Dr. Norbert Cheung's Series in Electrical Engineering

Level 2 Topic no: 32

Introduction to State Space Control

Contents

Reference:

Chapter 11 & 12 K. Ogata, "Modern Control Engineering"

Chapter 1 & 2, D Wiberg, "State Space and Linear Systems" Schaum's Outline Series

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1. Meaning of State Space

To introduce the subject, let's take an informal, physical approach to the idea of state. (An exact mathematical approach is taken in more advanced texts.) First, we make a distinction between physical and abstract objects. A physical object is an object perceived by our senses whose time behavior we wish to describe, and its abstraction is the mathematical relationships that give some expression for its behavior. This distinction is made because, in making an abstraction, it is possible to lose some of the relationships that make the abstraction behave similar to the physical object. Also, not all mathematical relationships can be realized by a physical object.

Definition 1.1: The *state of a physical object* is any property of the object which relates input to output such that knowledge of the input time function for $t \geqq t_0$ and state at time $t = t_0$ completely determines a unique output for $t \geqq t_0$.

Example 1.1.

Consider a black box, Fig. 1-1, containing a switch to one of two voltage dividers. Intuitively, the state of the box is the position of the switch, which agrees with Definition 1.1. This can be ascertained by the experiment of applying a voltage V to the input terminal. Natural laws (Ohm's law) dictate that if the switch is in the lower position A, the output voltage is $V/2$, and if the switch is in the upper position B, the output voltage is $V/4$. Then the state A determines the input-output pair to be $(V, V/2)$, and the state B corresponds to $(V, V/4)$.

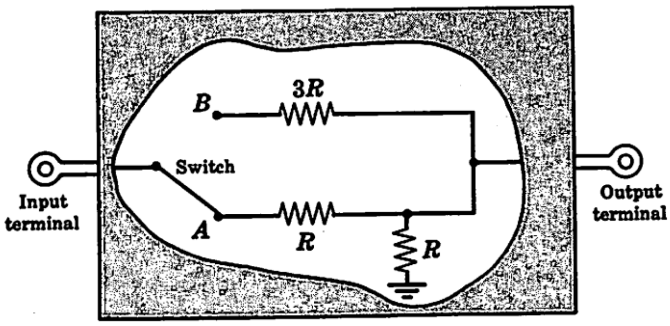


Fig. 1-1

Definition 1.2: An *abstract object* is the totality of input-output pairs that describe the behavior of a physical object.

Definition 1.3: The *state of an abstract object* is a collection of numbers which together with the input $\mathbf{u}(t)$ for all $t \geqq t_0$ uniquely determines the output $\mathbf{y}(t)$ for all $t \geqq t_0$.

Definition 1.4: A *state variable*, denoted by the vector $\mathbf{x}(t)$, is the time function whose value at any specified time is the state of the abstract object at that time.

Definition 1.5: The *state space*, denoted by Σ, is the set of all $\mathbf{x}(t)$.

2. Flow Diagrams

Flow diagrams are a simple diagrammatical means of obtaining the state equations. Because only linear differential or difference equations are considered here, only four basic objects are needed. The utility of flow diagrams results from the fact that no differentiating devices are permitted.

Definition 2.1: A *summer* is a diagrammatical abstract object having n inputs $u_1(t), u_2(t), \ldots, u_n(t)$ and one output $y(t)$ that obey the relationship

$$y(t) = \pm u_1(t) \pm u_2(t) \pm \cdots \pm u_n(t)$$

where the sign is positive or negative as indicated in Fig. 2-1, for example.

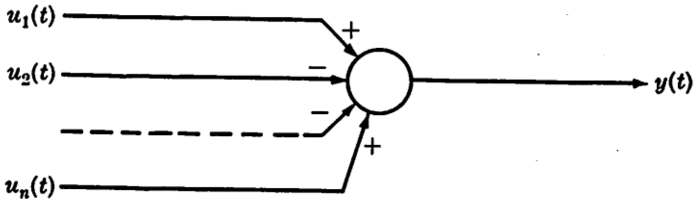


Fig. 2-1. Summer

Definition 2.2: A *scalor* is a diagrammatical abstract object having one input $u(t)$ and one output $y(t)$ such that the input is scaled up or down by the time function $\alpha(t)$ as indicated in Fig. 2-2. The output obeys the relationship $y(t) = \alpha(t)\,u(t)$.

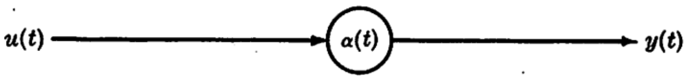


Fig. 2-2. Scalor

Definition 2.3: An *integrator* is a diagrammatical abstract object having one input $u(t)$, one output $y(t)$, and perhaps an initial condition $y(t_0)$ which may be shown or not, as in Fig. 2-3. The output obeys the relationship

$$y(t) = y(t_0) + \int_{t_0}^{t} u(\tau)\, d\tau$$

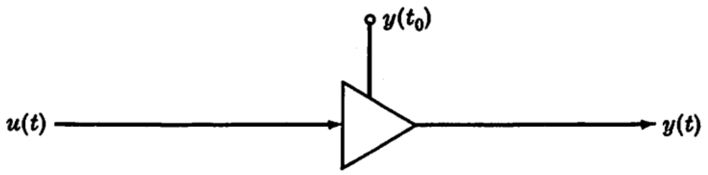


Fig. 2-3. Integrator at Time t

3. The Transfer Function

Transfer function. The *transfer function* of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

Consider the linear time-invariant system defined by the following differential equation:

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m) \qquad (3\text{–}1)$$

where y is the output of the system and x is the input. The transfer function of this system is obtained by taking the Laplace transforms of both sides of Equation (3–1), under the assumption that all initial conditions are zero, or

$$\text{Transfer function} = G(s) = \left.\frac{\mathcal{L}[\text{output}]}{\mathcal{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$

$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \qquad (3\text{–}2)$$

By using the concept of transfer function, it is possible to represent system dynamics by algebraic equations in s. If the highest power of s in the denominator of the transfer function is equal to n, the system is called an *nth-order system.*

State-space representation in canonical forms. Consider a system defined by

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u \qquad (11\text{–}1)$$

where u is the input and y is the output. This equation can also be written as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \qquad (11\text{–}2)$$

4. Observable Canonical Form

Consider a general time-invariant linear differential equation with one input and one output, with the letter p denoting the time derivative d/dt. Only the differential equations need be considered, because by Section 2.2 discrete time systems follow analogously.

$$p^n y + \alpha_1 p^{n-1} y + \cdots + \alpha_{n-1} p y + \alpha_n y = \beta_0 p^n u + \beta_1 p^{n-1} u + \cdots + \beta_{n-1} p u + \beta_n u \qquad (2.3)$$

This can be rewritten as

$$p^n(y - \beta_0 u) + p^{n-1}(\alpha_1 y - \beta_1 u) + \cdots + p(\alpha_{n-1} y - \beta_{n-1} u) + \alpha_n y - \beta_n u = 0$$

because $\alpha_i p^{n-i} y = p^{n-i} \alpha_i y$, which is not true if α_i depends on time. Dividing through by p^n and rearranging gives

$$y = \beta_0 u + \frac{1}{p}(\beta_1 u - \alpha_1 y) + \cdots + \frac{1}{p^{n-1}}(\beta_{n-1} u - \alpha_{n-1} y) + \frac{1}{p^n}(\beta_n u - \alpha_n y) \qquad (2.4)$$

from which the flow diagram shown in Fig. 2-9 can be drawn starting with the output y at the right and working to the left.

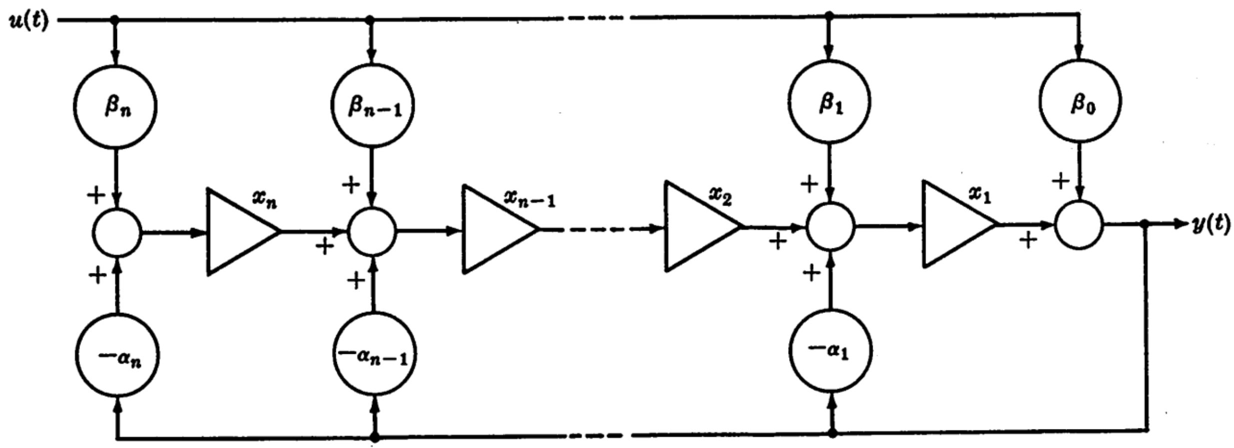


Fig. 2-9. Flow Diagram of the First Canonical Form

The output of each integrator is labeled as a state variable.

The summer equations for the state variables have the form

$$\begin{aligned}
y &= x_1 + \beta_0 u \\
\dot{x}_1 &= -\alpha_1 y + x_2 + \beta_1 u \\
\dot{x}_2 &= -\alpha_2 y + x_3 + \beta_2 u \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{x}_{n-1} &= -\alpha_{n-1} y + x_n + \beta_{n-1} u \\
\dot{x}_n &= -\alpha_n y + \beta_n u
\end{aligned} \qquad (2.5)$$

Using the first equation in (2.5) to eliminate y, the differential equations for the state variables can be written in the canonical matrix form

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -\alpha_1 & 1 & 0 & \dots & 0 \\ -\alpha_2 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\alpha_{n-1} & 0 & 0 & \dots & 1 \\ -\alpha_n & 0 & 0 & \dots & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} \beta_1 - \alpha_1\beta_0 \\ \beta_2 - \alpha_2\beta_0 \\ \vdots \\ \beta_{n-1} - \alpha_{n-1}\beta_0 \\ \beta_n - \alpha_n\beta_0 \end{pmatrix} u \qquad (2.6)$$

We will call this the first canonical form. Note the 1s above the diagonal and the α's down the first column of the $n \times n$ matrix. Also, the output can be written in terms of the state vector

$$y = (1\ 0\ \dots\ 0\ 0)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \beta_0 u \qquad (2.7)$$

Note this form can be written down directly from the original equation (2.3).

5. Controllable Canonical Form

Another useful form can be obtained by turning the first canonical flow diagram "backwards." This change is accomplished by reversing all arrows and integrators, interchanging summers and connection points, and interchanging input and output. This is a heuristic method of deriving a specific form that will be developed further in Chapter 7.

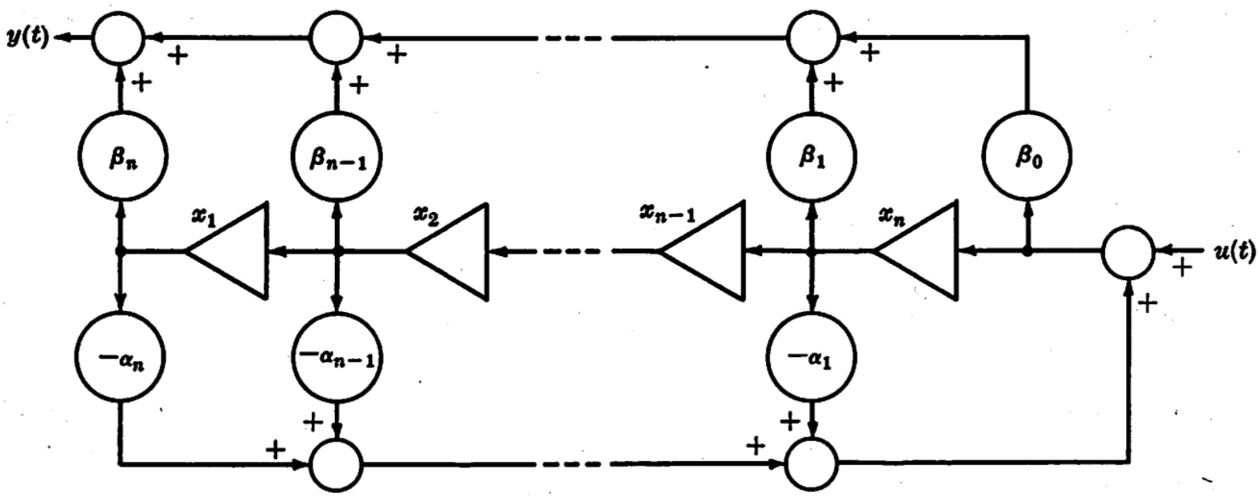


Fig. 2-10. Flow Diagram of the Second Canonical (Phase-variable) Form

Here the output of each integrator has been relabeled. The equations for the state variables are now

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\cdots\cdots\cdots\cdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -\alpha_1 x_n - \alpha_2 x_{n-1} - \cdots - \alpha_{n-1} x_2 - \alpha_n x_1 + u \\ y &= \beta_n x_1 + \beta_{n-1} x_2 + \cdots + \beta_1 x_n + \beta_0[u - \alpha_1 x_n - \cdots - \alpha_{n-1} x_2 - \alpha_n x_1] \end{aligned} \tag{2.8}$$

In matrix form, (*2.8*) may be written as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u \tag{2.9}$$

and

$$y = (\beta_n - \alpha_n\beta_0 \quad \beta_{n-1} - \alpha_{n-1}\beta_0 \quad \cdots \quad \beta_1 - \alpha_1\beta_0) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \beta_0 u \tag{2.10}$$

This will be called the second canonical form, or phase-variable canonical form. Here the 1s are above the diagonal but the α's go across the bottom row of the $n \times n$ matrix. By eliminating the state variables **x**, the general input-output relation (*2.3*) can be verified.

5. Jordan Canonical Form

The general time-invariant linear differential equation (*2.3*) for one input and one output can be written as

$$y = \frac{\beta_0 p^n + \beta_1 p^{n-1} + \cdots + \beta_{n-1} p + \beta_n}{p^n + \alpha_1 p^{n-1} + \cdots + \alpha_{n-1} p + \alpha_n} u \tag{2.11}$$

By dividing once by the denominator, this becomes

$$y = \beta_0 u + \frac{(\beta_1 - \alpha_1\beta_0)p^{n-1} + (\beta_2 - \alpha_2\beta_0)p^{n-2} + \cdots + (\beta_{n-1} - \alpha_{n-1}\beta_0)p + \beta_n - \alpha_n\beta_0}{p^n + \alpha_1 p^{n-1} + \cdots + \alpha_{n-1} p + \alpha_n} u \tag{2.12}$$

Consider first the case where the denominator polynomial factors into distinct poles λ_i, $i = 1, 2, \ldots, n$. Distinct means $\lambda_i \neq \lambda_j$ for $i \neq j$, that is, no repeated roots. Because most practical systems are stable, the λ_i usually have negative real parts.

$$p^n + \alpha_1 p^{n-1} + \cdots + \alpha_{n-1} p + \alpha_n = (p - \lambda_1)(p - \lambda_2)\cdots(p - \lambda_n) \tag{2.13}$$

A partial fraction expansion can now be made having the form

$$y = \beta_0 u + \frac{\rho_1}{p-\lambda_1}u + \frac{\rho_2}{p-\lambda_2}u + \cdots + \frac{\rho_n}{p-\lambda_n}u \qquad (2.14)$$

Here the residue ρ_i can be calculated as

$$\rho_i = \frac{(\beta_1-\alpha_1\beta_0)\lambda_i^{n-1} + (\beta_2-\alpha_2\beta_0)\lambda_i^{n-2} + \cdots + (\beta_{n-1}-\alpha_{n-1}\beta_0)\lambda_i + (\beta_n-\alpha_n\beta_0)}{(\lambda_i-\lambda_1)(\lambda_i-\lambda_2)\cdots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\cdots(\lambda_i-\lambda_n)} \qquad (2.15)$$

The partial fraction expansion (2.14) gives a very simple flow diagram, shown in Fig. 2-11 following.

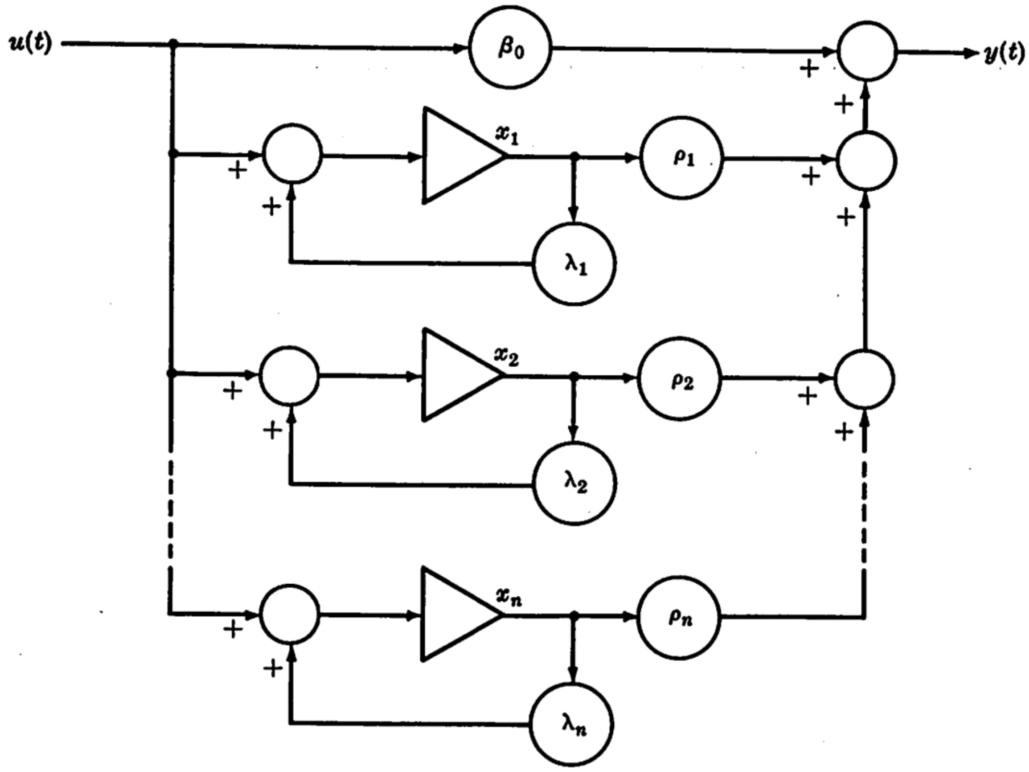


Fig. 2-11. Jordan Flow Diagram for Distinct Roots

Note that because ρ_i and λ_i can be complex numbers, the states x_i are complex-valued functions of time. The state equations assume the simple form

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + u \\ \dot{x}_2 &= \lambda_2 x_2 + u \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{x}_n &= \lambda_n x_n + u \\ y &= \beta_0 u + \rho_1 x_1 + \rho_2 x_2 + \cdots + \rho_n x_n \end{aligned} \qquad (2.16)$$

To Summarize:

Controllable canonical form. The following state-space representation is called a controllable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u \tag{11–3}$$

$$y = [b_n - a_n b_0 \;\vdots\; b_{n-1} - a_{n-1} b_0 \;\vdots\; \cdots \;\vdots\; b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u \tag{11–4}$$

The controllable canonical form is important in discussing the pole-placement approach to the control systems design. [The derivation of Equations (11–3) and (11–4) from Equation (11–1) or (11–2) is presented in Problem A-11-1.]

Observable canonical form. The following state-space representation is called an observable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u \tag{11–5}$$

$$y = [0 \quad 0 \quad \cdots \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u \tag{11–6}$$

Note that the $n \times n$ state matrix of the state equation given by Equation (11–5) is the transpose of that of the state equation defined by Equation (11–3).

Diagonal canonical form. Consider the transfer function system defined by Equation (11–2). Here we consider the case where the denominator polynomial involves only distinct roots. For the distinct roots case, Equation (11–2) can be written as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n} \tag{11–7}$$

The diagonal canonical form of the state-space representation of this system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & 0 \\ & -p_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u \tag{11–8}$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u \tag{11–9}$$

6. General State Equations

Multiple input–multiple output systems can be put in the same canonical forms as single input–single output systems. Due to complexity of notation, they will not be considered here. The input becomes a vector $\mathbf{u}(t)$ and the output a vector $\mathbf{y}(t)$. The components are the inputs and outputs, respectively. Inspection of matrix equations (*2.6*), (*2.9*), (*2.21*) and (*2.38*) indicates a similarity of form. Accordingly a general form for the state equations of a linear differential system of order n with m inputs and k outputs is

$$\begin{aligned} d\mathbf{x}/dt &= \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} \\ \mathbf{y} &= \mathbf{C}(t)\mathbf{x} + \mathbf{D}(t)\mathbf{u} \end{aligned} \tag{2.39}$$

where $\mathbf{x}(t)$ **is an** n**-vector,**
$\mathbf{u}(t)$ **is an** m**-vector,**
$\mathbf{y}(t)$ **is a** k**-vector,**
$\mathbf{A}(t)$ **is an** $n \times n$ **matrix,**
$\mathbf{B}(t)$ **is an** $n \times m$ **matrix,**
$\mathbf{C}(t)$ **is a** $k \times n$ **matrix,**
$\mathbf{D}(t)$ **is a** $k \times m$ **matrix.**

Specifically, if the system has only one input u and one output y, the differential equations for the system are

$$dx/dt = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t)u$$

$$y = \mathbf{c}^{\dagger}(t)\mathbf{x} + d(t)u$$

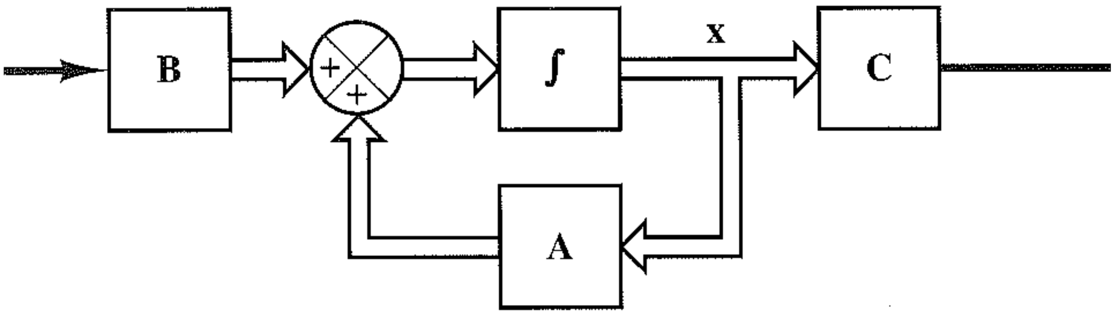
and similarly for discrete time systems. Here $\mathbf{c}(t)$ is taken to be a column vector, and $\mathbf{c}^{\dagger}(t)$ denotes the complex conjugate transpose of the column vector. Hence $\mathbf{c}^{\dagger}(t)$ is a row vector, and $\mathbf{c}^{\dagger}(t)\mathbf{x}$ is a scalar. Also, since u, y and $d(t)$ are not boldface, they are scalars.

In general case….

Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$



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