Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 03-e

Laplace Transform and LTI Systems 1

Contents

Reference:
Signals and Systems 2nd Edition – Oppenheim, Willsky

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1. The Laplace transform

In this chapter, the Laplace transform is introduced to represent continuous-time signals in the s-domain (s is a complex variable), and the concept of the system function for a continuous-time LTI system is described.

The Laplace Transform

For a general continuous-time signal x(t), the Laplace transform X(s) is defined as

$$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{3.3}$$

where:

$$s=\sigma+j\omega \tag{3.4}$$

The Laplace transform defined in Eq. (3.3) is often called the bilateral (or two-sided) Laplace transform in contrast to the unilateral (or one-sided) Laplace transform, which is defined as:

$$X_I(s)=\int_{0^-}^{\infty} x(t)e^{-st}dt \tag{3.5}$$

Equation (3.3) is sometimes considered an operator that transforms a signal x(t) into a function X(s) symbolically represented by

$$X(s)=\mathscr{L}\{x(t)\} \tag{3.6}$$

and the signal x(t) and its Laplace transform X(s) are said to form a Laplace transform pair denoted as:

$$x(t) \leftrightarrow X(s) \tag{3.7}$$

Region of Convergence (ROC)

Consider the signal:

$$x(t)=e^{-at}u(t) \qquad a \text{ real} \tag{3.8}$$

The Laplace Transform of *x(t)* is:

Example 1

$$X(s) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} dt = \int_{0^+}^{\infty} e^{-(s+a)t} dt$$

$$= -\frac{1}{s+a} e^{-(s+a)t} \Big|_{0^+}^{\infty} = \frac{1}{s+a} \qquad \mathrm{Re}(s) > -a \tag{3.9}$$

because $\lim_{t\to\infty} e^{-(s+a)t} = 0$ only if $Re(s+a) > 0$ or $Re(s) > -a$.

Thus, the ROC for this example is: as Re(s) > –a and is displayed in the complex plane as shown in Fig. 3-1 by the shaded area to the right of the line Re(s) = –a. In Laplace transform applications, the complex plane is commonly referred to as the s-plane. The horizontal and vertical axes are sometimes referred to as the σ-axis and the jω-axis, respectively.

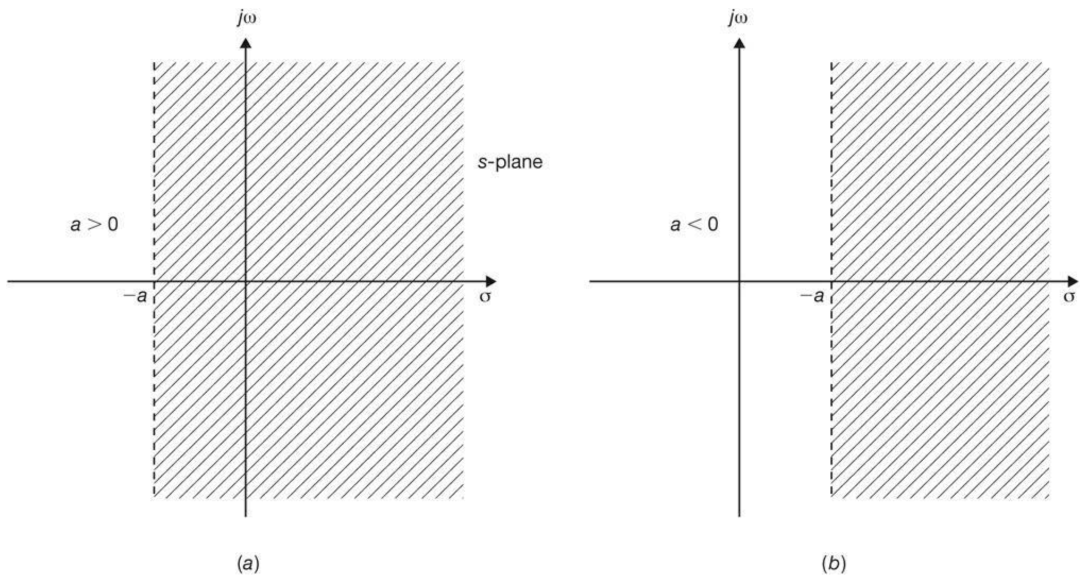


Fig. 3-1 ROC for Example 3.1

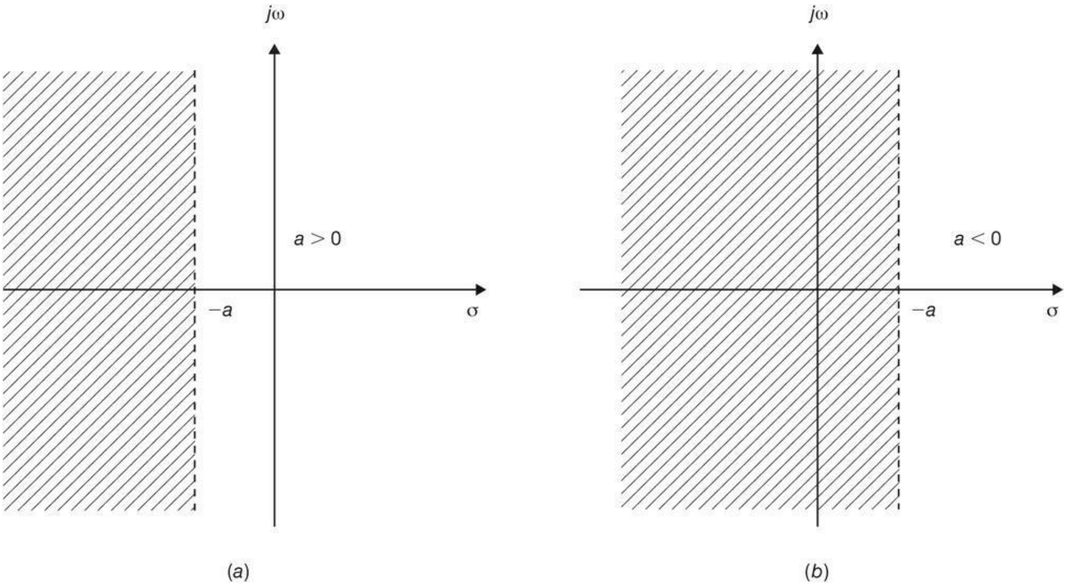
Example 2

$$x(t) = -e^{-at}u(-t) \qquad a \text{ real} \tag{3.10}$$

Laplace transform:

$$X(s) = \frac{1}{s+a} \qquad \mathrm{Re}(s) < -a \tag{3.11}$$

Re(s) < –a is displayed in the complex plane as shown in Fig. 3-2 by the shaded area to the left of the line Re(s) = –a.



we see that the algebraic expressions for X(s) for these two different signals are identical except for the ROCs. Therefore, in order for the Laplace transform to be unique for each signal x(t), the ROC must be specified as part of the transform.

Poles and Zeros

$$X(s) = \frac{a_0 s^m + a_1 s^{m-1} + \cdots + a_m}{b_0 s^n + b_1 s^{n-1} + \cdots + b_n} = \frac{a_0}{b_0} \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} \tag{3.12}$$

The coefficients a_k and b_k are real constants, and m and n are positive integers. The *X(s)* is called a proper rational function if $n > m$, and an improper rational function if $n \leq m$. The roots of the numerator polynomial, p_k, are called the zeros of *X(s)* because $X(s) = 0$ for those values of s.

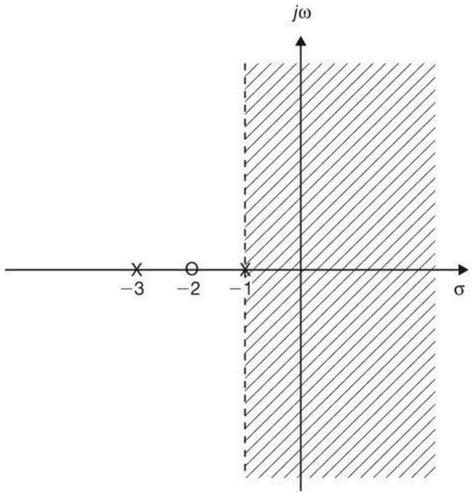
Similarly, the roots of the denominator polynomial, p_k, are called the poles of *X(s)* because *X(s)* is infinite for those values of *s*. Therefore, the poles of *X(s)* lie outside the ROC since *X(s)* does not converge at the poles, by definition.

The zeros, on the other hand, may lie inside or outside the ROC. Except for a scale factor a_0/b_0, *X(s)* can be completely specified by its zeros and poles. Thus, a very compact representation of *X(s)* in the s-plane is to show the locations of poles and zeros in addition to the ROC. Traditionally, an “×” is used to indicate each pole location and an “o” is used to indicate each zero.

Example 3

$$X(s)=\frac{2s+4}{s^2+4s+3}=2\frac{s+2}{(s+1)(s+3)} \qquad \mathrm{Re}(s)>-1$$

Note that X(s) has one zero at s = –2 and two poles at s = –1 and s = –3 with scale factor 2.



Properties of ROC

Property 1: The ROC does not contain any poles.

Property 2: If *x(t)* is a finite-duration signal, that is, *x(t)=0* except in a finite interval $t_1 \le t \le t_2$ $(-\infty < t_1 \text{ and } t_2 < \infty)$ then the ROC is the entire s-plane except possibly s = 0 or s = ∞.

Property 3: If x(t) is a right-sided signal, that is, *x(t)=0* for $t< t_1<\infty$,

then the ROC is of the form *Re(s)* < σ_{min} where σ_{min} equals the minimum real part of any of the poles of *X(s)*. Thus, the ROC is a half-plane to the left of the vertical line *Re(s)* = σ_{min} in the s-plane and thus to the left of all of the poles of *X(s)*.

Property 4: If x(t) is a left-sided signal, that is, *x(t)=0* for *t >t2>* $-\infty$

then the ROC is of the form *Re(s)* < σ_{min} where σ_{min} equals the minimum real part of any of the poles of *X(s)*. Thus, the ROC is a half-plane to the left of the vertical line *Re(s)* = σ_{min} in the s-plane and thus to the left of all of the poles of X(s).

Property 5: If *x(t)* is a two-sided signal, that is, *x(t)* is an infinite duration signal that is neither right-sided nor left-sided, then the ROC is of the form $\sigma_1 < Re(s) < \sigma_2$ where σ_1 and σ_2 are the real parts of the two poles of *X(s)*. Thus, the ROC is a vertical strip in the s-plane between the vertical lines *Re(s) = σ1* and *Re(s) = σ2*.

2. Laplace Transform of Some Common Signals

Unit Impulse Function

$$\mathscr{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1 \qquad \text{all } s \tag{3.13}$$

Unit Step Function

$$\mathscr{L}[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-st} dt = \int_{0^+}^{\infty} e^{-st} dt$$

$$= -\frac{1}{s} e^{-st} \Big|_{0^+}^{\infty} = \frac{1}{s} \qquad \text{Re}(s) > 0 \tag{3.14}$$

Laplace Transform for Some Common Signals

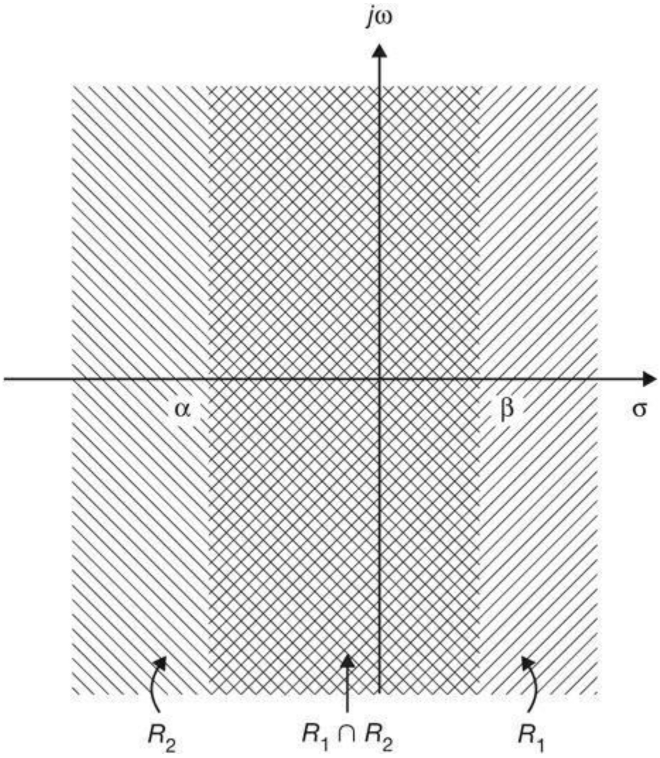
$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
$tu(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$\mathrm{Re}(s) > 0$
$e^{-at} u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$
$te^{-at} u(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-te^{-at}u(-t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$
$\cos \omega_0 t u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$\sin \omega_0 t u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$e^{-at} \cos \omega_0 t u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$e^{-at} \sin \omega_0 t u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$

3. Properties of Laplace Transform

Linearity

$$x_1(t) \leftrightarrow X_1(s) \qquad \text{ROC} = R_1$$
$$x_2(t) \leftrightarrow X_2(s) \qquad \text{ROC} = R_2$$
$$a_1x_1(t) + a_2x_2(t) \leftrightarrow a_1X_1(s) + a_2X_2(s) \qquad R' \supset R_1 \cap R_2 \qquad (3.15)$$

The set notation $A \supset B$ means that set A contains set B, while $A \cap B$ denotes the intersection of sets A and B, that is, the set containing all elements in both A and B. Thus, Eq. (3.15) indicates that the ROC of the resultant Laplace transform is at least as large as the region in common between R_1 and R_2. Usually we have simply $R' = R_1 \cap R_2$. This is illustrated in Fig. 3-4.



Time Shifting

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $$x(t-t_0) \leftrightarrow e^{-st_0}X(s) \qquad R' = R \qquad (3.16)$$

Equation (3.16) indicates that the ROCs before and after the time-shift operation are the same.

Shifting in the S Domian

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $$e^{s_0 t} x(t) \leftrightarrow X(s - s_0) \qquad R' = R + \text{Re}(s_0) \tag{3.17}$$

Equation (3.17) indicates that the ROC associated with *X(s–s₀)* is that of *X(s)* shifted by *Re(s₀)*. This is illustrated in Fig. 3-5.

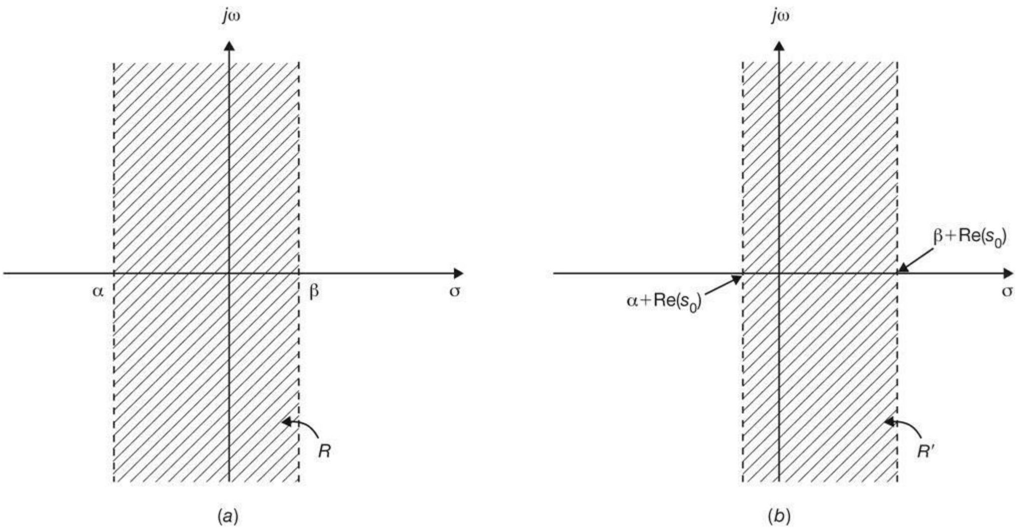


Fig. 3-5 Effect on the ROC of shifting in the *s*-domain. (*a*) ROC of *X*(*s*); (*b*) ROC of $X(s - s_0)$.

Time Scaling

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right) \qquad R' = aR \tag{3.18}$$

Equation (3.18) indicates that scaling the time variable t by the factor a causes an inverse scaling of the variable s by 1/a as well as an amplitude scaling of X(s/a) by 1/|a|. The corresponding effect on the ROC is illustrated in Fig. 3-6.

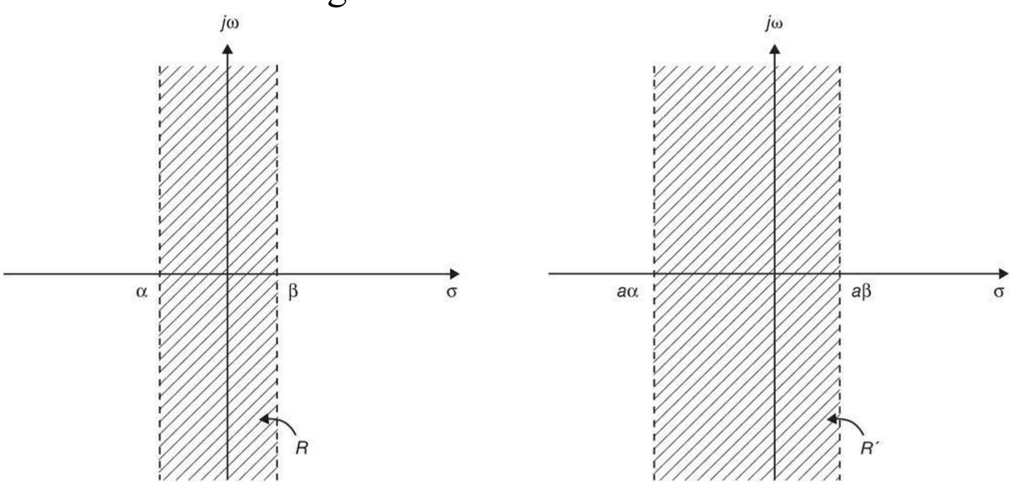


Fig. 3-6 Effect on the ROC of time scaling. (a) ROC of X(s); (b) ROC of X(s/a).

Time Reversal

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $$x(-t) \leftrightarrow X(-s) \qquad R' = -R \tag{3.19}$$

Thus, time reversal of *x(t)* produces a reversal of both the *σ*- and *jω*-axes in the s-plane. Eqn (3.19) is readily obtained by setting *a = –1* in Eq. (3.18).

Differentiation in the Time Domain

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $$\frac{dx(t)}{dt} \leftrightarrow sX(s) \qquad R' \supset R \tag{3.20}$$

Equation (3.20) shows that the effect of differentiation in the time domain is multiplication of the corresponding Laplace transform by s. The associated ROC is unchanged unless there is a pole-zero cancellation at s = 0.

Differentiation in the S Domain

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $$-tx(t) \leftrightarrow \frac{dX(s)}{ds} \qquad R' = R \tag{3.21}$$

Integration in the Time Domain

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}X(s) \qquad R' = R \cap \{\text{Re}(s) > 0\} \tag{3.22}$$

Equation (3.22) shows that the Laplace transform operation corresponding to time-domain integration is multiplication by 1/s, and this is expected since integration is the inverse operation of differentiation. The form of R′ follows from the possible introduction of an additional pole at s = 0 by the multiplication by 1

Convolution

$$x_1(t) \leftrightarrow X_1(s) \qquad \text{ROC} = R_1$$
$$x_2(t) \leftrightarrow X_2(s) \qquad \text{ROC} = R_2$$

then
$$x_1(t) * x_2(t) \leftrightarrow X_1(s)X_2(s) \qquad R' \supset R_1 \cap R_2 \tag{3.23}$$

Table of Properties of Laplace transform

PROPERTY	SIGNAL	TRANSFORM	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(s) + a_2X_2(s)$	$R' \supset R_1 \cap R_2$
Time shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	$R'=R$
Shifting in s	$e^{s_0t}x(t)$	$X(s-s_0)$	$R'=R+\text{Re}(s_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(a)$	$R'=aR$
Time reversal	$x(-t)$	$X(-s)$	$R'=-R$
Differentiation in t	$\frac{dx(t)}{dt}$	$sX(s)$	$R'\supset R$
Differentiation in s	$-tx(t)$	$\frac{dX(s)}{ds}$	$R'=R$
Integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s)$	$R'\supset R\cap\{\text{Re}(s)>0\}$
Convolution	$x_1(t)*x_2(t)$	$X_1(s)\,X_2(s)$	$R'\supset R_1\cap R_2$

3. Glossary – English/Chinese Translation

English	Chinese
laplace transform	拉普拉斯变换
s domain	S 域
complex variable	复变量
continuous time LTI system	连续时间 LTI 系统
unilateral and bilateral	单边和双边
region of convergence	收敛区域
s-plane	S-平面
complex plane	复平面
poles and zeros	极点和零点
numerator and denominator	分子和分母
set notation	设置符号。

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