Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 03-c

Linear Time Invariant Systems -1 (Convolution)

Contents

Reference:

Signals and Systems 2nd Edition – Oppenheim, Willsky

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1. Introduction

- One of the primary reasons LTI systems are important to analysis is the superposition property
- If we can represent the input to an LTI system in terms of a linear combination of a set of basic signals, we can then use superposition to compute the output of the system in terms of its responses to these basic signals.
- The unit impulse, both in discrete time and in continuous time, is that very general signals can be represented as linear combinations of delayed impulses.
- Together with the properties of superposition and time invariance, will allow us to develop a complete characterization of any LTI system in terms of its response to a unit impulse.
- Such a representation, referred to as the convolution sum in the discrete-time case, and the convolution integral in continuous time.

2. Discrete Time LTI Systems – The Convolution Sum

In Fig 2.1a, we have five time-shifted, scaled unit impulse sequences, where the scaling on each impulse equals the value of *x[n]* at the particular instant the unit sample occurs. For example,

$$x[-1]\delta[n+1] = \begin{cases} x[-1], & n = -1 \\ 0, & n \neq -1 \end{cases},$$

$$x[0]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n \neq 0 \end{cases},$$

$$x[1]\delta[n-1] = \begin{cases} x[1], & n = 1 \\ 0, & n \neq 1 \end{cases}.$$

Therefore, the sum of the five-sequences in the figure equals *x[n]* for - 2 :s *n* :s 2. More generally, by including additional shifted, scaled impulses, we can write

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots \quad (2.1)$$

Or in a compact form:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]. \quad (2.2)$$

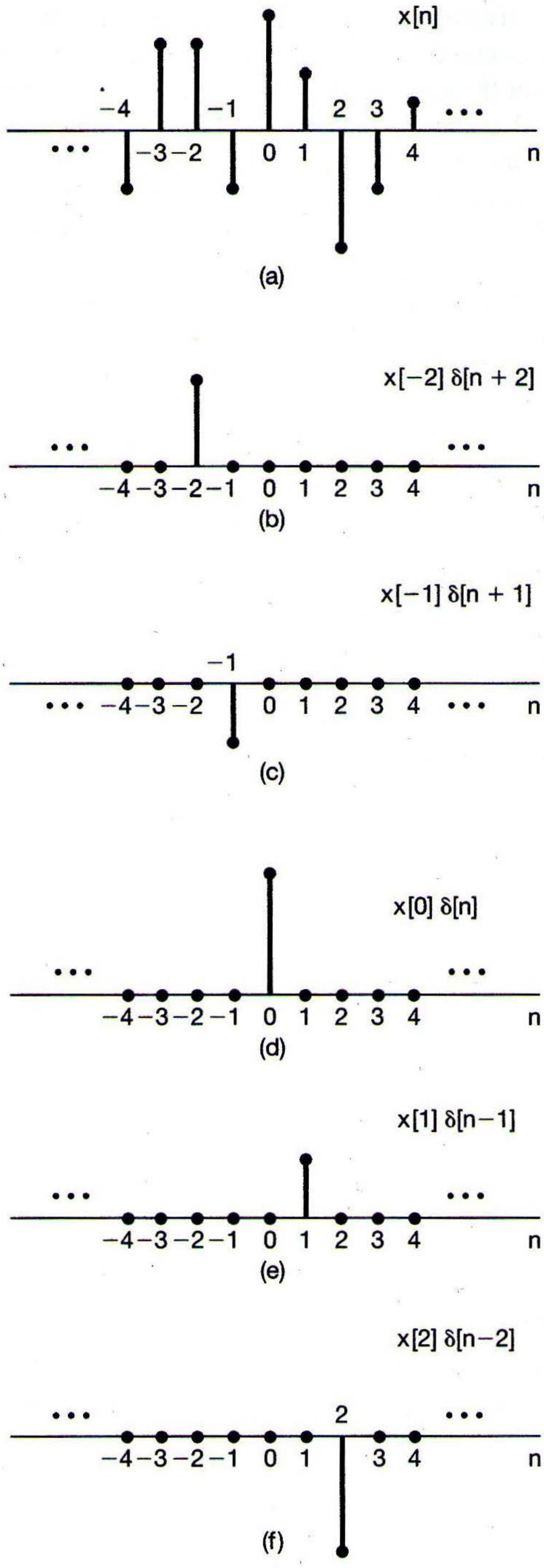


Figure 2.1 Decomposition of a discrete-time signal into a weighted sum of shifted impulses.

For k> 0, eq. (2.2) becomes:

$$u[n] = \sum_{k=0}^{+\infty} \delta[n-k],$$

Equation (2.2) is called the sifting property of the discrete-time unit impulse. For input *x[n]* to a linear system expressed in the form of eq. (2.2), the output *y[n]* can be expressed as:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]. \tag{2.3}$$

That is, $h[n]$ is the output of the LTI system when $\delta[n]$ is the input. Then for an LTI system, eq. (2.3) becomes

$$\boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].} \tag{2.6}$$

This result is referred to as the *convolution sum* or *superposition sum*, and the operation on the right-hand side of eq. (2.6) is known as the *convolution* of the sequences $x[n]$ and $h[n]$. We will represent the operation of convolution symbolically as

$$y[n] = x[n] * h[n]. \tag{2.7}$$

Let us look at some examples…. (next page)

Example 2.1:

Consider an LTI system with impulse response $h[n]$ and input $x[n]$, as illustrated in Figure 2.3(a). For this case, since only $x[0]$ and $x[1]$ are nonzero, eq. (2.6) simplifies to the expression

$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1]. \tag{2.8}$$

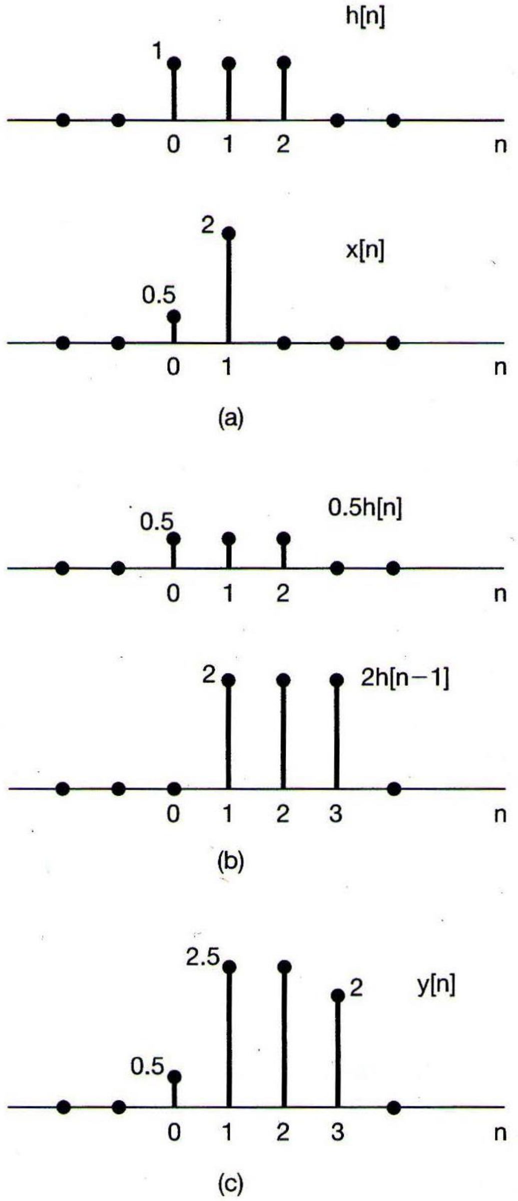


Figure 2.3 (a) The impulse response $h[n]$ of an LTI system and an input $x[n]$ to the system; (b) the responses or "echoes," $0.5h[n]$ and $2h[n-1]$, to the nonzero values of the input, namely, $x[0] = 0.5$ and $x[1] = 2$; (c) the overall response $y[n]$, which is the sum of the echos in (b).

Example 2.2

Consider Fig. 2.4a. Find out the values of *y[n]* for 0,1,2,3,4

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = 0.5. \tag{2.9}$$

The product of the sequence $x[k]$ with the sequence $h[1-k]$ has two nonzero samples, which may be summed to obtain

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 0.5 + 2.0 = 2.5. \tag{2.10}$$

Similarly,

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 0.5 + 2.0 = 2.5, \tag{2.11}$$

and

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = 2.0. \tag{2.12}$$

Therefore, the output y[n] is…….

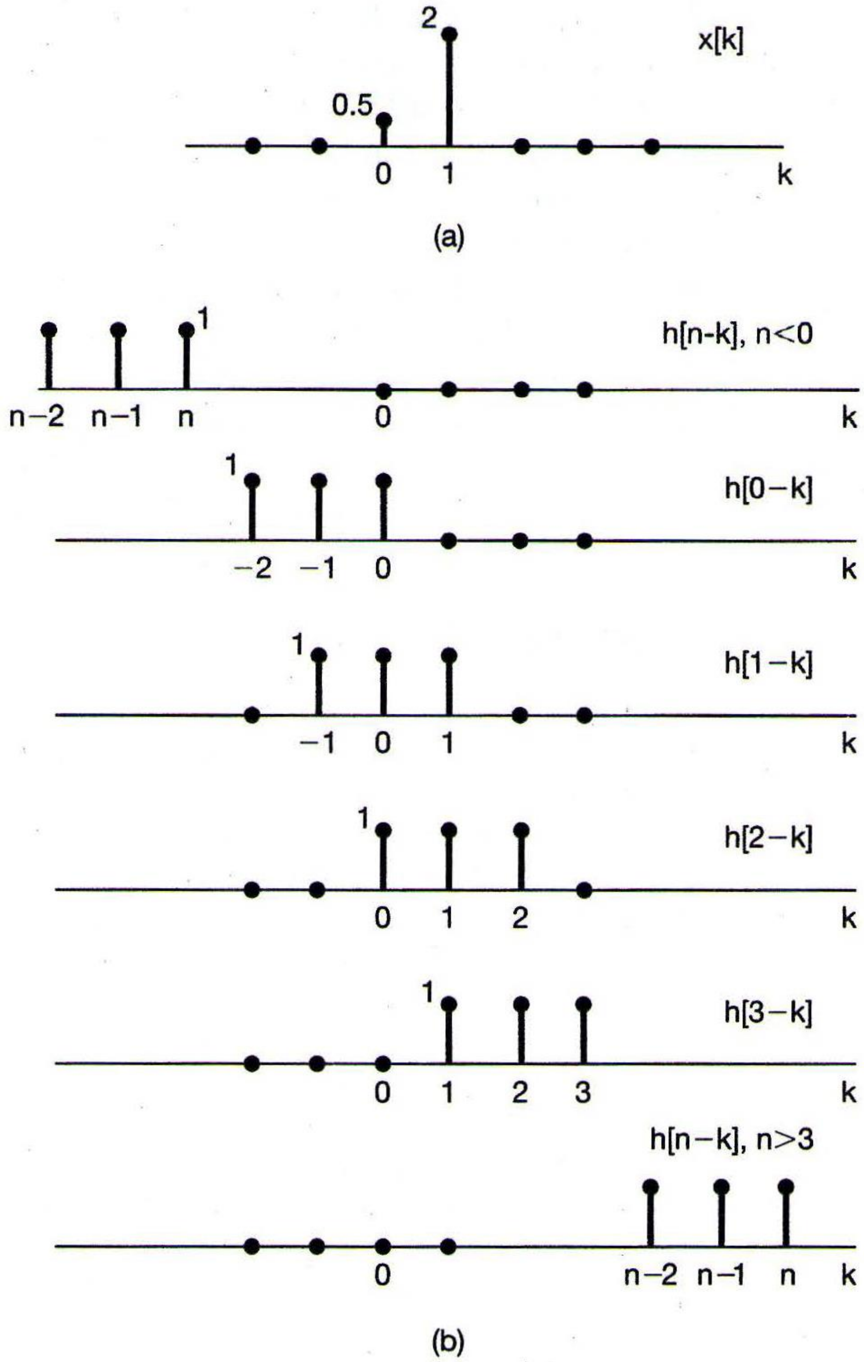


Figure 2.4 Interpretation of eq. (2.6) for the signals $h[n]$ and $x[n]$ in Figure 2.3; (a) the signal $x[k]$ and (b) the signal $h[n-k]$ (as a function of k with n fixed) for several values of n ($n < 0$; $n = 0, 1, 2, 3$; $n > 3$). Each of these signals is obtained by reflection and shifting of the unit impulse response $h[k]$. The response $y[n]$ for each value of n is obtained by multiplying the signals $x[k]$ and $h[n-k]$ in (a) and (b) and then summing the products over all values of k. The calculation for this example is carried out in detail in Example 2.2.

Example 2.3

Consider an input x[n] and a unit impulse response h[n] given by:

$$x[n] = \alpha^n u[n],$$

$$h[n] = u[n], \qquad \text{with } 0 < \alpha < 1.$$

$$x[k]h[n-k] = \begin{cases} \alpha^k, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}.$$

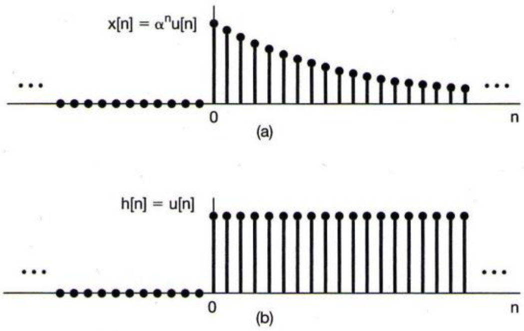


Figure 2.5 The signals $x[n]$ and $h[n]$ in Example 2.3.

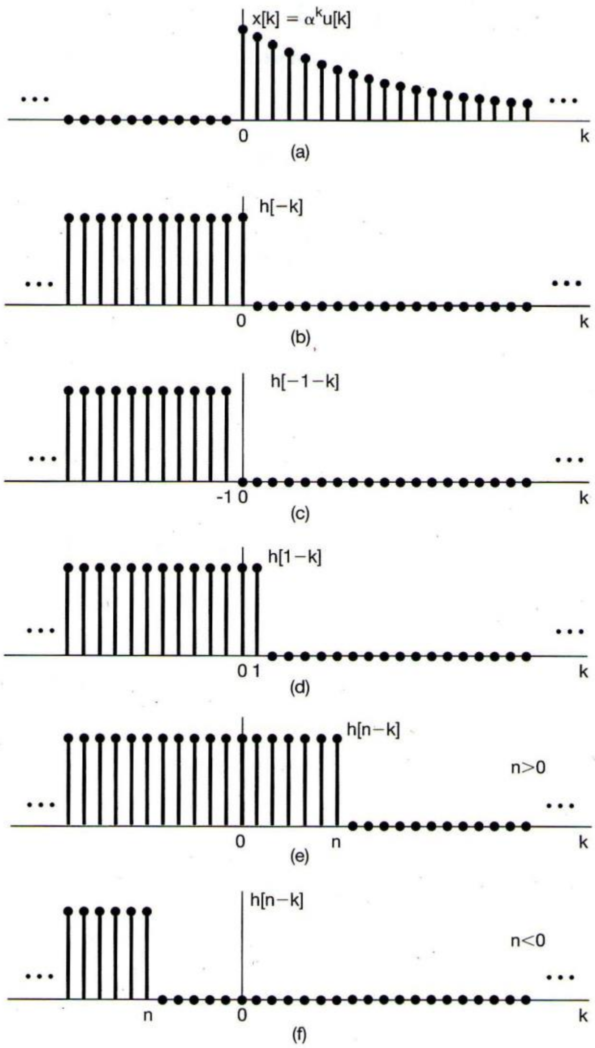


Figure 2.6 Graphical interpretation of the calculation of the convolution sum for Example 2.3.

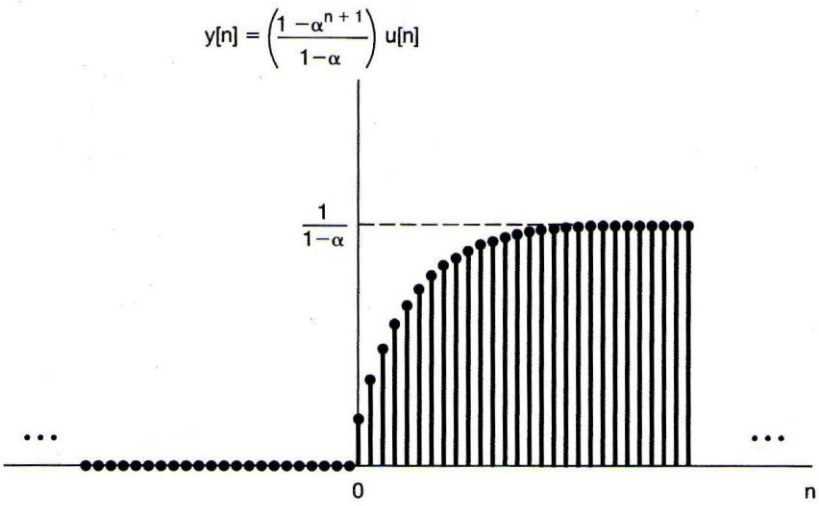


Figure 2.7 Output for Example 2.3.

Thus, for $n \geq 0$,

$$y[n] = \sum_{k=0}^{n} \alpha^k,$$

and using the result of Problem 1.54 we can write this as

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \qquad \text{for } n \geq 0. \tag{2.13}$$

Thus, for all n,

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n].$$

The signal $y[n]$ is sketched in Figure 2.7.

Example 2.4

As a further example, consider the two sequences

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

and

$$h[n] = \begin{cases} \alpha^n, & 0 \leq n \leq 6 \\ 0, & \text{otherwise} \end{cases}.$$

These signals are depicted in Figure 2.8 for a positive value of $\alpha > 1$. In order to calculate the convolution of the two signals, it is convenient to consider five separate intervals for n. This is illustrated in Figure 2.9.

Interval 1. For $n < 0$, there is no overlap between the nonzero portions of $x[k]$ and $h[n-k]$, and consequently, $y[n] = 0$.

Interval 2. For $0 \leq n \leq 4$,

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}.$$

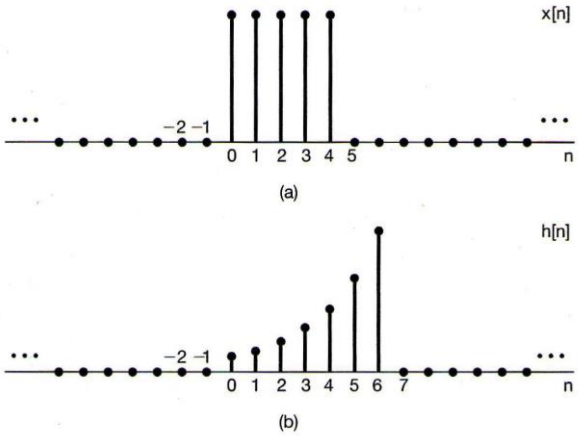


Figure 2.8 The signals to be convolved in Example 2.4.

Thus, in this interval,

$$y[n] = \sum_{k=0}^{n} \alpha^{n-k}. \tag{2.14}$$

We can evaluate this sum using the finite sum formula, eq. (2.13). Specifically, changing the variable of summation in eq. (2.14) from k to $r = n - k$, we obtain

$$y[n] = \sum_{r=0}^{n} \alpha^{r} = \frac{1-\alpha^{n+1}}{1-\alpha}.$$

Interval 3. For $n > 4$ but $n - 6 \leq 0$ (i.e., $4 < n \leq 6$),

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \leq k \leq 4 \\ 0, & \text{otherwise} \end{cases}.$$

Thus, in this interval,

$$y[n] = \sum_{k=0}^{4} \alpha^{n-k}. \tag{2.15}$$

Once again, we can use the geometric sum formula in eq. (2.13) to evaluate eq. (2.15). Specifically, factoring out the constant factor of α^n from the summation in eq. (2.15) yields

$$y[n] = \alpha^{n} \sum_{k=0}^{4} (\alpha^{-1})^{k} = \alpha^{n} \frac{1-(\alpha^{-1})^{5}}{1-\alpha^{-1}} = \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}. \tag{2.16}$$

Interval 4. For $n > 6$ but $n - 6 \leq 4$ (i.e., for $6 < n \leq 10$),

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k}, & (n-6) \leq k \leq 4 \\ 0, & \text{otherwise} \end{cases},$$

so that

$$y[n] = \sum_{k=n-6}^{4} \alpha^{n-k}.$$

We can again use eq. (2.13) to evaluate this summation. Letting $r = k - n + 6$, we obtain

$$y[n] = \sum_{r=0}^{10-n} \alpha^{6-r} = \alpha^{6} \sum_{r=0}^{10-n} (\alpha^{-1})^{r} = \alpha^{6} \frac{1-\alpha^{n-11}}{1-\alpha^{-1}} = \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}.$$

Interval 5. For $n - 6 > 4$, or equivalently, $n > 10$, there is no overlap between the nonzero portions of $x[k]$ and $h[n-k]$, and hence,

$$y[n] = 0.$$

Summarizing, then, we obtain

$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n \le 4 \\ \dfrac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, & 4 < n \le 6 \\ \dfrac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}, & 6 < n \le 10 \\ 0, & 10 < n \end{cases},$$

which is pictured in Figure 2.10.

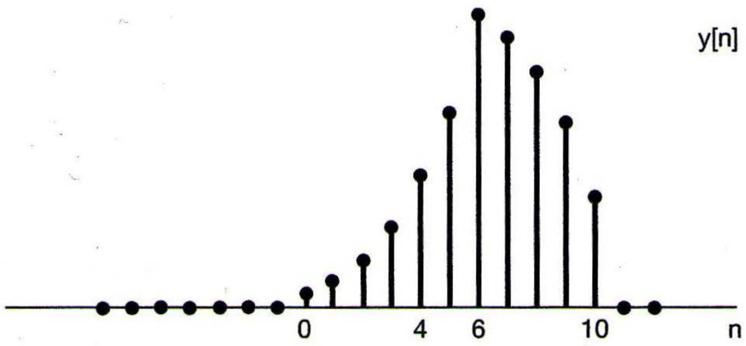


Figure 2.10 Result of performing the convolution in Example 2.4.

3. Continuous Time LTI Systems: The Convolution Integral

In analogy with the results derived and discussed in the preceding section, the goal of this section is to obtain a complete characterization of a continuous-time LTI system in terms of its unit impulse response.

Representation of analogue signals by staircase waveform
We begin by considering a pulse or "staircase" approximation, x(t), to a continuous-time signal x(t), as illustrated in Figure 2.12(a). In Figure 2.12 (a) –(e):

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}, \tag{2.24}$$

then, since $\Delta\delta_\Delta(t)$ has unit amplitude, we have the expression

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta. \tag{2.25}$$

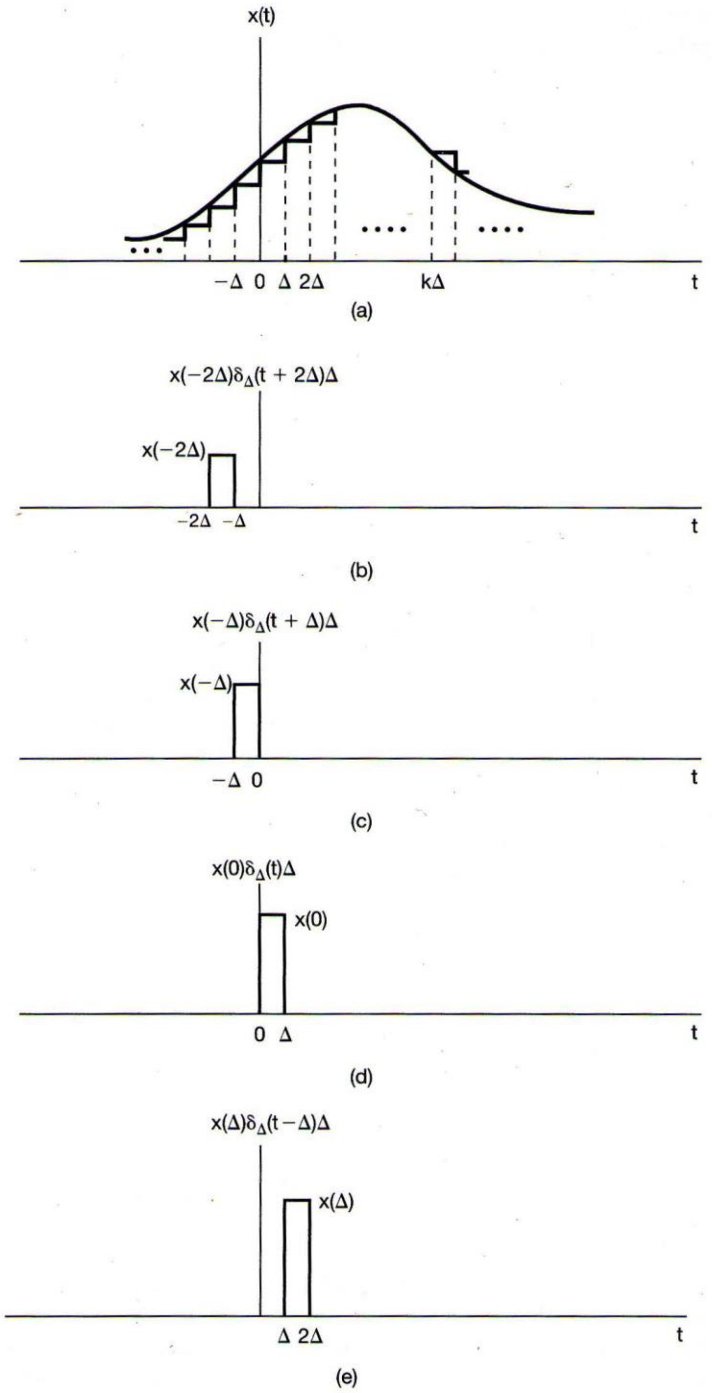


Figure 2.12 Staircase approximation to a continuous-time signal.

As we let Δ approach 0, the approximation $\hat{x}(t)$ becomes better and better, and in the limit equals $x(t)$. Therefore,

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta. \tag{2.26}$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau. \tag{2.27}$$

As in discrete time, we refer to eq. (2.27) as the *sifting property* of the continuous-time impulse. We note that, for the specific example of $x(t) = u(t)$, eq. (2.27) becomes

$$u(t) = \int_{-\infty}^{+\infty} u(\tau)\delta(t-\tau)d\tau = \int_{0}^{\infty} \delta(t-\tau)d\tau, \tag{2.28}$$

since $u(\tau) = 0$ *for* $\tau < 0$ *and* $u(\tau) = 1$ *for* $\tau > 0$.

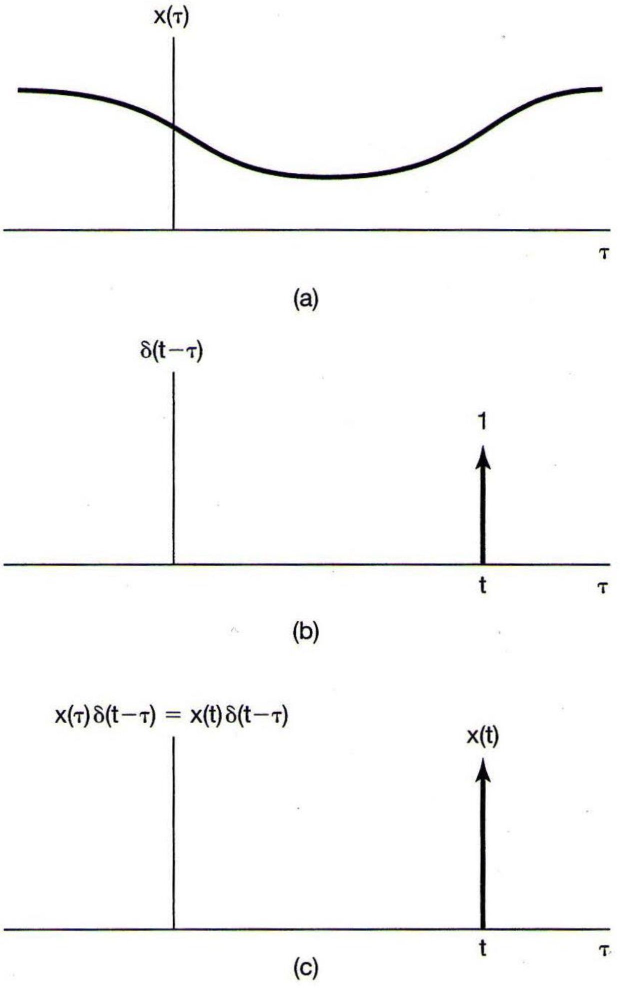


Figure 2.14 (a) Arbitrary signal $x(\tau)$; (b) impulse $\delta(t-\tau)$ as a function of τ with t fixed; (c) product of these two signals.

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau. \tag{2.31}$$

The interpretation of eq. (2.31) is analogous to the one for eq. (2.29). As we showed in Section 2.2.1, any input $x(t)$ can be represented as

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau.$$

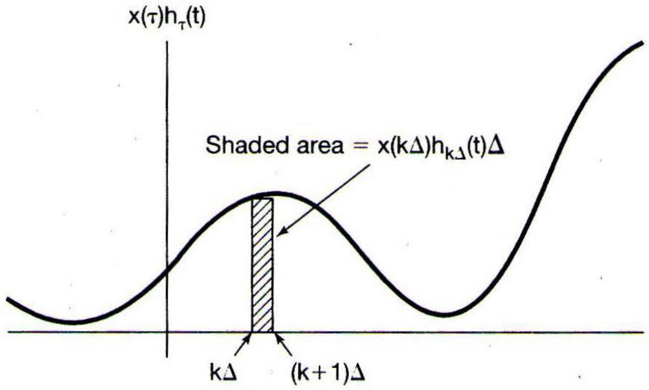


Figure 2.16 Graphical illustration of eqs. (2.30) and (2.31).

and define the *unit impulse response* $h(t)$ as

$$h(t) = h_0(t); \tag{2.32}$$

i.e., $h(t)$ is the response to $\delta(t)$. In this case, eq. (2.31) becomes

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.} \tag{2.33}$$

The convolution of 2 signals is:

$$y(t) = x(t) * h(t). \tag{2.34}$$

<u>Example 2.6</u>

Let x(t) be the input to an LTI system with unit impulse response h(t), where:

$x(t) = e^{-at}u(t), \quad a > 0$ and $h(t) = u(t).$

<u>Solution</u>: For t > 0,

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & \text{otherwise} \end{cases}.$$

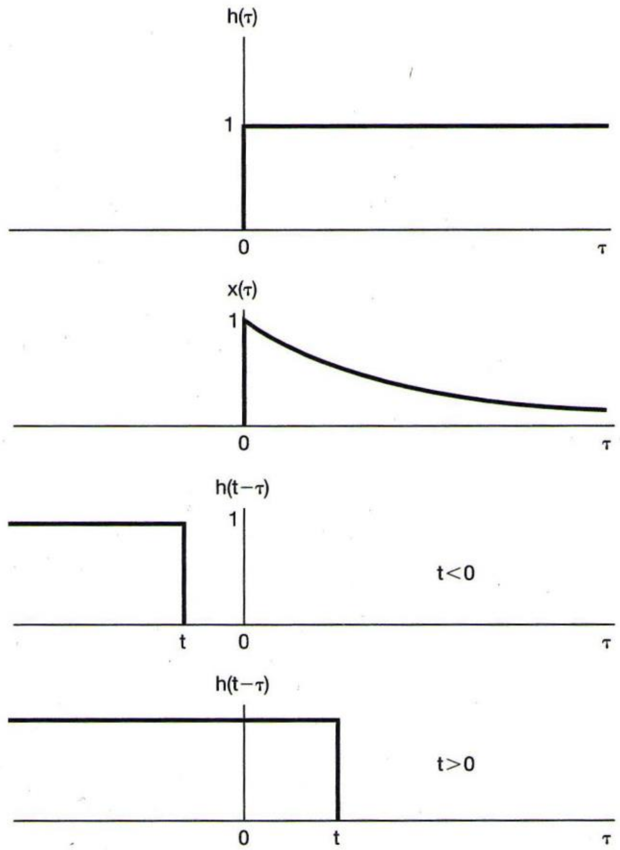


Figure 2.17 Calculation of the convolution integral for Example 2.6.

From this expression, we can compute $y(t)$ for $t > 0$:

$$y(t) = \int_0^t e^{-a\tau} d\tau = -\frac{1}{a} e^{-a\tau}\Big|_0^t$$
$$= \frac{1}{a}(1 - e^{-at}).$$

Thus, for all t, $y(t)$ is

$$y(t) = \frac{1}{a}(1 - e^{-at})u(t),$$

which is shown in Figure 2.18.

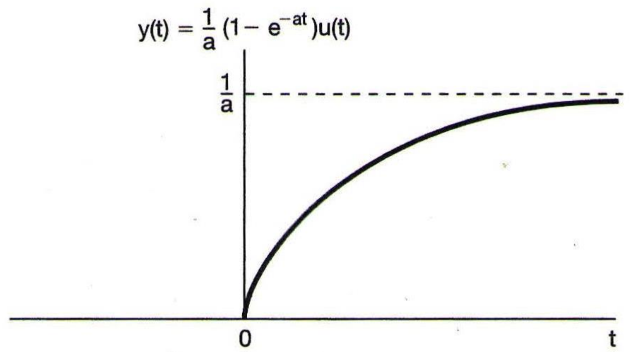


Figure 2.18 Response of the system in Example 2.6 with impulse response $h(t) = u(t)$ to the input $x(t) = e^{-at}u(t)$.

Example 2.7

Consider the convolution of the following two signals:

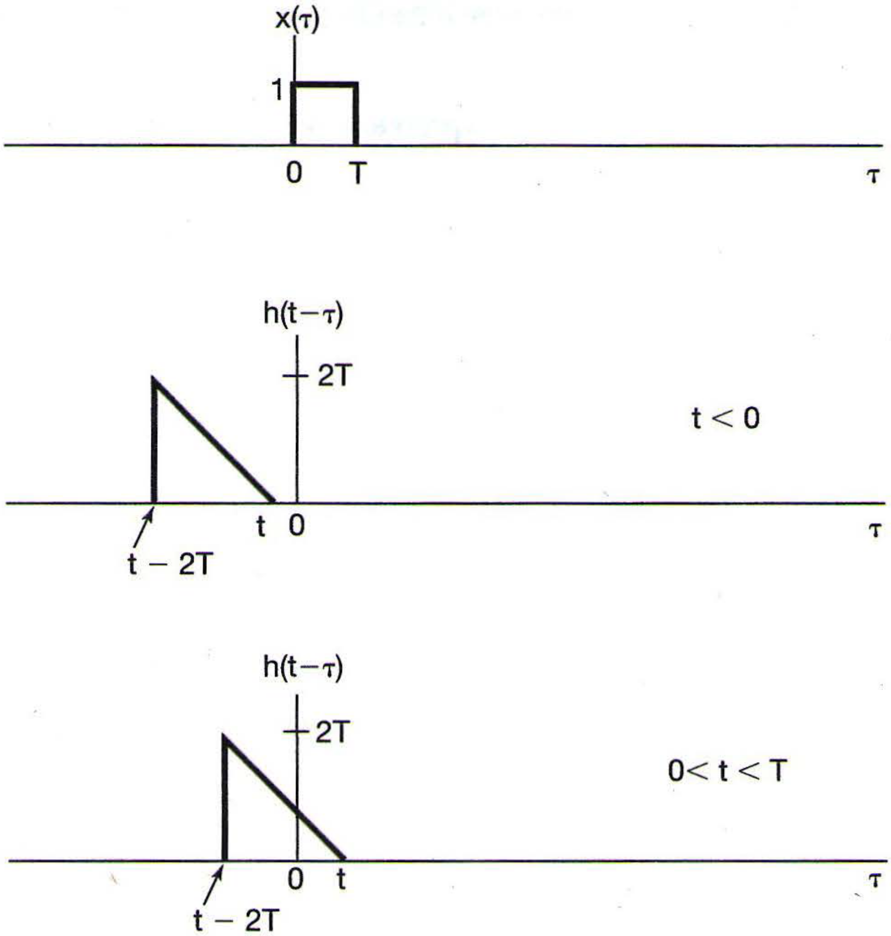
$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases},$$

$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}.$$

Solution: for these three intervals, the integration can be carried out graphically, with the result that

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases},$$

which is depicted in Figure 2.21.



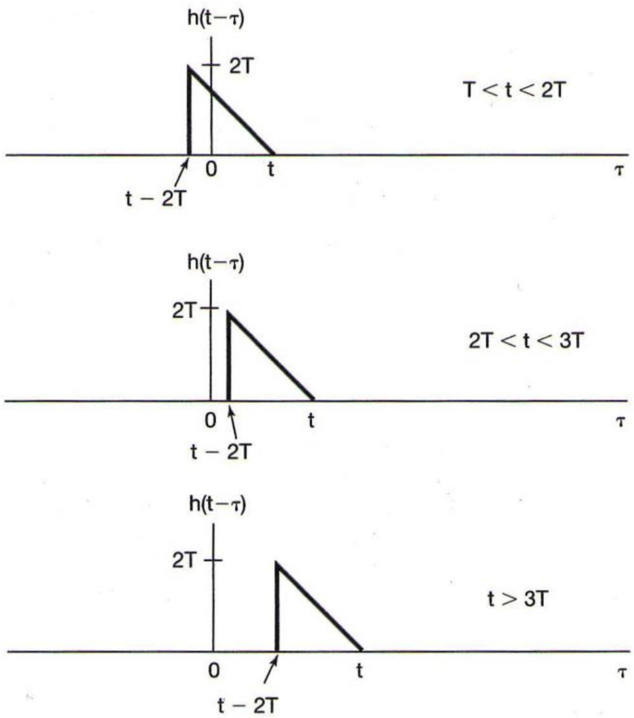


Figure 2.19 Signals $x(\tau)$ and $h(t-\tau)$ for different values of t for Example 2.7.

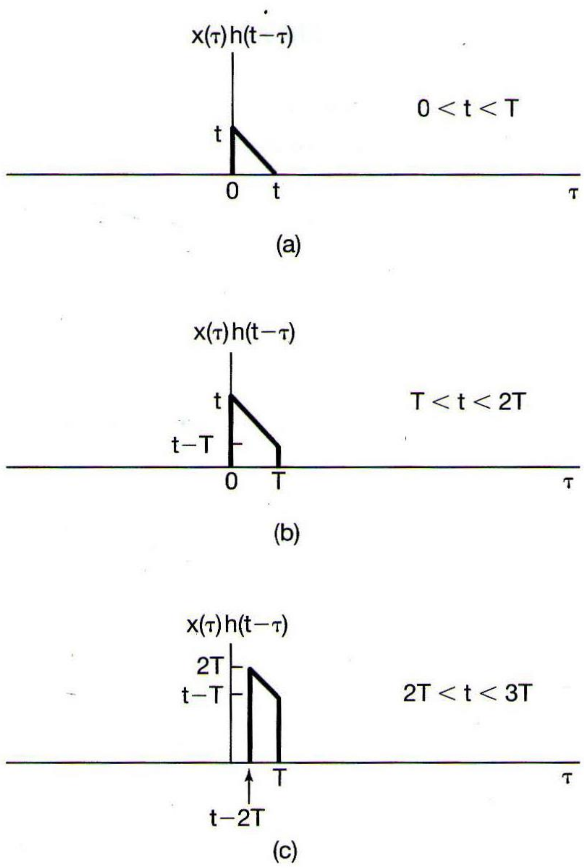


Figure 2.20 Product $x(\tau)h(t-\tau)$ for Example 2.7 for the three ranges of values of t for which this product is not identically zero. (See Figure 2.19.)

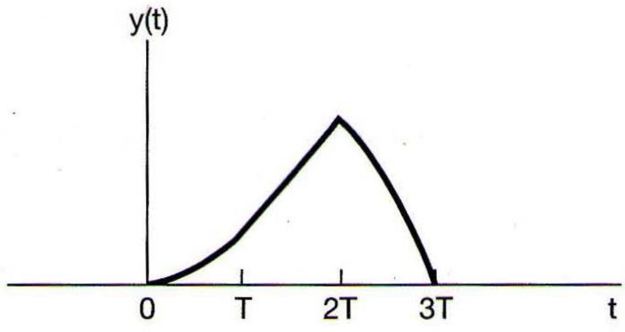


Figure 2.21 Signal $y(t) = x(t) * h(t)$ for Example 2.7.

Example 2.8

Let y(t) denote the convolution of the following two signals:

$$x(t) = e^{2t}u(-t), \tag{2.35}$$
$$h(t) = u(t-3). \tag{2.36}$$

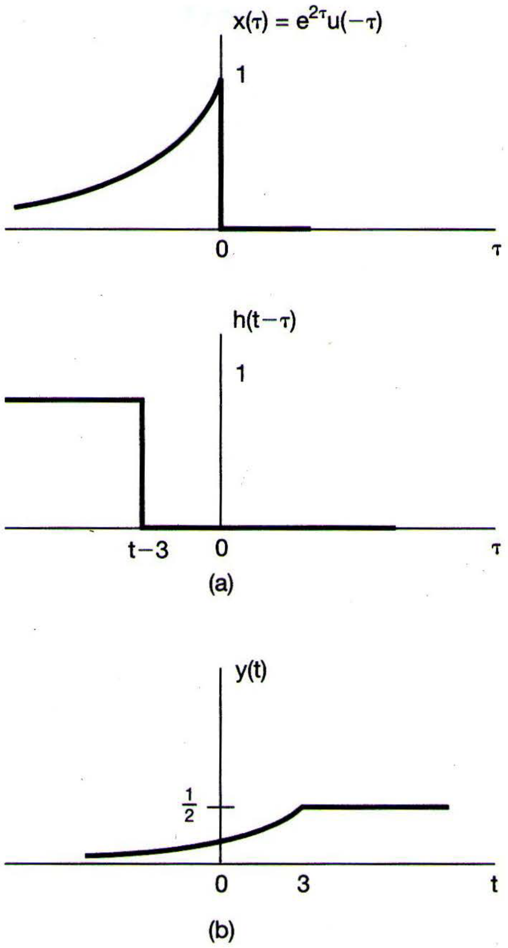


Figure 2.22 The convolution problem considered in Example 2.8.

We see that these signals have regions of nonzero overlap, regardless of t. When $t - 3 \leq 0$, the product of $x(\tau)$ and $h(t - \tau)$ is nonzero for $-\infty < \tau < t - 3$, and the convolution integral becomes

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \frac{1}{2} e^{2(t-3)}. \tag{2.37}$$

For $t - 3 \geq 0$, the product $x(\tau)h(t-\tau)$ is nonzero for $-\infty < \tau < 0$, so that the convolution integral is

$$y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau = \frac{1}{2}. \tag{2.38}$$

The resulting signal $y(t)$ is plotted in Figure 2.22(b).

Glossary – English/Chinese Translation

English	Chinese
Linear Time Invariant System	线性时不变系统
Convolution Sum	卷积和
Superposition Sum	叠加总和
Convolution Integral	卷积积分
Discrete time	离散时间
Continuous Time	连续时间
Superposition Property	叠加属性
Time Shifted	时移
Scaled Unit	缩放单位
Impulse Sequence	脉冲序列
Unit Impulse Response	单元脉冲响应
Geometric Sum Formula	几何求和公式
Analogue Signals	模拟信号
Staircase Waveform	楼梯波形