Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 03-f

Laplace Transform and LTI Systems 2

Contents

Reference:
Signals and Systems 2nd Edition – Oppenheim, Willsky

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Last Updated: 2024-04

1. The Inverse Laplace Transform

Inversion of the Laplace transform to find the signal x(t) from its Laplace transform X(s) is called the inverse Laplace transform:

$$x(t) = \mathscr{L}^{-1}\{X(s)\} \tag{3.24}$$

Inversion Formula

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds \tag{3.25}$$

In this integral, the real c is to be selected such that if the ROC of X(s) is $\sigma 1 < \mathrm{Re}(s) < \sigma 2$, then $\sigma 1 < c < \sigma 2$. The evaluation of this inverse Laplace transform integral requires understanding of complex variable theory.

Partial Fraction Expansion

If X(s) is a rational function, that is, of the form:

$$X(s) = \frac{N(s)}{D(s)} = k\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} \tag{3.28}$$

a simple technique based on partial-fraction expansion can be used for the inversion of X(s).

Simple Pole Case

If all poles of X(s), that is, all zeros of D(s), are simple (or distinct), then X(s) can be written as:

$$X(s) = \frac{c_1}{s-p_1} + \cdots + \frac{c_n}{s-p_n} \tag{3.29}$$

where coefficients c_k are given by:

$$c_k = (s - p_k)X(s)\Big|_{s=p_k} \tag{3.30}$$

Multiple Pole Case

If D(s) has multiple roots, that is, if it contains factors of the form: $(s - p_i)^r$, we say that pi is the multiple pole of X(s) with multiplicity r. Then the expansion of X(s) will consist of terms of the form:

$$\frac{\lambda_1}{s - p_i} + \frac{\lambda_2}{(s - p_i)^2} + \cdots + \frac{\lambda_r}{(s - p_i)^r} \tag{3.31}$$

where:

$$\lambda_{r-k} = \frac{1}{k!}\frac{d^k}{ds^k}[(s - p_i)^r X(s)]\Big|_{s=p_i} \tag{3.32}$$

2. The System Function

System Function

The output y(t) of a continuous-time LTI system equals the convolution of the input x(t) with the impulse response h(t):

$$y(t) = x(t) * h(t) \tag{3.35}$$

Applying the convolution property (3.23), we obtain:

$$Y(s) = X(s)H(s) \tag{3.36}$$

where Y(s), X(s), and H(s) are the Laplace transforms of y(t), x(t), and h(t), respectively. Equation (3.36) can be expressed as:

$$H(s) = \frac{Y(s)}{X(s)} \tag{3.37}$$

The Laplace transform H(s) of h(t) is referred to as the system function (or the transfer function) of the system. By Eq. (3.37), the system function H(s) can also be defined as the ratio of the Laplace transforms of the output y(t) and the input x(t). The system function H(s) completely characterizes the system because the impulse response h(t) completely characterizes the system.

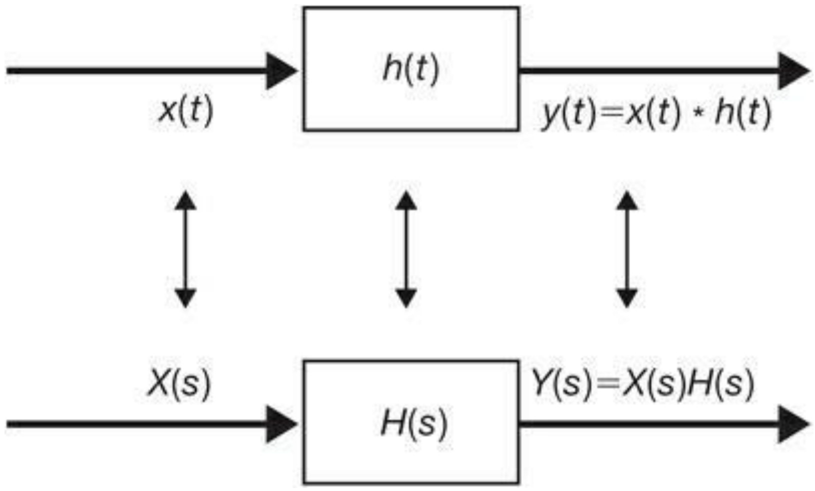

Fig. 3-7 Impulse response and system function.

Characterization of LTI Systems:

1. Causality:

For a causal continuous-time LTI system, we have:

$$h(t) = 0 \qquad t < 0$$

Since h(t) is a right-sided signal, the corresponding requirement on H(s) is that the ROC of H(s) must be of the form:

$$\mathrm{Re}(s) > \sigma_{\max}$$

That is, the ROC is the region in the s-plane to the right of all of the system poles. Similarly, if the system is anti-causal, then:

$$h(t) = 0 \qquad t > 0$$

and h(t) is left-sided. Thus, the ROC of H(s) must be of the form

$$\mathrm{Re}(s) < \sigma_{\min}$$

That is, the ROC is the region in the s-plane to the left of all of the system poles.

2. Stability

A continuous-time LTI system is BIBO stable if and only if:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

The corresponding requirement on H(s) is that the ROC of H(s) contains the jω-axis (that is, s = jω)

3. Causal and Stable Systems:

If the system is both causal and stable, then all the poles of H(s) must lie in the left half of the s-plane; that is, they all have negative real parts because the ROC is of the form Re(s) > σ_{max}, and since the jω axis is included in the ROC, we must have $\sigma_{max} < 0$.

<u>System Function for LTI Systems described by Linear Constant Coefficient Differential Equations:</u>

Considered a continuous-time LTI system for which input x(t) and output y(t):

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad (3.38)$$

Applying the Laplace transform and using the differentiation property (3.20) of the Laplace transform, we obtain:

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

or

$$Y(s)\sum_{k=0}^{N} a_k\, s^k = X(s)\sum_{k=0}^{M} b_k\, s^k \tag{3.39}$$

Thus:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k\, s^k}{\sum_{k=0}^{N} a_k\, s^k} \tag{3.40}$$

System Interconnection

For two LTI systems [with h1(t) and h2(t), respectively] in cascade, the overall impulse response h(t) is given by:

$$h(t) = h_1(t) * h_2(t)$$

Then:

$$H(s) = H_1(s)H_2(s) \tag{3.41}$$

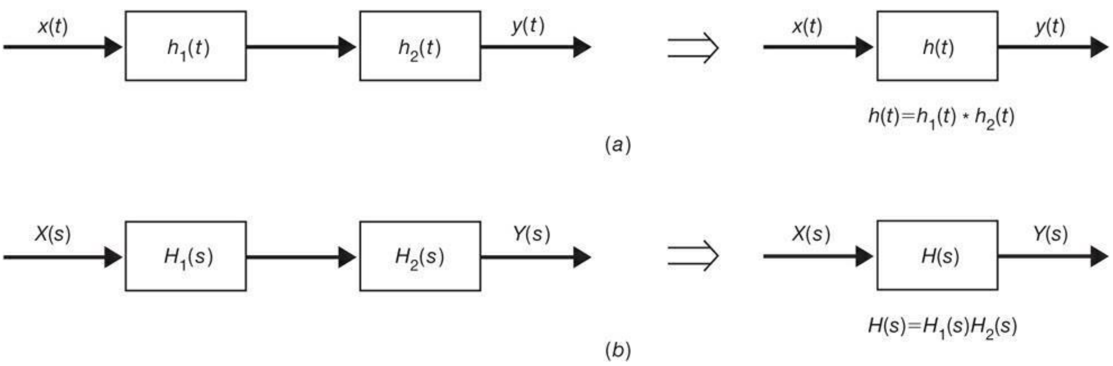

Fig. 3-8 Two systems in cascade. (a) Time-domain representation; (b) s-domain representation.

Similarly, the impulse response of a parallel combination of two LTI Systems:

$$h(t) = h_1(t) + h_2(t)$$

becomes:

$$H(s) = H_1(s) + H_2(s) \tag{3.42}$$

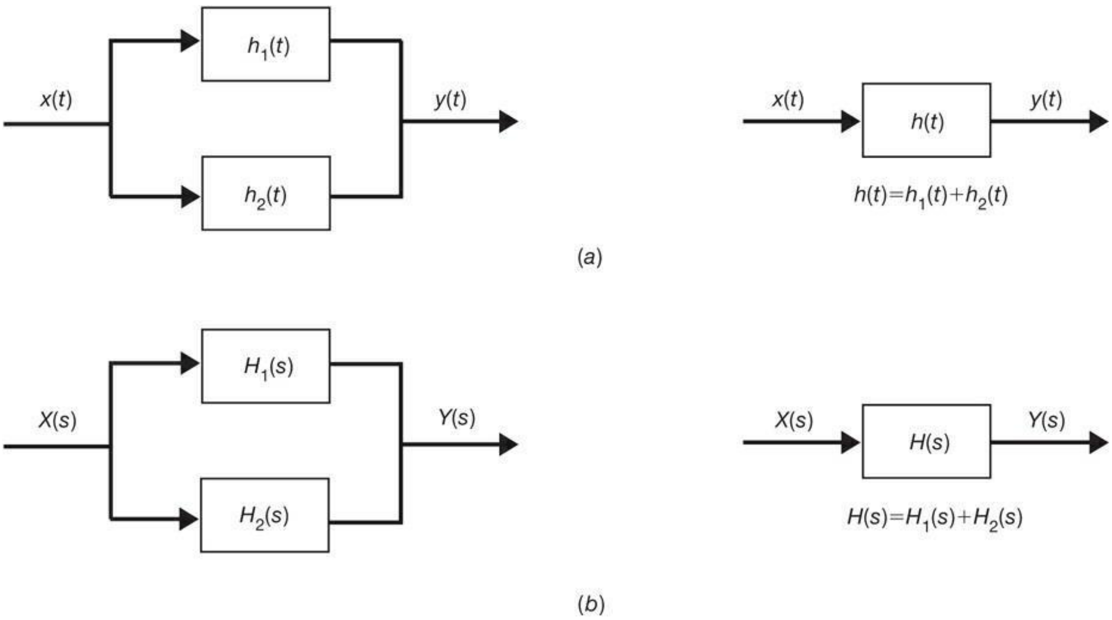

Fig. 3-9 Two systems in parallel. (a) Time-domain representation; (b) s-domain representation.

4. The Unilateral Laplace Transform

Definition

The unilateral (or one-sided) Laplace transform $X_I(s)$ of a signal x(t) is defined as:

$$X_I(s) = \int_{0^-}^{\infty} x(t) s^{-st} dt \tag{3.43}$$

The lower limit of integration is chosen to be 0^-(rather than 0 or 0^+) to permit x(t) to include δ(t) or its derivatives. Thus, we note immediately that the integration from 0^- to 0^+ is zero except when there is an impulse function or its derivative at the origin. The unilateral Laplace transform ignores x(t) for t < 0. Since x(t) is a right-sided signal, the ROC of $X_1(s)$ is always of the form Re(s) > σ_{max}, that is, a right half-plane in the s-plane.

Basic Properties

Most of the properties of the unilateral Laplace transform are the same as for the bilateral transform. The unilateral Laplace transform is useful for calculating the response of a causal system to a causal input when the system is described by a linear constant-coefficient differential equation.

The basic properties of the unilateral Laplace transform that are useful in this application are the time-differentiation and time-integration with non-zero initial condition properties which are different from those of the bilateral transform.

1. Differentiation in the Time Domain:

$$\frac{dx(t)}{dt} \leftrightarrow sX_I(s) - x(0^-) \tag{3.44}$$

provided that $\lim_{t\to\infty} x(t)e^{st-} = 0$. Repeated application of this property yields

$$\frac{d^2x(t)}{dt^2} \leftrightarrow s^2X_I(s) - sx(0^-) - x'(0^-) \tag{3.45}$$

$$\frac{d^nx(t)}{dt^n} \leftrightarrow s^nX_I(s) - s^{n-1}x(0^-) - s^{n-2}x'(0^-) - \cdots - x^{(n-1)}(0^-) \tag{3.46}$$

Where

$$x^{(r)}(0^-) = \frac{d^rx(t)}{dt^r}\bigg|_{t=0^-}$$

2. Integration in the Time Domain:

$$\int_{0^-}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}X_I(s) \tag{3.47}$$

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}X_I(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\tau)d\tau \tag{3.48}$$

Transform Circuits

The solution for signals in an electric circuit can be found without writing differential equations if the circuit operations and signals are represented with their Laplace transform equivalents. [In this subsection the Laplace transform means the unilateral Laplace transform].

The second model of the capacitance C in Fig. 3-10 is obtained by rewriting Eq. (3.52) as

$$v(t) \leftrightarrow V(s) = \frac{1}{sC} I(s) + \frac{1}{s} v(0^-) \tag{3.53}$$

1. Signal Sources:

$$v(t) \leftrightarrow V(s) \qquad i(t) \leftrightarrow I(s)$$

where $v(t)$ and $i(t)$ are the voltage and current source signals, respectively.

2. Resistance *R*:

$$v(t) = Ri(t) \leftrightarrow V(s) = RI(s) \tag{3.49}$$

3. Inductance *L*:

$$v(t) = L\frac{di(t)}{dt} \leftrightarrow V(s) = sLI(s) - Li(0^-) \tag{3.50}$$

The second model of the inductance L in Fig. 3-10 is obtained by rewriting Eq. (3.50) as

$$i(t) \leftrightarrow I(s) = \frac{1}{sL} V(s) + \frac{1}{s} i(0^-) \tag{3.51}$$

4. Capacitance *C*:

$$i(t) = C\frac{dv(t)}{dt} \leftrightarrow I(s) = sCV(s) - Cv(0^-) \tag{3.52}$$

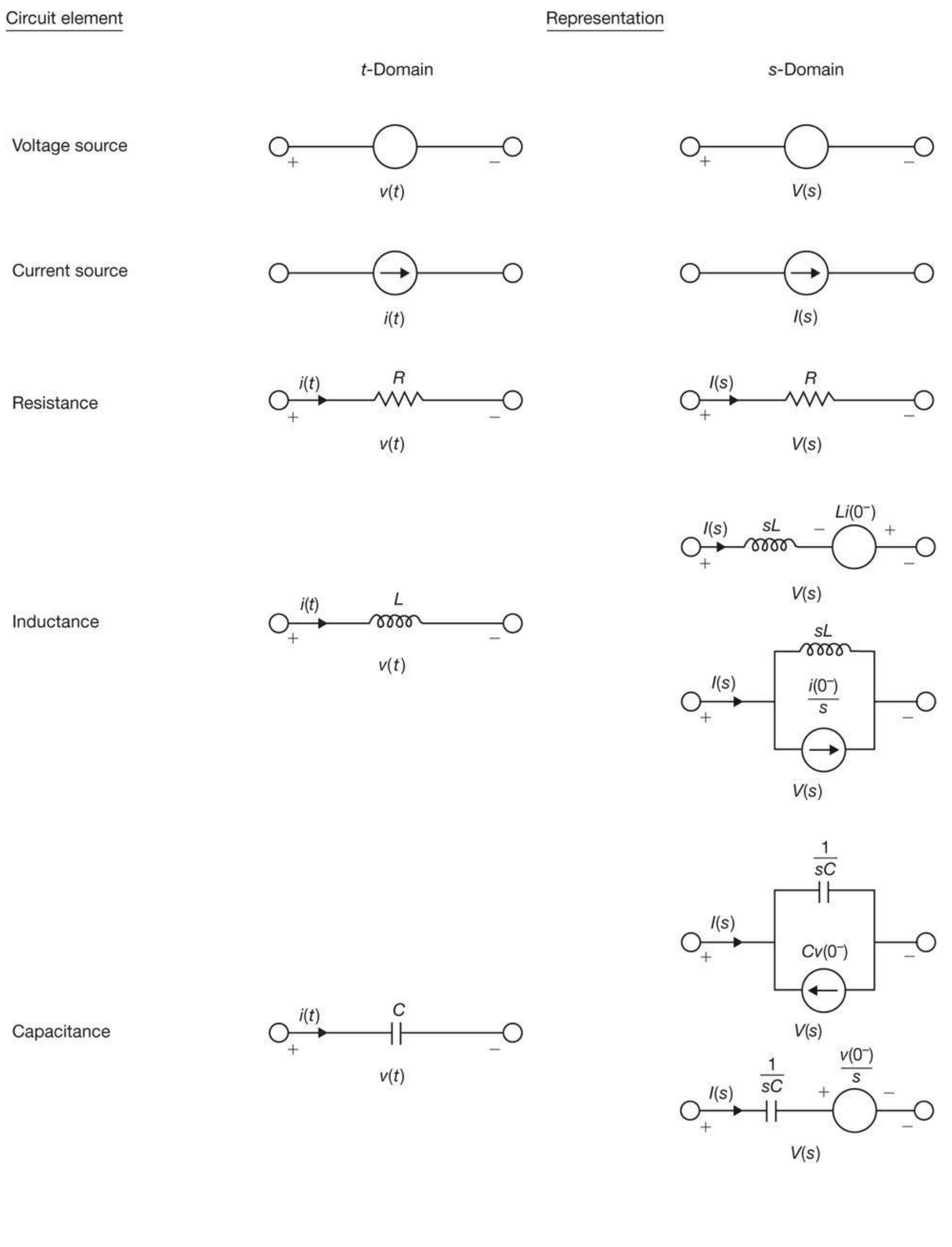

Fig. 3-10 Representation of Laplace transform circuit-element models.

3. Glossary – English/Chinese Translation

English	Chinese
Inverse Laplace transform	逆拉普拉斯变换
System Function	系统功能
Unilateral Laplace transform	单边拉普拉斯变换
Partial Fraction Expansion	部分馏分展开
Cascade	级联
Transform Circuits	变换电路

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Your Notes