Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 03-g

Fourier Series of Continuous Time Systems

Contents

Reference:
Signals and Systems 2nd Edition – Oppenheim, Willsky

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1. Combination of Harmonically Related Signals

As defined in Chapter 1, a signal is periodic if, for some positive value of T,

$$x(t) = x(t+T) \quad \text{for all } t. \tag{3.21}$$

The fundamental period of $x(t)$ is the minimum positive, nonzero value of T for which eq. (3.21) is satisfied, and the value $\omega_0 = 2\pi/T$ is referred to as the fundamental frequency.

In Chapter 1 we also introduced two basic periodic signals, the sinusoidal signal

$$x(t) = \cos \omega_0 t \tag{3.22}$$

and the periodic complex exponential

$$x(t) = e^{j\omega_0 t}. \tag{3.23}$$

Both of these signals are periodic with fundamental frequency ω_0 and fundamental period $T = 2\pi/\omega_0$. Associated with the signal in eq. (3.23) is the set of *harmonically related* complex exponentials

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{3.24}$$

Each of these signals has a fundamental frequency that is a multiple of ω_0, and therefore, each is periodic with period T (although for $|k| \geq 2$, the fundamental period of $\phi_k(t)$ is a fraction of T). Thus, a linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \tag{3.25}$$

Example 3.2

Consider a periodic signal $x(t)$, with fundamental frequency 2π, that is expressed in the form of eq. (3.25) as

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}, \tag{3.26}$$

where

$$a_0 = 1,$$

$$a_1 = a_{-1} = \frac{1}{4},$$

$$a_2 = a_{-2} = \frac{1}{2},$$

$$a_3 = a_{-3} = \frac{1}{3}.$$

Rewriting eq. (3.26) and collecting each of the harmonic components which have the same fundamental frequency, we obtain

$$x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t}). \tag{3.27}$$

Equivalently, using Euler's relation, we can write $x(t)$ in the form

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t. \tag{3.28}$$

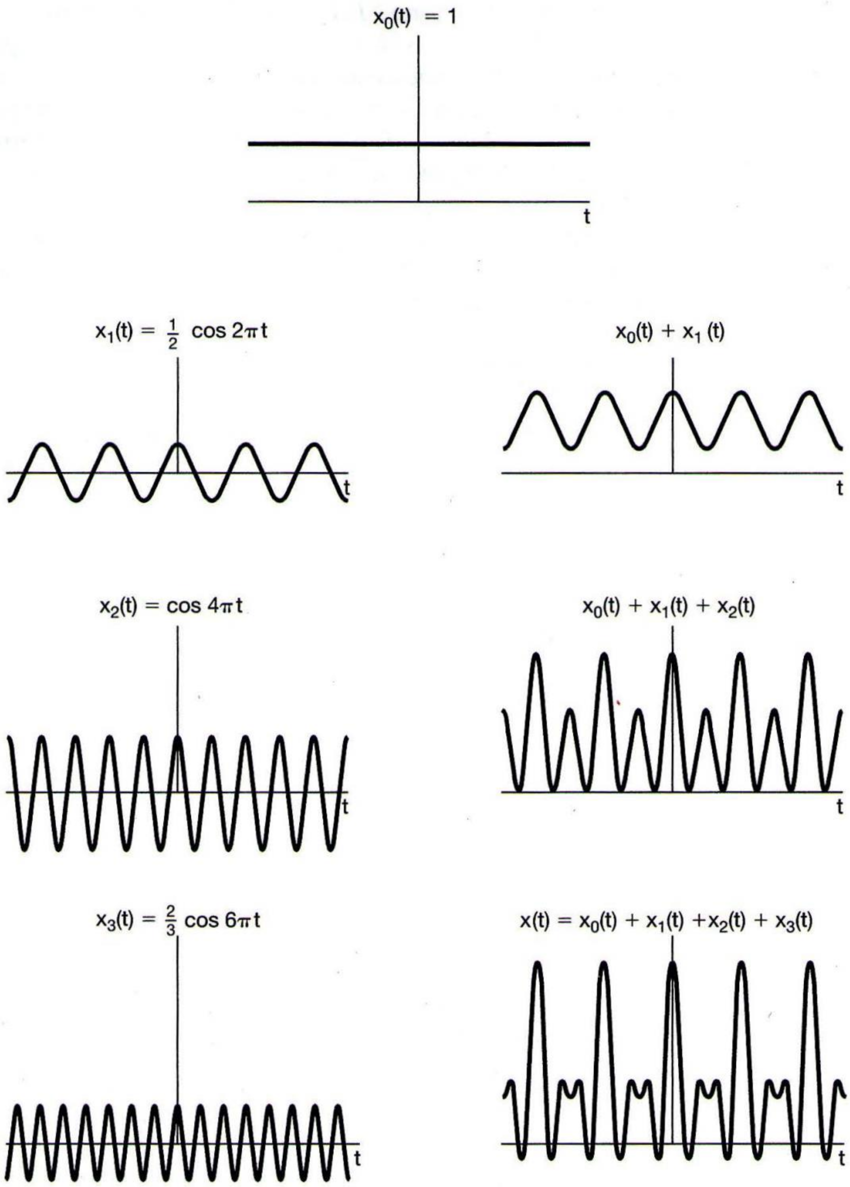

Figure 3.4 Construction of the signal $x(t)$ in Example 3.2 as a linear combination of harmonically related sinusoidal signals.

2. The Fourier Series Representation

To summarize, if $x(t)$ has a Fourier series representation [i.e., if it can be expressed as a linear combination of harmonically related complex exponentials in the form of eq. (3.25)], then the coefficients are given by eq. (3.37). This pair of equations, then, defines the Fourier series of a periodic continuous-time signal:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}, \tag{3.38}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt. \tag{3.39}$$

Here, we have written equivalent expressions for the Fourier series in terms of the fundamental frequency ω_0 and the fundamental period T. Equation (3.38) is referred to as the *synthesis* equation and eq. (3.39) as the *analysis* equation. The set of coefficients $\{a_k\}$ are often called the *Fourier series coefficients* or the *spectral coefficients* of $x(t)$.[8] These complex coefficients measure the portion of the signal $x(t)$ that is at each harmonic of the fundamental component. The coefficient a_0 is the dc or constant component of $x(t)$ and is given by eq. (3.39) with $k = 0$. That is,

$$a_0 = \frac{1}{T}\int_T x(t)\,dt, \tag{3.40}$$

which is simply the average value of $x(t)$ over one period.

Equations (3.38) and (3.39) were known to both Euler and Lagrange in the middle of the 18th century. However, they discarded this line of analysis without having

Example 3.3

Consider the signal

$$x(t) = \sin\omega_0 t,$$

whose fundamental frequency is ω_0. One approach to determining the Fourier series coefficients for this signal is to apply eq. (3.39). For this simple case, however, it is easier to expand the sinusoidal signal as a linear combination of complex exponentials and identify the Fourier series coefficients by inspection. Specifically, we can express $\sin\omega_0 t$ as

$$\sin\omega_0 t = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}.$$

Comparing the right-hand sides of this equation and eq. (3.38), we obtain

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j},$$
$$a_k = 0, \qquad k \neq +1 \text{ or } -1.$$

Example 3.4

Let

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

which has fundamental frequency ω_0. As with Example 3.3, we can again expand $x(t)$ directly in terms of complex exponentials, so that

$$x(t) = 1 + \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2}[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}].$$

Collecting terms, we obtain

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}.$$

Thus, the Fourier series coefficients for this example are

$$a_0 = 1,$$

$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j,$$

$$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j,$$

$$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j),$$

$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1 - j),$$

$$a_k = 0, \ |k| > 2.$$

In Figure 3.5, we show a bar graph of the magnitude and phase of a_k.

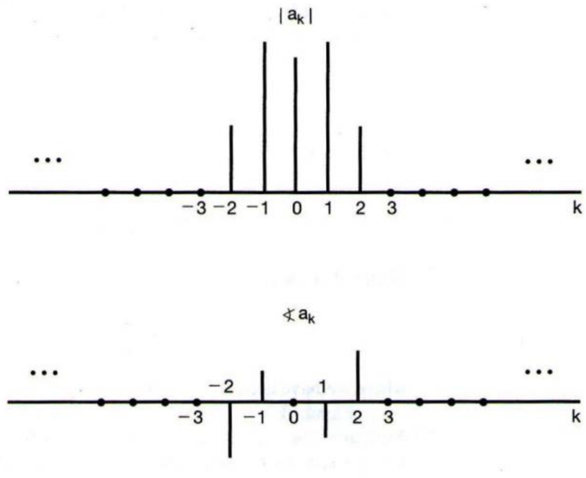

Figure 3.5 Plots of the magnitude and phase of the Fourier coefficients of the signal considered in Example 3.4.

3. Convergence of the Fourier Series – The Dirichlet Conditions

Condition 1. Over any period, $x(t)$ must be *absolutely integrable*; that is,

$$\int_T |x(t)|\,dt < \infty. \tag{3.56}$$

As with square integrability, this guarantees that each coefficient a_k will be finite, since

$$|a_k| \le \frac{1}{T}\int_T |x(t)e^{-jk\omega_0 t}|\,dt = \frac{1}{T}\int_T |x(t)|\,dt.$$

So if

$$\int_T |x(t)|\,dt < \infty,$$

then

$$|a_k| < \infty.$$

A periodic signal that violates the first Dirichlet condition is

$$x(t) = \frac{1}{t}, \quad 0 < t \le 1;$$

that is, $x(t)$ is periodic with period 1. This signal is illustrated in Figure 3.8(a).

Condition 2. In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

An example of a function that meets Condition 1 but not Condition 2 is

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \le 1, \tag{3.57}$$

as illustrated in Figure 3.8(b). For this function, which is periodic with $T = 1$,

$$\int_0^1 |x(t)|\,dt < 1.$$

The function has, however, an infinite number of maxima and minima in the interval.

Condition 3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

An example of a function that violates Condition 3 is illustrated in Figure 3.8(c). The signal, of period $T = 8$, is composed of an infinite number of sections, each of which is half the height and half the width of the previous section. Thus, the area under one period of the function is clearly less than 8. However, there are an infinite number of discontinuities in each period, thereby violating Condition 3.

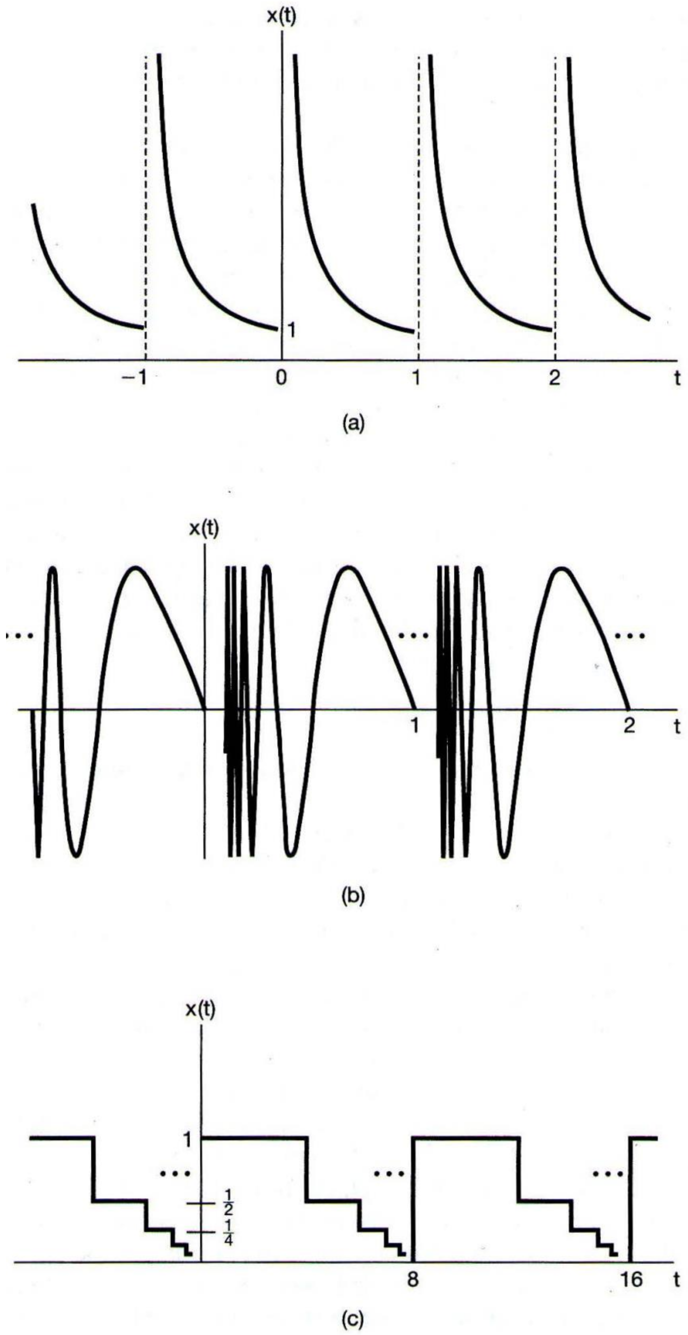

Figure 3.8 Signals that violate the Dirichlet conditions: (a) the signal $x(t) = 1/t$ for $0 < t \le 1$, a periodic signal with period 1 (this signal violates the first Dirichlet condition); (b) the periodic signal of eq. (3.57), which violates the second Dirichlet condition; (c) a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \le t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1,\ 0 \le t < 4,\ x(t) = 1/2,\ 4 \le t < 6,\ x(t) = 1/4,\ 6 \le t < 7,\ x(t) = 1/8,\ 7 \le t < 7.5$, etc.].

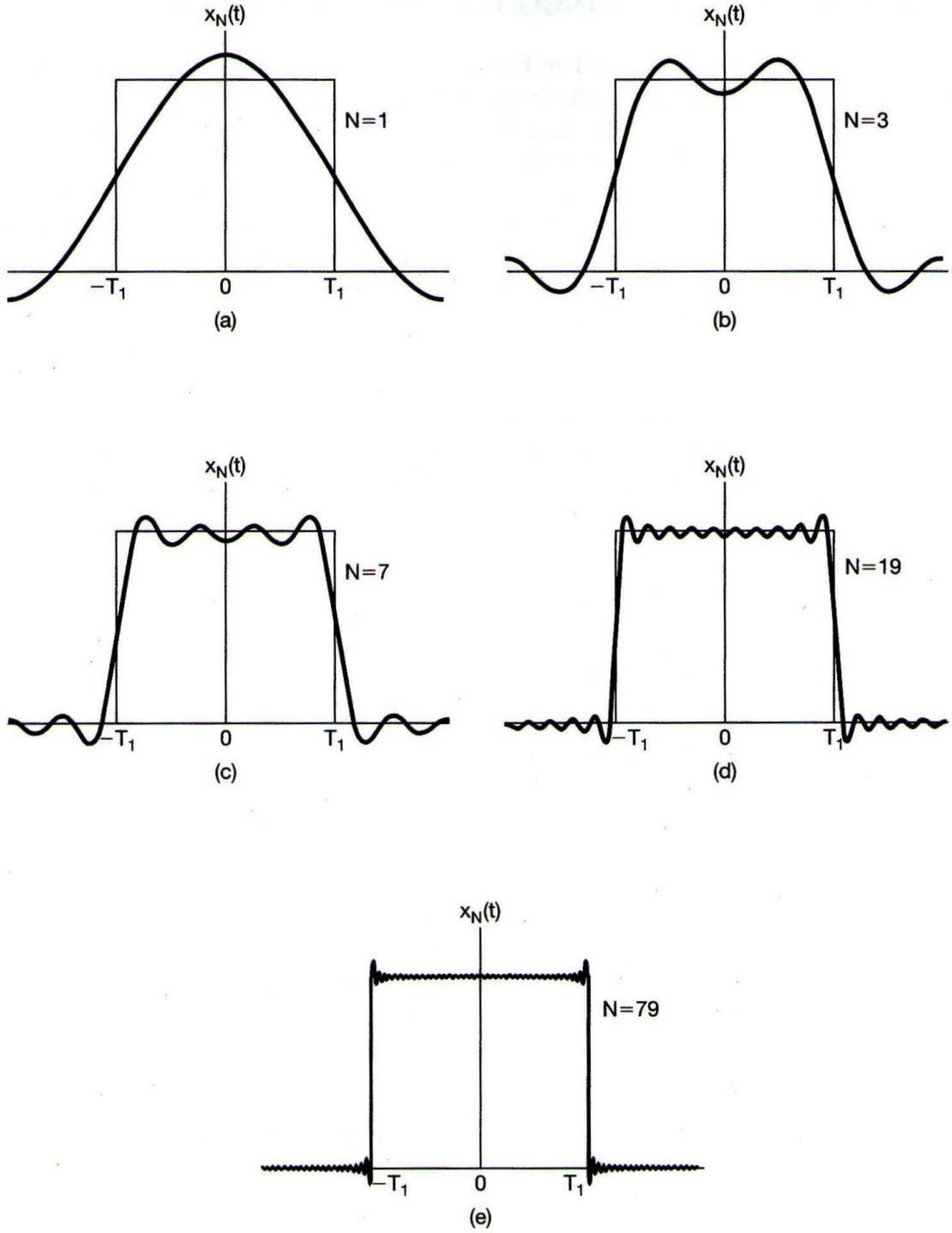

Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

3 types of Fourier Series – A Summary

A. Periodic Signals:

In Chap. 1 we defined a continuous-time signal $x(t)$ to be periodic if there is a positive nonzero value of T for which

$$x(t+T) = x(t) \qquad \text{all } t \tag{5.1}$$

The fundamental period T_0 of $x(t)$ is the smallest positive value of T for

which Eq. (5.1) is satisfied, and $1/T_0 = f_0$ is referred to as the *fundamental frequency*.

Two basic examples of periodic signals are the real sinusoidal signal

$$x(t) = \cos(\omega_0 t + \phi) \tag{5.2}$$

and the complex exponential signal

$$x(t) = e^{j\omega_0 t} \tag{5.3}$$

where $\omega_0 = 2\pi/T_0 = 2\pi f_0$ is called the *fundamental angular frequency*.

B. Complex Exponential Fourier Series Representation:

The complex exponential Fourier series representation of a periodic signal $x(t)$ with fundamental period T_0 is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0} \tag{5.4}$$

where c_k are known as the *complex Fourier coefficients* and are given by

$$c_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt \tag{5.5}$$

where $\int_{T_0}$ denotes the integral over any one period and 0 to T_0 or $-T_0/2$ to $T_0/2$ is commonly used for the integration. Setting $k = 0$ in Eq. (5.5), we have

$$c_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt \tag{5.6}$$

which indicates that c_0 equals the average value of $x(t)$ over a period.

When $x(t)$ is real, then from Eq. (5.5) it follows that

$$c_{-k} = c_k^* \tag{5.7}$$

where the asterisk indicates the complex conjugate.

C. Trigonometric Fourier Series:

The trigonometric Fourier series representation of a periodic signal $x(t)$ with fundamental period T_0 is given by

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \qquad \omega_0 = \frac{2\pi}{T_0} \tag{5.8}$$

where a_k and b_k are the Fourier coefficients given by

$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos k\omega_0 t \, dt \tag{5.9a}$$

$$b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin k\omega_0 t \, dt \tag{5.9b}$$

The coefficients a_k and b_k and the complex Fourier coefficients c_k are related by (Prob. 5.3)

$$\frac{a_0}{2} = c_0 \qquad a_k = c_k + c_{-k} \qquad b_k = j(c_k - c_{-k}) \tag{5.10}$$

From Eq. (5.10) we obtain

$$c_k = \frac{1}{2}(a_k - jb_k) \qquad c_{-k} = \frac{1}{2}(a_k + jb_k) \tag{5.11}$$

When $x(t)$ is real, then a_k and b_k are real and by Eq. (5.10) we have

$$a_k = 2\,\mathrm{Re}[c_k] \qquad b_k = -2\,\mathrm{Im}[c_k] \tag{5.12}$$

Even and Odd Signals:

If a periodic signal $x(t)$ is even, then $b_k = 0$ and its Fourier series (5.8) contains only cosine terms:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0} \tag{5.13}$$

If $x(t)$ is odd, then $a_k = 0$ and its Fourier series contains only sine terms:

$$x(t) = \sum_{k=1}^{\infty} b_k \sin k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0} \tag{5.14}$$

D. Harmonic Form Fourier Series:

Another form of the Fourier series representation of a real periodic signal $x(t)$ with fundamental period T_0 is

$$x(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(k\omega_0 t - \theta_k) \qquad \omega_0 = \frac{2\pi}{T_0} \tag{5.15}$$

Equation (5.15) can be derived from Eq. (5.8) and is known as the *harmonic form* Fourier series of $x(t)$. The term C_0 is known as the *dc component*, and the term $C_k \cos(k\omega_0 t - \theta_k)$ is referred to as the *kth harmonic component* of $x(t)$. The first harmonic component $C_1 \cos(\omega_0 t - \theta_1)$ is commonly called the *fundamental component* because it has the same fundamental period as $x(t)$. The coefficients C_k and the angles θ_k are called the *harmonic amplitudes* and *phase angles*, respectively, and they are related to the Fourier coefficients a_k and b_k by

$$C_0 = \frac{a_0}{2} \qquad C_k = \sqrt{a_k^2 + b_k^2} \qquad \theta_k = \tan^{-1}\frac{b_k}{a_k} \tag{5.16}$$

3. Glossary – English/Chinese Translation

English	Chinese
Fourier series	傅里叶级数
convergence	收敛
harmonics	谐波
sinusoidal signal	正弦信号
fundamental frequency	基频
Euler's rule	欧拉法则
magnitude and phase	幅度和相位
Dirichlet conditions	狄利克雷条件
exponential Fourier series	复指数傅里叶级数
trigonometric Fourier series	三角傅里叶级数
harmonic form Fourier series	谐波形式傅里叶级数

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Your Notes