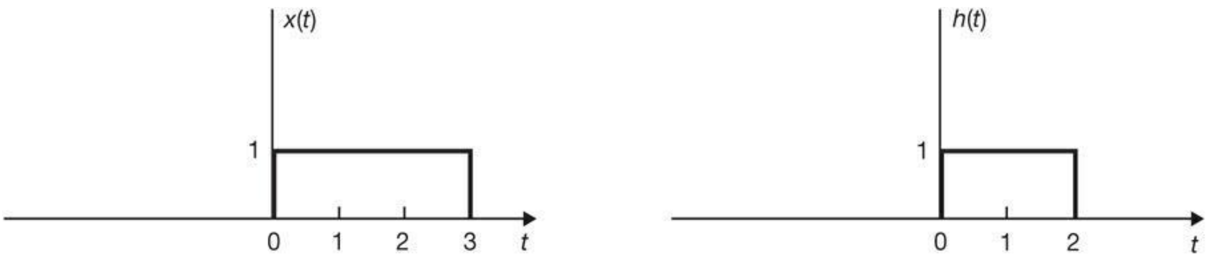
1-03-d tutorial solution

Question 1

2.6 Evaluate y(t) = x(t) * h(t), where x(t) and h(t) are shown in Fig. 2-6, (a) by an analytical technique, and (b) by a graphical method.



SOLUTION

(a)

We first express x(t) and h(t) in functional form: x(t) = u(t) – u(t – 3) h(t) = u(t) – u(t – 2)

$$\begin{aligned} y(t) = x(t) * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} [u(\tau) - u(\tau-3)]\,[u(t-\tau) - u(t-\tau-2)]\,d\tau \\ &= \int_{-\infty}^{\infty} u(\tau)u(t-\tau)\,d\tau - \int_{-\infty}^{\infty} u(\tau)u(t-2-\tau)\,d\tau \\ &\quad - \int_{-\infty}^{\infty} u(\tau-3)u(t-\tau)\,d\tau + \int_{-\infty}^{\infty} u(\tau-3)u(t-2-\tau)\,d\tau \end{aligned}$$

$$u(\tau)u(t-\tau) = \begin{cases} 1 & 0 < \tau < t, t > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$u(\tau)u(t-2-\tau) = \begin{cases} 1 & 0 < \tau < t-2, t > 2 \\ 0 & \text{otherwise} \end{cases}$$

$$u(\tau-3)u(t-\tau) = \begin{cases} 1 & 3 < \tau < t, t > 3 \\ 0 & \text{otherwise} \end{cases}$$

$$u(\tau-3)u(t-2-\tau) = \begin{cases} 1 & 3 < \tau < t-2, t > 5 \\ 0 & \text{otherwise} \end{cases}$$

We can express y(t) as:

$$\begin{aligned} y(t) &= \left(\int_0^t d\tau\right)u(t) - \left(\int_0^{t-2} d\tau\right)u(t-2) \\ &\quad - \left(\int_3^t d\tau\right)u(t-3) + \left(\int_3^{t-2} d\tau\right)u(t-5) \\ &= tu(t) - (t-2)u(t-2) - (t-3)u(t-3) + (t-5)u(t-5) \end{aligned}$$

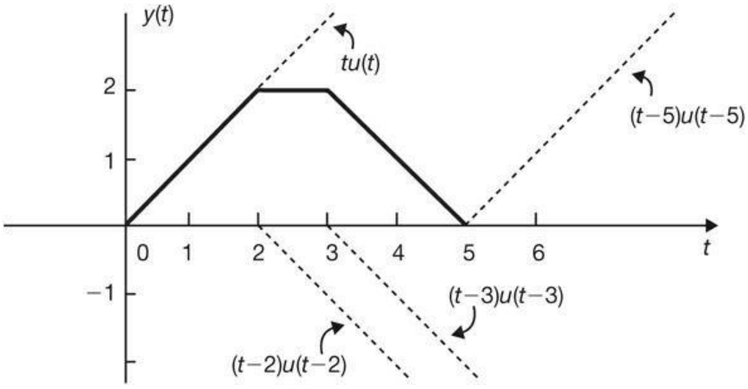


Fig 2-7

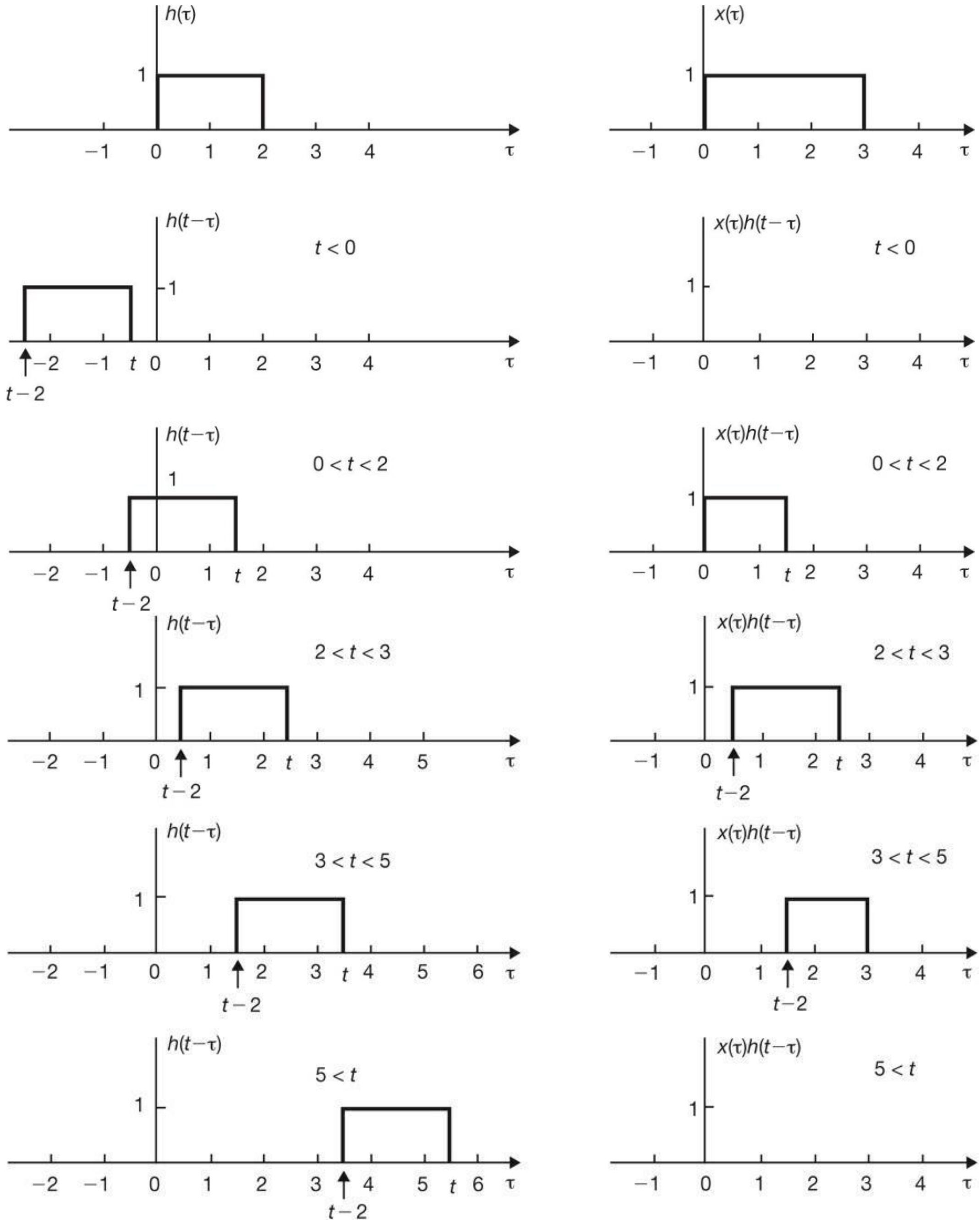


Fig 2-8

(b)

Functions h(τ), x(τ) and h(t – τ), x(τ)h(t – τ) for different values of t are sketched in Fig. 2-8. From Fig. 2-8 we see that x(τ) and h(t – τ) do not overlap for t < 0 and t > 5, and hence, y(t) = 0 for t < 0 and t >

5. For the other intervals, x(τ) and h(t – τ) overlap. Thus, computing the area under the rectangular pulses for these intervals, we obtain:

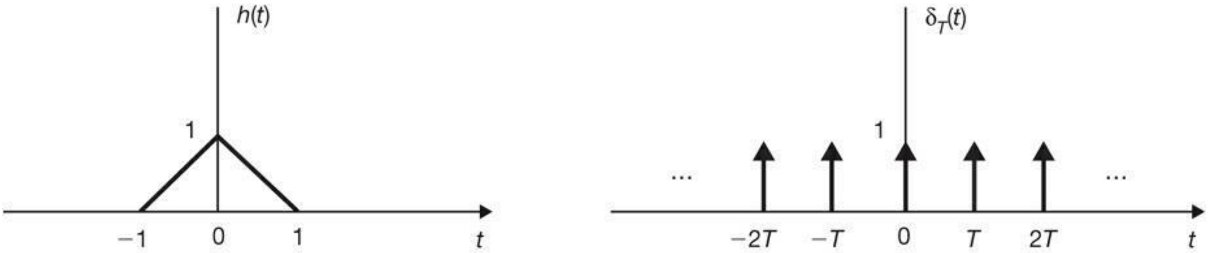
$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t \le 2 \\ 2 & 2 < t \le 3 \\ 5 - t & 3 < t \le 5 \\ 0 & 5 < t \end{cases}$$

Question 2

Let h(t) be the triangular pulse shown in Fig. 2-10(a) and let x(t) be the unit impulse train [Fig. 2-10(b)] expressed as

$$x(t) = \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{2.68}$$

Determine and sketch y(t) = h(t) * x(t) for the following values of T: (a)T = 3, (b) T = 2, (c) T = 1.5.



(a) (b)

SOLUTION

According to the property of the sampling signal:

$$\begin{aligned} y(t) &= h(t) * \delta_T(t) = h(t) * \left[\sum_{n=-\infty}^{\infty} \delta(t - nT) \right] \\ &= \sum_{n=-\infty}^{\infty} h(t) * \delta(t - nT) = \sum_{n=-\infty}^{\infty} h(t - nT) \end{aligned} \tag{2.69}$$

which is sketched in Fig. 2-11(c). Note that when T < 2, the triangular pulses are no longer separated and they overlap.

(a) For T = 3, Eq. (2.69) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} h(t-3n)$$

(b) For T = 2, Eq. (2.69) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} h(t-2n)$$

(c) For T = 1.5, Eq. (2.69) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} h(t-1.5n)$$

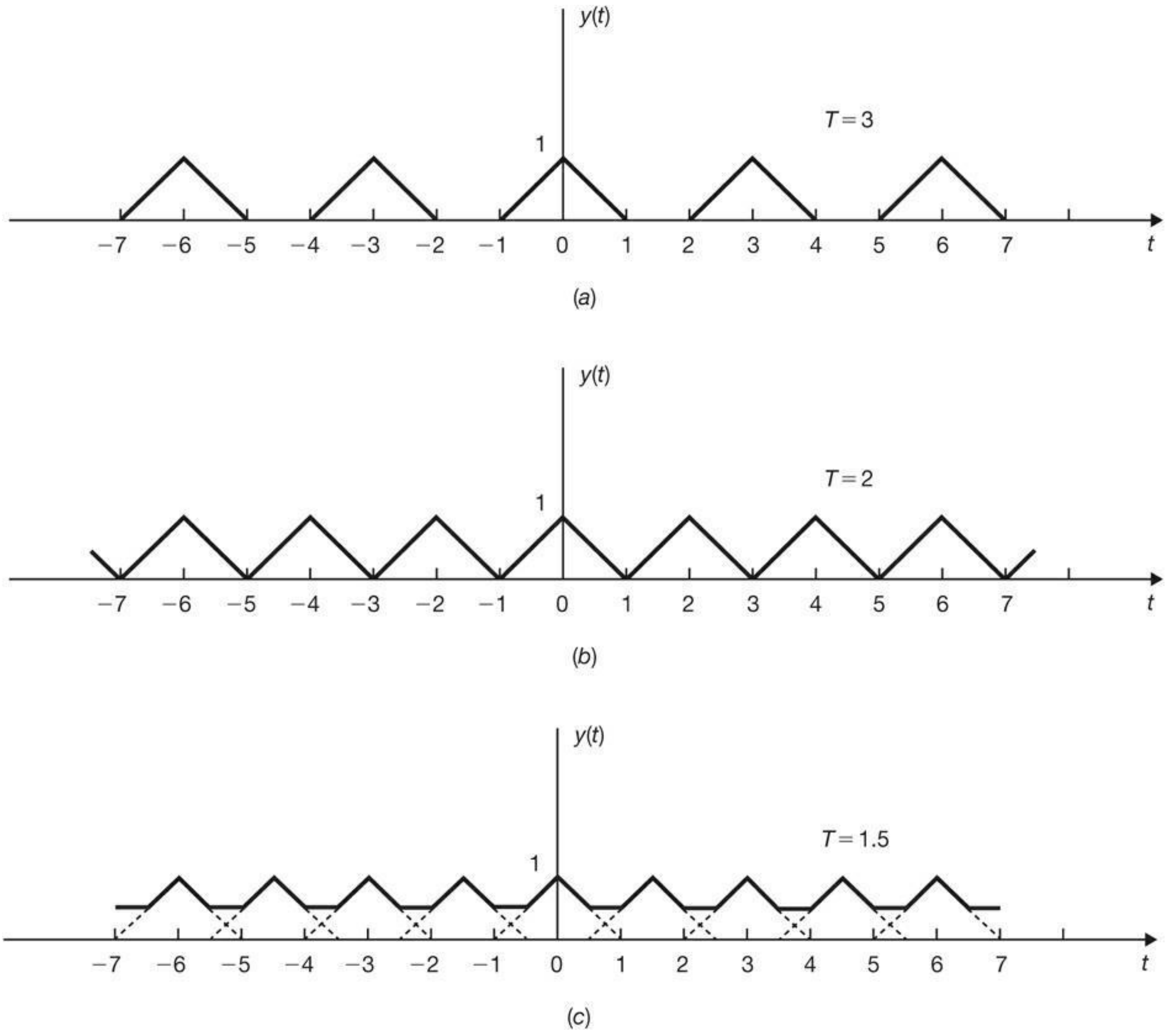


Fig 2-11

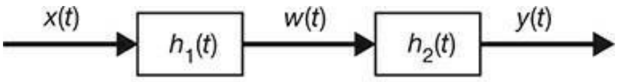
Question 3

2.14. The system shown in Fig. 2-17(a) is formed by connecting two systems in cascade. The impulse responses of the systems are given by h1(t) and h2(t), respectively, and

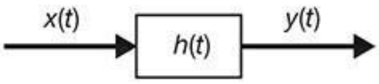
$h1(t)= e^{-2t}\, u(t)$ $h2(t) = 2e^{-t}\, u(t)$

(a) Find the impulse response h(t) of the overall system shown in Fig. 2-17(b).

(b) Determine if the overall system is BIBO stable.



(a)



(b)

SOLUTION

(*a*) Let $w(t)$ be the output of the first system. By Eq. (2.6)

$$w(t) = x(t) * h_1(t) \tag{2.78}$$

Then we have

$$y(t) = w(t) * h_2(t) = [x(t) * h_1(t)] * h_2(t) \tag{2.79}$$

But by the associativity property of convolution (2.8), Eq. (2.79) can be rewritten as

$$y(t) = x(t) * [h_1(t) * h_2(t)] = x(t) * h(t) \tag{2.80}$$

Therefore, the impulse response of the overall system is given by

$$h(t) = h_1(t) * h_2(t) \tag{2.81}$$

Thus, with the given $h_1(t)$ and $h_2(t)$, we have

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} h_1(\tau) h_2(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-2\tau} u(\tau)\, 2e^{-(t-\tau)} u(t-\tau)\, d\tau \\ &= 2e^{-t} \int_{-\infty}^{\infty} e^{-\tau} u(\tau) u(t-\tau)\, d\tau = 2e^{-t} \left[\int_0^t e^{-\tau}\, d\tau \right] u(t) \\ &= 2(e^{-t} - e^{-2t}) u(t) \end{aligned}$$

(b) Using the above h(t), we have

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = 2\int_0^{\infty} (e^{-\tau} - e^{-2\tau})\, d\tau = 2\left[\int_0^{\infty} e^{-\tau}\, d\tau - \int_0^{\infty} e^{-2\tau}\, d\tau\right]$$
$$= 2\left(1 - \frac{1}{2}\right) = 1 < \infty$$

Thus, the system is BIBO stable.

Question 4

2.15. Consider a continuous-time LTI system with the input-output relation given by

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau)\, d\tau \qquad (2.82)$$

(a) Find the impulse response h(t) of this system.

(b) Show that the complex exponential function e^{st} is an eigenfunction of the system.

(c) Find the eigenvalue of the system corresponding to e^{st} by using the impulse response h(t) obtained in part (a).

SOLUTION

(*a*) From Eq. (2.82), definition (2.1), and Eq. (1.21) we get

$$h(t) = \int_{-\infty}^{t} e^{-(t-\tau)} \delta(\tau)\, d\tau = e^{-(t-\tau)}\Big|_{\tau=0} = e^{-t} \qquad t > 0$$

$$h(t) = e^{-t} u(t) \qquad (2.83)$$

Thus,

(*b*) Let $x(t) = e^{st}$. Then

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}\, e^{s\tau}\, d\tau = e^{-t} \int_{-\infty}^{t} e^{(s+1)\tau}\, d\tau$$
$$= \frac{1}{s+1} e^{st} = \lambda e^{st} \qquad \text{if } \operatorname{Re} s > -1 \qquad (2.84)$$

Thus, by definition (2.22) e^{st} is the eigenfunction of the system and the associated eigenvalue is