Dr. Norbert Cheung's
Lecture Series

Level 1 Topic no: 03-c

Linear Time Invariant Systems -2
(Properties)

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Reference:
Signals and Systems 2nd Edition – Oppenheim, Willsky

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1. Properties of LTI Systems

Repeat the previous notes on convolution

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] * h[n] \quad (2.39)$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t) \quad (2.40)$$

The characteristics of an LTI system are completely determined by its impulse response. It is important to emphasize that this property holds *only* for LTI systems.

The Commutative Property

For discrete time and continuous time systems:

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k], \quad (2.43)$$

and

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau. \quad (2.44)$$

The Distributive Property

For discrete time and continuous time systems:

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n], \quad (2.46)$$

and

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t). \quad (2.47)$$

It can be represented in block diagram as:

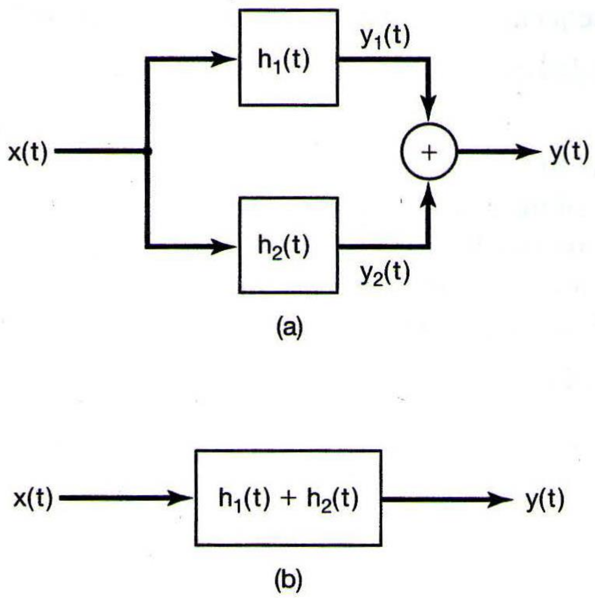

Figure 2.23 Interpretation of the distributive property of convolution for a parallel interconnection of LTI systems.

The two systems, with impulse responses $h_1(t)$ and $h_2(t)$, have identical inputs, and their outputs are added. Since

$$y_1(t) = x(t) * h_1(t)$$

and

$$y_2(t) = x(t) * h_2(t),$$

the system of Figure 2.23(a) has output

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t), \tag{2.48}$$

corresponding to the right-hand side of eq. (2.47). The system of Figure 2.23(b) has output

$$y(t) = x(t) * [h_1(t) + h_2(t)], \tag{2.49}$$

Also, as a consequence of both the commutative and distributive properties, we have

$$[x_1[n] + x_2[n]] * h[n] = x_1[n] * h[n] + x_2[n] * h[n] \tag{2.50}$$

and

$$[x_1(t) + x_2(t)] * h(t) = x_1(t) * h(t) + x_2(t) * h(t), \tag{2.51}$$

Example

Let $y[n]$ denote the convolution of the following two sequences:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n], \tag{2.52}$$

$$h[n] = u[n]. \tag{2.53}$$

Find y[n].

Solution

if we let $x_1[n] = (1/2)^n u[n]$ and $x_2[n] = 2^n u[-n]$, it follows that

$$y[n] = (x_1[n] + x_2[n]) * h[n]. \tag{2.54}$$

Using the distributive property of convolution, we may rewrite eq. (2.54) as

$$y[n] = y_1[n] + y_2[n], \tag{2.55}$$

where

$$y_1[n] = x_1[n] * h[n] \tag{2.56}$$

and

$$y_2[n] = x_2[n] * h[n]. \tag{2.57}$$

After that, it is much easier to evaluate $y_1[n]$ and $y_2[n]$

Answer

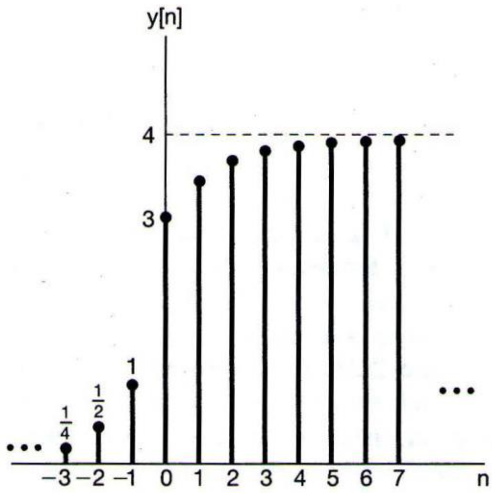

Figure 2.24 The signal $y[n] = x[n] * h[n]$ for Example 2.10.

<u>The Associative Property</u>

discrete time

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n], \tag{2.58}$$

and in continuous time

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t). \tag{2.59}$$

Hence it does not matter in what order we convolve these signals.

$$y[n] = x[n] * h_1[n] * h_2[n] \tag{2.60}$$

and

$$y(t) = x(t) * h_1(t) * h_2(t) \tag{2.61}$$

An interpretation of the associative property is illustrated for discrete-time systems in Figures 2.25(a) and (b). In Figure 2.25(a),

$$\begin{aligned} y[n] &= w[n] * h_2[n] \\ &= (x[n] * h_1[n]) * h_2[n]. \end{aligned}$$

In Figure 2.25(b),

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] * (h_1[n] * h_2[n]). \end{aligned}$$

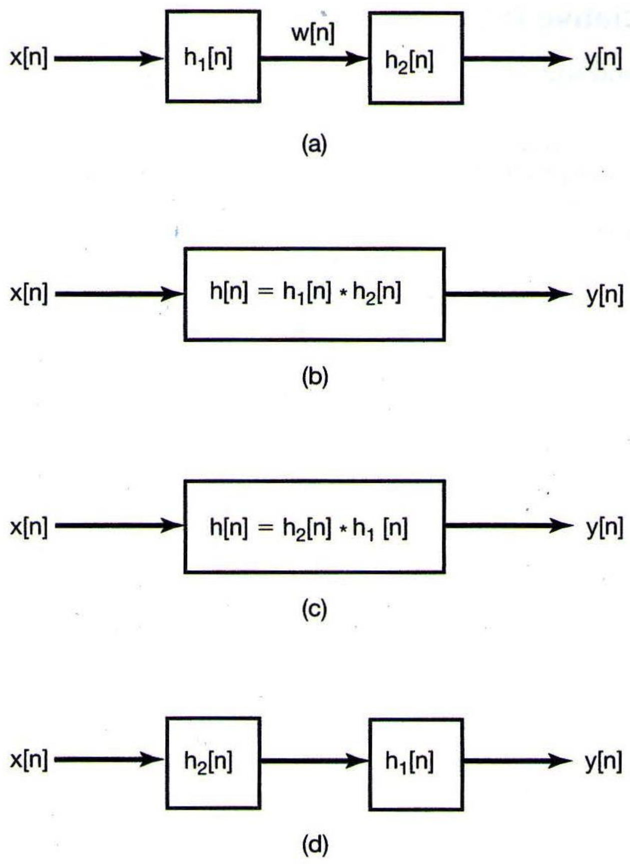

Figure 2.25 Associative property of convolution and the implication of this and the commutative property for the series interconnection of LTI systems.

LTI System with and without memory

For a discrete-time LTI system is if $h[n] = 0$ for $n \neq 0$. In this case the impulse response has the form:

$$h[n] = K\delta[n], \tag{2.62}$$

where $K = h[0]$ is a constant, and the convolution sum reduces to the relation

$$y[n] = Kx[n]. \tag{2.63}$$

In particular, a continuous-time LTI system is memoryless if $h(t) = 0$ for $t \neq 0$, and such a memoryless LTI system has the form

$$y(t) = Kx(t) \tag{2.64}$$

for some constant K and has the impulse response

$$h(t) = K\delta(t). \tag{2.65}$$

If $K=1$, the convolution sum and integral formulae…..

$$x[n] = x[n] * \delta[n]$$

and

$$x(t) = x(t) * \delta(t),$$

Will reduce to the shifting properties of the discrete time and continuous time unit impulses:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau.$$

Therefore, if $h(t_0) \neq 0$ for $t_0 \neq 0$, then continuous-time LTI system has memory.

Invertibility of LTI Systems

Consider a continuous-time LTI system with impulse response h(t). This system is invertible only if an inverse system exists that, when connected in series with the original system, produces an output equal to the input to the first system. Furthermore, if an LTI system is invertible, then it has an LTI inverse.

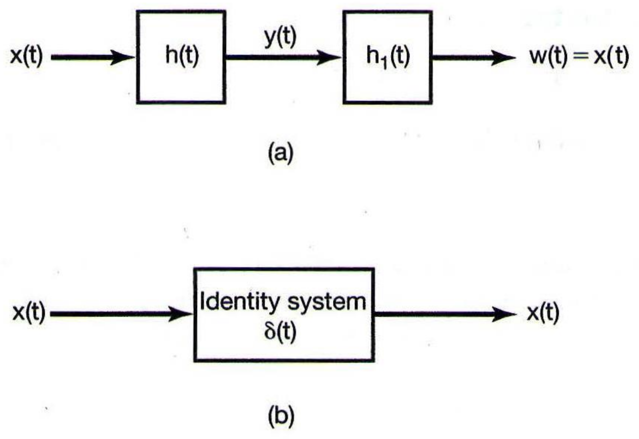

Figure 2.26 Concept of an inverse system for continuous-time LTI systems. The system with impulse response $h_1(t)$ is the inverse of the system with impulse response $h(t)$ if $h(t) * h_1(t) = \delta(t)$.

$$h(t) * h_1(t) = \delta(t). \tag{2.66}$$

Similarly, in discrete time, the impulse response $h_1[n]$ of the inverse system for an LTI system with impulse response $h[n]$ must satisfy

$$h[n] * h_1[n] = \delta[n]. \tag{2.67}$$

Continuous time

Discrete Time

Causality for LTI Systems

The output of a causal system depends only on the present and past values of the input to the system. Specifically, in order for a discrete-time LTI system to be causal, y[n] must not depend on x[k] for k > n.

From eq. (2.39), we see that for this to be true, all of the coefficients h[n- k] that multiply values of x[k] for k > n must be zero.

A causal discrete-time LTI system satisfy the condition:

$$h[n] = 0 \quad \text{for } n < 0. \tag{2.77}$$

For a causal discrete-time LTI system, the condition in eq. (2.77) implies that the convolution sum representation in eq. (2.39) becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k], \tag{2.78}$$

and the alternative equivalent form, eq. (2.43), becomes

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]. \tag{2.79}$$

Similarly, a continuous-time LTI system is causal if

$$h(t) = 0 \quad \text{for } t < 0, \tag{2.80}$$

and in this case the convolution integral is given by

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau. \tag{2.81}$$

Stability for LTI Systems

A system is stable if every bounded input produces a bounded output. Consider an input *x[n]* that is bounded in magnitude:

$$|x[n]| < B \quad \text{for all } n. \tag{2.82}$$

$$|y[n]| \leq B \sum_{k=-\infty}^{+\infty} |h[k]| \quad \text{for all } n. \tag{2.85}$$

From eq. (2.85), we can conclude that if the impulse response is *absolutely summable*, that is, if

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty, \tag{2.86}$$

then $y[n]$ is bounded in magnitude, and hence, the system is stable.

Similarly, in continuous time:

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \right| \\ &\leq \int_{-\infty}^{+\infty} |h(\tau)||x(t-\tau)|d\tau \\ &\leq B \int_{-\infty}^{+\infty} |h(\tau)|d\tau. \end{aligned}$$

Therefore, the system is stable if the impulse response is *absolutely integrable*, i.e., if

$$\int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty. \tag{2.87}$$

2. Eigenfunctions and Eigenvalues

Question: What are Eigenvalues and Eigenfunctions?
Answer: Watch video 11 in Video Gallery web page.

For Continuous Time LTI Systems

LTI systems represented by **T** are the complex exponentials e^{st}, with s a complex variable. That is,

$$\mathbf{T}\{e^{st}\} = \lambda e^{st} \tag{2.22}$$

where λ is the eigenvalue of **T** associated with e^{st}. Setting $x(t) = e^{st}$ in Eq. (2.10), we have

$$\begin{aligned} y(t) = \mathbf{T}\{e^{st}\} &= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau = \left[\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau\right] e^{st} \\ &= H(s)\, e^{st} = \lambda e^{st} \end{aligned} \tag{2.23}$$

where

$$\lambda = H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau \tag{2.24}$$

Thus Eigenvalue: λ given by *H(s)*
Eigenfunction: e^{st}

The above results underlie the definitions of the Laplace transform and Fourier transform, which will be discussed later.

For Discrete Time LTI Systems

The eigenfunctions of discrete-time LTI systems represented by T are the complex exponentials z^n, with z a complex variable. That is,

$$\mathbf{T}\{z^n\} = \lambda z^n \tag{2.50}$$

where λ is the eigenvalue of $\mathbf{T}$ associated with z^n. Setting $x[n] = \mathbf{z^n}$ in Eq. (2.39), we have

$$y[n] = \mathbf{T}\{z^n\} = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} = \left[\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}\right] z^n$$
$$= H(z)\, z^n = \lambda z^n \tag{2.51}$$

where

$$\lambda = H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k} \tag{2.52}$$

Thus Eigenvalue: λ given by *H(z)*
Eigenfunction: z^n

The above results underlie the definitions of the Z transform and Discrete Fourier transform, which will be discussed later.

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3. Glossary – English/Chinese Translation

English	Chinese
Linear Time Invariant Systems	线性时间不变系统
Eigenfunction and Eigenvalue	特征函数和特征值
Commutative Property	交换特性
Distributive Property	分配律
Associative Property	关联属性
Unit Impulse	单位脉冲
Invertibility	可逆
Inverse System	逆系统
Causal and Non-Causal System	因果和非因果系统
Discrete Fourier Transform	离散傅里叶变换

Your Notes