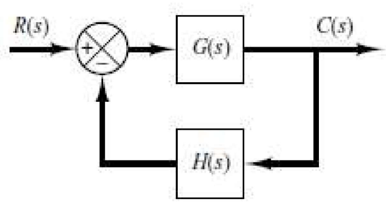
Root Locus Method

Angle and Magnitude Conditions

- Consider the negative feedback system below, the closed-l



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

- The characteristic equation is obtained by setting the denominator of the above function equal to zero:

$$\Delta(s) = 1 + G(s)H(s) = 0 \quad \text{or } G(s)H(s) = -1$$

- The values of *s* that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles,

Angle Condition: $\angle G(s)H(s) = \pm 180°(2k+1), \quad k = 0, 1, 2, \ldots$

Magnitude Condition: $|G(s)H(s)| = 1$

Root Locus Method

Angle and Magnitude Conditions

- A locus of the points in the complex plane satisfying the angle condition alone is the root locus
- The roots of the characteristic equation (the closed-loop poles) corresponding to a given value of the gain can be determined from the magnitude condition
- In many cases, $G(s)H(s)$ involves a **gain parameter** *K*, and the characteristic equation may be written as,

$$1 + \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0$$

- The root loci for the system are the loci of the closed-loop poles as the gain *K* is varied from zero to infinity
- The root loci are always symmetrical about the real axis
- Remember that the angles of the complex quantities originating from the open-loop poles and open-loop zeros to the test point *s* are measured in the counterclockwise direction

Root Locus Method

Angle and Magnitude Conditions

- For example, if $G(s)H(s)$ is given by,

$$G(s)H(s) = \frac{K(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$

(how about the root nature of the above equation?)

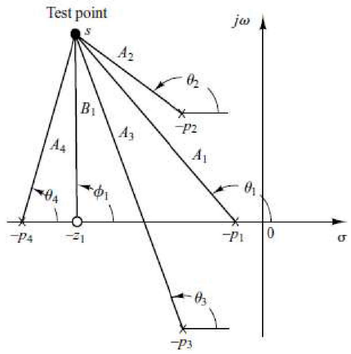
- The angle of $G(s)H(s)$ is then,

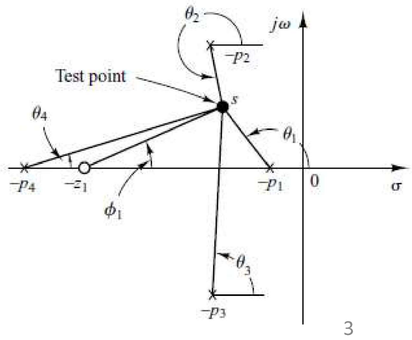
$$\angle G(s)H(s) = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$$

- The magnitude of $G(s)H(s)$ for this system is

$$|G(s)H(s)| = \frac{KB_1}{A_1 A_2 A_3 A_4}$$

where A_1, A_2, A_3, A_4, and B_1 are the magnitudes of the complex quantities $s+p_1$, $s+p_2$, $s+p_3$, $s+p_4$, and $s+z_1$, respectively





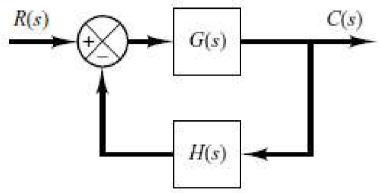
General Rules for Constructing Root Loci

- First, obtain the characteristic equation
  $1 + G(s)H(s) = 0$

- Then rearrange this equation in the form of

$$1 + \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0$$

For $K > 0$



1. Locate the poles and zeros of $G(s)H(s)$ on the s plane

   The root-locus branches start from open-loop poles and terminate at zeros (finite zeros or zeros at infinity), as K increases from zero to infinity

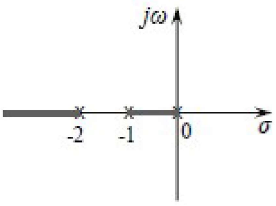
   Number of branches are equal to the number of roots of the characteristic equation

General Rules for Constructing Root Loci

2. Determine the root loci on the real axis

If the total number of real poles and real zeros to the right of this test point is odd, then this point lies on a root locus

Example:



- Select a test point, s, in each interval
- If select a test point on the negative real axis between 0 and –1,

$$\angle s = 180°, \qquad \angle(s+1) = \angle(s+2) = 0°$$

$$\therefore -\angle s - \angle(s+1) - \angle(s+2) = \pm 180°(2k+1), k = 0, 1, 2,3 \ldots$$

- satisfies angle condition and this range forms a portion of the root locus

General Rules for Constructing Root Loci

3. Determine the asymptotes of root loci

The root loci for very large values of s must be asymptotic to straight lines whose angles (slopes) are given by

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{n-m} \quad (k = 0, 1, 2, \ldots)$$

where n = no. of finite poles of $G(s)H(s)$ and m = no. of finite zeros of $G(s)H(s)$

All the asymptotes intersect at a point on the real axis. The point at which they do so is obtained by

$$s = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}$$

General Rules for Constructing Root Loci

4. Find the breakaway and break-in points

   If a root locus lies between two adjacent open-loop poles on the real axis, then there

   exists at least one breakaway point between the two poles

   If the root locus lies between two adjacent zeros on the real axis, then there always exists at least one break-in point between the two zeros

   Suppose that the characteristic equation is given by $B(s) + KA(s) = 0$

   The breakaway and break-in points can be determined from the roots of

$$\frac{dK}{ds} = -\frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)}$$

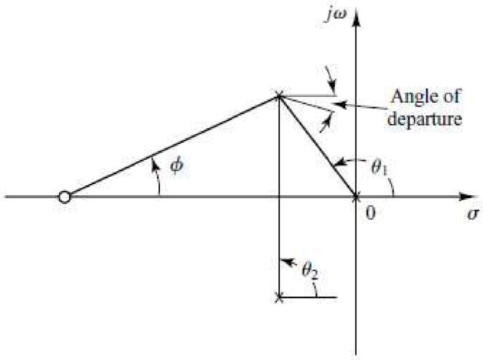
   If a real root is NOT on the root locus portion of the real axis, this root is NOT the actual breakaway or break-in point

General Rules for Constructing Root Loci

5. Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero)

   Angle of departure from a complex pole = 180°
   – (sum of the angles of vectors to a complex pole in question from other poles)
   + (sum of the angles of vectors to a complex pole in question from zeros)

   Angle of arrival at a complex zero = 180°
   – (sum of the angles of vectors to a complex zero in question from other zeros)
   + (sum of the angles of vectors to a complex zero in question from poles)



Angle of departure = ?

General Rules for Constructing Root Loci

6. Find the points where the root loci may cross the imaginary axis
   (a) Use of Routh's stability criterion; OR
   (b) Put $s = j\omega$ in the characteristic equation, equating both the real part and the imaginary part to zero, and solving for ω and K

7. Determine the closed-loop poles or Draw the root locus

A particular point on each root-locus branch will be a closed-loop pole if the value of K at that point satisfies the magnitude condition, $|G(s)H(s)| = 1$

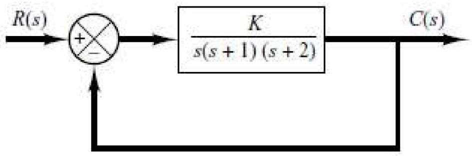
The value of K corresponding to any point s on a root locus can be obtained using the magnitude condition, or

$$K = \frac{\text{product of lengths between point } s \text{ to poles}}{\text{product of lengths between point } s \text{ to zeros}}$$

Example 6

Consider the negative feedback system shown below. For this system,

$$G(s) = \frac{K}{s(s+1)(s+2)} \quad , H(s) = 1$$

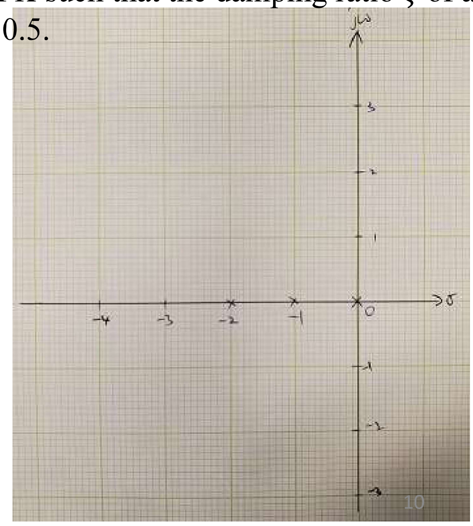


Sketch the root-locus plot and then determine the value of K such that the damping ratio ζ of a pair of dominant complex-conjugate closed-loop poles is 0.5.

Answer:

1. Locate the poles and zeros of $G(s)H(s)$ on the s plane

Pole: $s = 0, s = -1, s = -2$

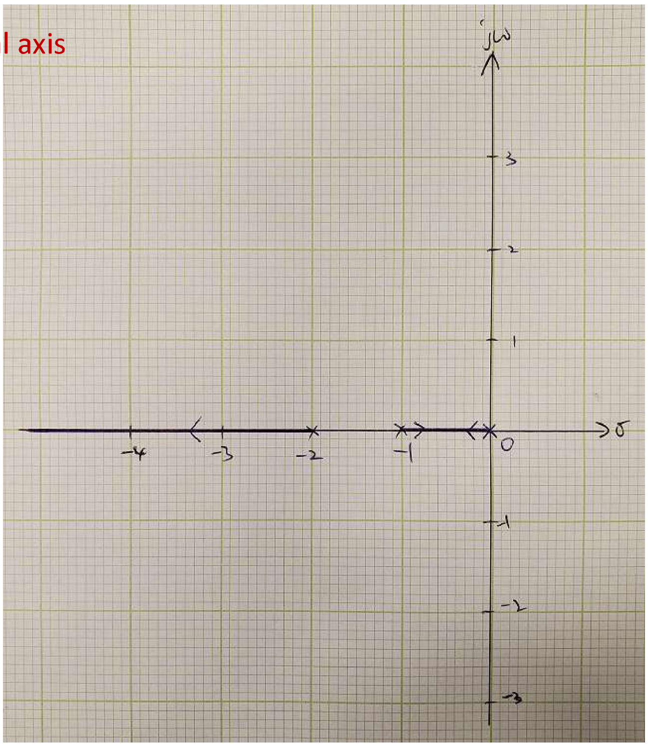


Example 6 (continued)

2. Determine the root loci on the real axis

$(-1, 0)$

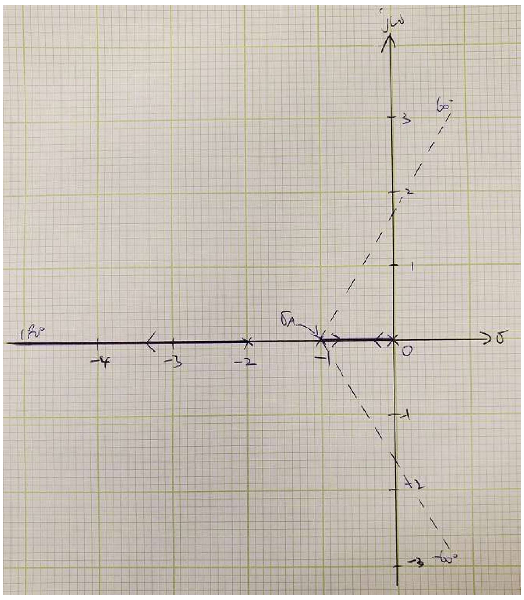
$(-\infty, -2)$



Example 6 (continued)

3. Determine the asymptotes of root loci

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{n-m} \quad (k = 0, 1, 2, \dots)$$



$$= \frac{\pm 180°(2k+1)}{3-0} = \pm 60°(2\text{k}+1)$$

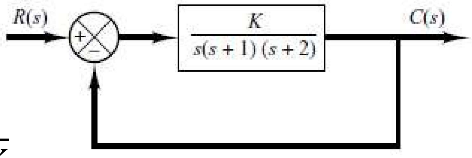
$$= \pm 60°, \pm 180°$$

$$s = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}$$

$$s = \frac{(0) + (-1) + (-2)}{3-0} = -1$$

Example 6 (continued)

4. Find the breakaway and break-in points



$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} = \frac{K}{s(s+1)(s+2)+K}$$

$$\because \frac{C(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

$$\Delta(s) = s(s+1)(s+2)+K = 0$$

$$\therefore \Delta(s) = 1 + G(s)H(s)$$

$$= s^3 + 3s^2 + 2s + K = 0$$

$$\therefore K = -s^3 - 3s^2 - 2s$$

Rejected and why?

$$\frac{dK}{ds} = -3s^2 - 6s - 2 = 0$$

$s = -0.423, -1.58$

Only breakaway point

$K = -(-0.423)^3 - 3(-0.423)^2 - 2(-0.423) = 0.385$

Example 6 (continued)

5. Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero)

Since there are neither complex pole(s) nor complex zero(s), this step can be omitted

Example 6 (continued)

6. Find the points where the root loci may cross the imaginary axis

The characteristic equation, $\Delta(s) = s^3 + 3s^2 + 2s + K = 0$

Routh Array Method

Using Routh array,

s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	
s^0	K	

For stability: $0 < K < 6$

The crossing points on the imaginary axis can then be found by solving the auxiliary equation obtained from the s^2 row; that is,

$$3s^2 + K = 3s^2 + 6 = 0 \implies s = \pm j\sqrt{2}$$

The frequencies at the crossing points on the imaginary axis are thus $\omega = \pm\sqrt{2}$.
The gain value corresponding to the crossing points is $K = 6$.

Example 6 (continued)

6. Find the points where the root loci may cross the imaginary axis

Equating term Method

Put $s = j\omega$ into $\Delta(s)$,

$$\Delta(j\omega) = (j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + K = 0$$

$$= -j\omega^3 - 3\omega^2 + 2j\omega + K = 0$$

Equating terms, we have

$$-j\omega^3 + 2j\omega = 0$$

$$-j\omega(\omega^2 - 2) = 0$$

$$\therefore \omega = 0, \pm\sqrt{2}$$

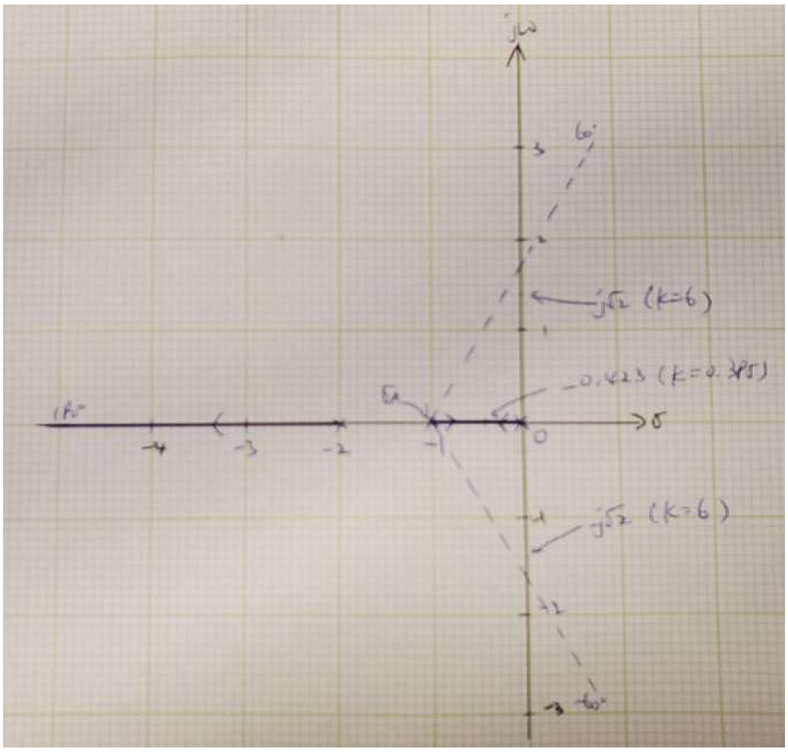
$$-3\omega^2 + K = 0$$

$$3\left(\sqrt{2}\right)^2 = K$$

$$\therefore K = 6$$

Example 6 (continued)

6. Find the points where the root loci may cross the imaginary axis



Example 6 (continued)

7. Determine the closed-loop poles or Draw the root locus