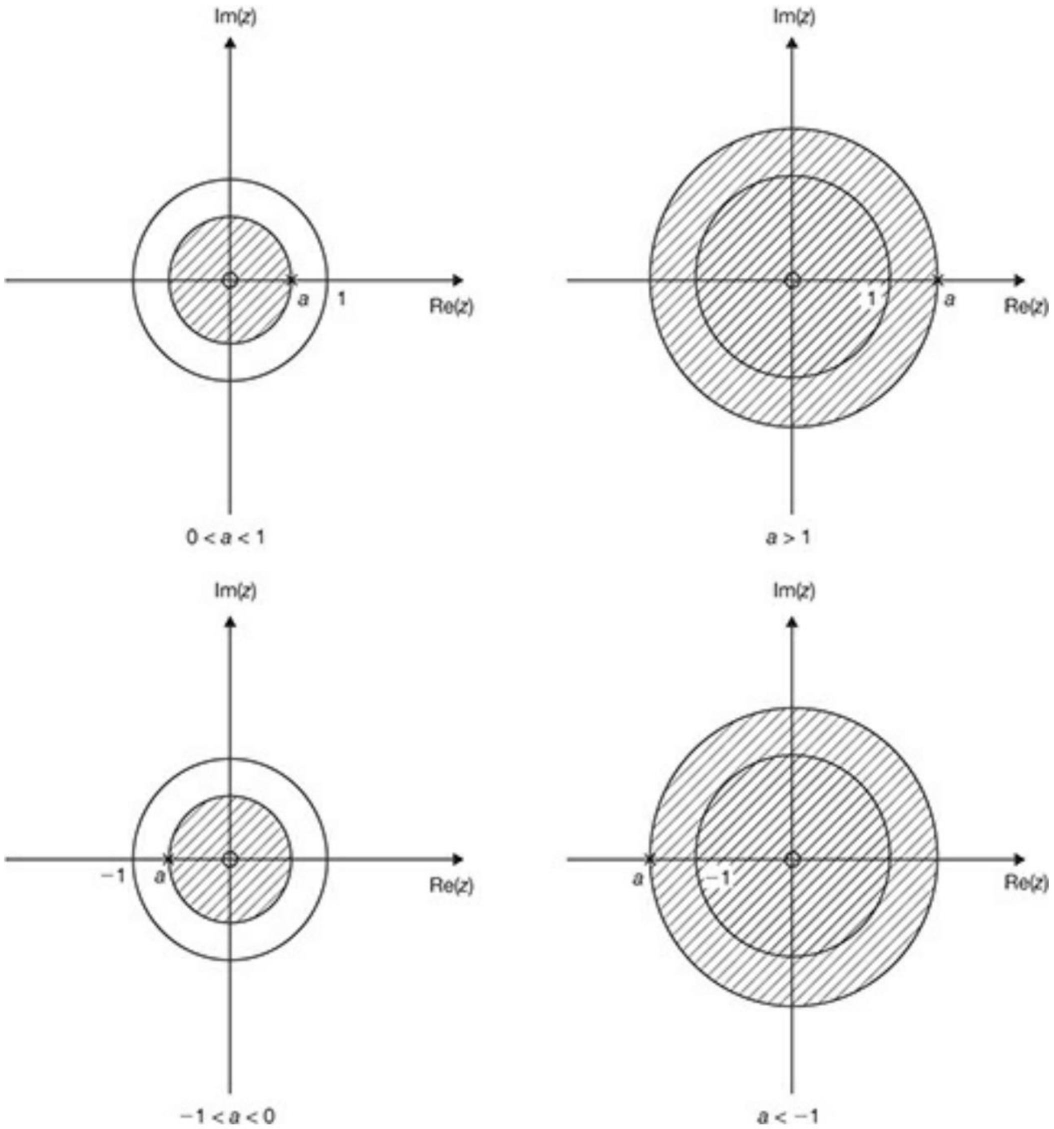


Fig. 4-2 ROC of the form $|z| < |a|$.

C. Properties of the ROC:

Property 1: The ROC does not contain any poles.

Property 2: If $x[n]$ is a finite sequence (that is, $x[n] = 0$ except in a finite interval $N_1 \le n \le N_2$, where N_1 and N_2 are finite) and $X(z)$ converges for some value of z, then the ROC is the entire z-plane except possibly $z = 0$ or $z = \infty$.

Property 3: If $x[n]$ is a right-sided sequence (that is, $x[n] = 0$ for $n < N_1 < \infty$) and $X(z)$ converges for some value of z, then the ROC is of the form

$$|z| > r_{\max} \qquad \text{or} \qquad \infty > |z| > r_{\max}$$

where r_{max} equals the largest magnitude of any of the poles of $X(z)$. Thus, the ROC is the exterior of the circle $|z| = r_{max}$ in the z-plane with the possible exception of $z = \infty$.

Property 4: If $x[n]$ is a left-sided sequence (that is, $x[n] = 0$ for $n > N_2 > -\infty$) and $X(z)$ converges for some value of z, then the ROC is of the form

$$|z| < r_{\min} \qquad \text{or} \qquad 0 < |z| < r_{\min}$$

where r_{min} is the smallest magnitude of any of the poles of $X(z)$. Thus, the ROC is the interior of the circle $|z|$ r_{min} in the z-plane with the possible exception of $z = 0$.

Property 5: If $x[n]$ is a two-sided sequence (that is, $x[n]$ is an infinite-duration sequence that is neither right-sided nor left-sided) and $X(z)$ converges for some value of z, then the ROC is of the form

$$r_1 < |z| < r_2$$

where r_1 and r_2 are the magnitudes of the two poles of $X(z)$. Thus, the ROC is an annular ring in the z-plane between the circles $|z| = r_1$ and $|z| = r_2$ not containing any poles.

2. Z Transform of some common sequences

A. Unit Impulse Sequence $\delta[n]$:

From definitions (1.45) and (4.3)

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = z^{-0} = 1 \qquad \text{all } z \tag{4.14}$$

Thus,

$$\delta[n] \leftrightarrow 1 \qquad \text{all } z \tag{4.15}$$

B. Unit Step Sequence $u[n]$:

Setting $a = 1$ in Eqs. (4.8) to (4.10), we obtain

$$u[n] \leftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad |z| > 1 \tag{4.16}$$

TABLE 4-1 Some Common z-Transform Pairs

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 if $(m > 0)$ or ∞ if $(m < 0)$
$a^n u[n]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\|z\| < \|a\|$
$(n+1)a^n u[n]$	$\frac{1}{(1-az^{-1})^2}, \left[\frac{z}{z-a}\right]^2$	$\|z\| > \|a\|$
$(\cos \Omega_0 n)u[n]$	$\frac{z^2 - (\cos \Omega_0)z}{z^2 - (2\cos \Omega_0)z + 1}$	$\|z\| > 1$
$(\sin \Omega_0 n)u[n]$	$\frac{(\sin \Omega_0)z}{z^2 - (2\cos \Omega_0)z + 1}$	$\|z\| > 1$

Examples

Assumptions:

- Only interesting in the positive time values (i.e. one-sided z-transform)
- There is some region of the complex z-plane where the series of F(z) will converge to a limit value.
- z^{-1} is the previous value, z^{-2} is the previous previous value.
- Under the above assumptions, the z-transform is denoted by:

$$F(z) = \mathscr{Z}[f(k)] = f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots$$

or

$$F(z) = \mathscr{Z}[f(k)] = \sum_{k=0}^{\infty} f(k)z^{-k}$$

Example 1: unit step sequence

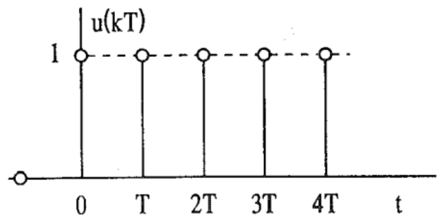
Consider the unit-step sequence of Fig. 2.2a. The function is defined as

$$u(kT) = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases} \tag{2.3.3}$$

By application of the definition of the z-transform (2.3.2) and that of the function (2.3.3) which defines the samples, we get

$$U(z) = \mathscr{Z}[u(kT)] = \sum_{k=0}^{\infty} z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \tag{2.3.4}$$

It is clear that this series converges for $|z| > 1$, and a glance at a set of



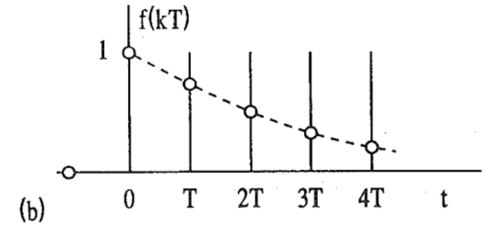
(a)

ordinary math tables will give the limiting form of such a convergent geometric series as

$$\mathscr{Z}[u(kT)] = \frac{z}{z-1} \qquad \text{for } |z| > 1 \tag{2.3.5}$$

which can easily be verified by long division. The requirement that $|z| > 1$ defines what is known as the region of convergence, which in this case is the area of the complex z-plane exterior to the unit circle.

Example 2: exponential function



Consider now the sampled exponential function illustrated in Fig. 2.2b. The sequence is defined by

$$f(k) = f(kT) = \begin{cases} 0 & k < 0 \\ e^{-akT} & k \geq 0 \end{cases} \qquad (2.3.6)$$

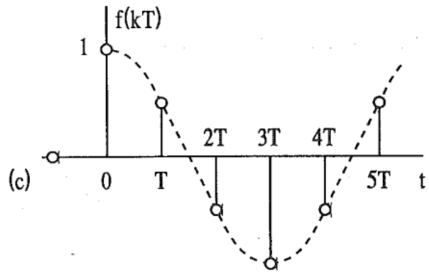
Substitution of (2.3.6) into (2.3.3) yields

$$\mathcal{Z}[e^{-akT}] = \sum_{k=0}^{\infty} (e^{-aT}z^{-1})^k \qquad (2.3.7)$$

and from the previous transform the limit of the series is

$$\mathcal{Z}[e^{-akT}] = \frac{z}{z - e^{-aT}} \qquad \text{for } |z| > e^{-aT} \qquad (2.3.8)$$

Example 3: cosine function



Consider the sampled cosine function of radian frequency Ω which is shown in Fig. 2.2c. The sequence is defined by

$$f(k) = f(kT) = \begin{cases} 0 & k < 0 \\ \cos k\Omega T & k \geq 0 \end{cases} \qquad (2.3.13)$$

The cosine function can be rewritten using the Euler identity as

$$\cos k\Omega T = \frac{1}{2}(e^{jk\Omega T} + e^{-jk\Omega T}) \qquad (2.3.14)$$

Since the z-transform of a sum is the sum of individual z-transforms, the result of (2.3.8) can be used to give

$$\mathcal{Z}[\cos k\Omega T] = \frac{1}{2}\left(\frac{z}{z - e^{j\Omega T}} + \frac{z}{z - e^{-j\Omega T}}\right) \qquad (2.3.15)$$

and finding a common denominator yields

$$\mathcal{Z}[\cos k\Omega T] = \frac{z^2 - z\cos\Omega T}{z^2 - z \cdot 2\cos\Omega T + 1} \qquad (2.3.16)$$

The region of convergence is the region of the z-plane exterior to the unit circle. The sampled sine function will be left as an exercise for the reader but is given in Table 2.1.

Example 4: impulse function

Consider a sequence $\delta(k)$ that is defined by

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{2.3.17}$$

Using the definition of (2.3.2) the result is

$$\mathscr{Z}[\delta(k)] = 1 \tag{2.3.18}$$

In a similar fashion we can show that a delayed impulse function defined by

$$\delta(k - n) = \begin{cases} 0 & k \neq n \\ 1 & k = n > 0 \end{cases} \tag{2.3.19}$$

has a z-transform

$$\mathscr{Z}[\delta(k - n)] = z^{-n} \tag{2.3.20}$$

Example 5: ramp function

The sampled ramp function is defined by

$$f(k) = kT \qquad k = 0, 1, 2, \ldots \tag{2.3.21}$$

and application of definition (2.3.2) yields

$$\mathscr{Z}[kT] = T \sum_{k=0}^{\infty} kz^{-k} \tag{2.3.22}$$

Again consulting a table of mathematical functions, the limit of the series is

$$\mathscr{Z}[kT] = \frac{Tz}{(z - 1)^2} \qquad \text{for } |z| > 1 \tag{2.3.23}$$

3. Properties of Z Transform

A. Linearity:

If

$$x_1[n] \leftrightarrow X_1(z) \qquad \text{ROC} = R_1$$
$$x_2[n] \leftrightarrow X_2(z) \qquad \text{ROC} = R_2$$

then

$$a_1x_1[n] + a_2x_2[n] \leftrightarrow a_1X_1(z) + a_2X_2(z) \qquad R' \supset R_1 \cap R_2 \tag{4.17}$$

where a_1 and a_2 are arbitrary constants.

B. Time Shifting:

If

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

then

$$x[n-n_0] \leftrightarrow z^{-n_0}X(z) \qquad R' = R \cap \{0 < |z| < \infty\} \tag{4.18}$$

Special Cases:

$$x[n-1] \leftrightarrow z^{-1}X(z) \qquad R' = R \cap \{0 < |z|\} \tag{4.19}$$

$$x[n+1] \leftrightarrow zX(z) \qquad R' = R \cap \{|z| < \infty\} \tag{4.20}$$

Because of these relationship [Eqs. (4.19) and (4.20)], z^{-1} is often called the *unit-delay operator* and z is called the *unit-advance operator*. Note that in the Laplace transform the operators $s^{-1} = 1/s$ and s correspond to time-domain integration and differentiation, respectively [Eqs. (3.22) and (3.20)].

C. Multiplication by z_0^n:

If

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

then

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \qquad R' = |z_0|R \tag{4.21}$$

In particular, a pole (or zero) at $z = z_k$ in $X(z)$ moves to $z = z_0z_k$ after multiplication by z_0^n and the ROC expands or contracts by the factor $|z_0|$.

If

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

then

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \qquad R' = |z_0| R \tag{4.21}$$

In particular, a pole (or zero) at $z = z_k$ in $X(z)$ moves to $z = z_0 z_k$ after multiplication by z_0^n and the ROC expands or contracts by the factor $|z_0|$.

Special Case:

$$e^{j\Omega_0 n} x[n] \leftrightarrow X(e^{-j\Omega_0} z) \qquad R' = R \tag{4.22}$$

In this special case, all poles and zeros are simply rotated by the angle Ω_0 and the ROC is unchanged.

D. Time Reversal:

If

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

then

$$x[-n] \leftrightarrow X\left(\frac{1}{z}\right) \qquad R' = \frac{1}{R} \tag{4.23}$$

Therefore, a pole (or zero) in $X(z)$ at $z = z_k$ moves to $1/z_k$ after time reversal. The relationship $R' = 1/R$ indicates the inversion of R, reflecting the fact that a right-sided sequence becomes left-sided if time-reversed, and vice versa.

E. Multiplication by *n* (or Differentiation in *z*):

If

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

then

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz} \qquad R' = R \tag{4.24}$$

F. Accumulation:

If

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

then

$$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \frac{1}{1-z^{-1}}X(z) = \frac{z}{z-1}X(z) \qquad R' \supset R \cap \{|z| > 1\} \tag{4.25}$$

Note that $\sum_{k=-\infty}^{n} x[k]$ is the discrete-time counterpart to integration in the time domain and is called the *accumulation*. The comparable Laplace transform operator for integration is $1/s$.

G. Convolution:

If

$$x_1[n] \leftrightarrow X_1(z) \qquad \text{ROC} = R_1$$
$$x_2[n] \leftrightarrow X_2(z) \qquad \text{ROC} = R_2$$

then

$$x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z) \qquad R' \supset R_1 \cap R_2 \tag{4.26}$$

This relationship plays a central role in the analysis and design of discrete-time LTI systems, in analogy with the continuous-time case.

H. Summary of Some *z*-transform Properties:

For convenient reference, the properties of the *z*-transform presented above are summarized in Table 4-2.

TABLE 4-2. Some Properties of the z-Transform

PROPERTY	SEQUENCE	TRANSFORM	ROC
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(z) + a_2X_2(z)$	$R' \supset R_1 \cap R_2$
Time shifting	$x[n-n_0]$	$z^{-n_0} X(z)$	$R' \supset R \cap \{0 < \|z\| < \infty\}$
Multiplication by z_0^n	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$R' = \|z_0\| R$
Multiplication by $e^{j\Omega_0 n}$	$e^{j\Omega_0 n} x[n]$	$X(e^{-j\Omega_0} z)$	$R' = R$
Time reversal	$x[-n]$	$X\left(\frac{1}{z}\right)$	$R' = \frac{1}{R}$
Multiplication by n	$nx[n]$	$-z\frac{dX(z)}{dz}$	$R' = R$
Accumulation	$\sum_{k=-\infty}^{n} x[n]$	$\frac{1}{1-z^{-1}} X(z)$	$R' \supset R \cap \{\|z\| > 1\}$
Convolution	$x_1[n] * x_2[n]$	$X_1(z) X_2(z)$	$R' \supset R_1 \cap R_2$

S3. Glossary – English/Chinese Translation

English	Chinese
difference equation	差分方程
z-transform	z 变换
rectangular approximation	矩形近似
discrete time LTI system	离散时间 LTI 系统
bilateral z-transform	双边 z 变换
unilateral z-transform	单边 z 变换
region of convergence	收敛区域
unit impulse sequence	单位脉冲序列
unit step sequence	单位步长序列

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