Question

State Space Analysis

7. Consider a system defined by the following state-space equations:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u \quad \text{and} \quad y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Obtain the transfer function $G(s)$ of the system.

(Ans: $G(s) = \frac{12s+59}{s^2+6s+8}$)

Solution

7. $$\frac{Y(s)}{X(s)} = C(sI - A)^{-1}B = \begin{bmatrix} 1 & 2 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

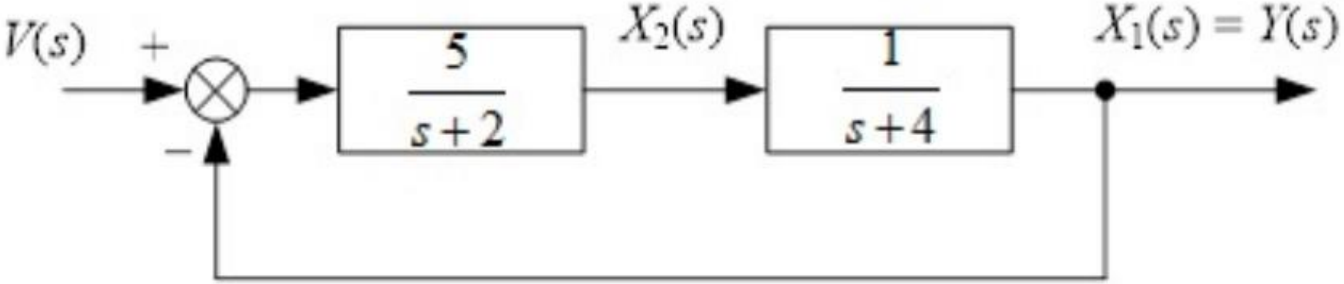
$$\frac{Y(s)}{X(s)} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1} = \frac{1}{(s+5)(s+1) - (-3)(1)} \begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}$$

$$\frac{Y(s)}{X(s)} = \begin{bmatrix} 1 & 2 \end{bmatrix} \frac{1}{(s+5)(s+1) - (-3)(1)} \begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \frac{1}{s^2 + 6s + 8} \begin{bmatrix} s+7 & 2s+9 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \frac{12s + 59}{s^2 + 6s + 8}$$

Question

8. For the system shown below, find the state-space equations and calculate the state transition matrix in time domain.



(Ans: $\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}-4 & 1\\ -5 & -2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 5\end{bmatrix} v(t),\ y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix},$

$\phi(t) = \begin{bmatrix} e^{-3t}\cos(2t) - \frac{1}{2}e^{-3t}\sin(2t) & \frac{1}{2}e^{-3t}\sin(2t) \\ -\frac{5}{2}e^{-3t}\sin(2t) & e^{-3t}\cos(2t) + \frac{1}{2}e^{-3t}\sin(2t) \end{bmatrix}$)

Solution

8. $X_1(s) = \frac{1}{s+4} X_2(s) \;\Rightarrow\; sX_1(s) = -4X_1(s) + X_2(s)$

$X_2(s) = \frac{5}{s+2}[V(s) - X_1(s)] \;\Rightarrow sX_2(s) = 5V(s) - 5X_1(s) - 2X_2(s)$

Taking inverse Laplace Transform, we have

$\dot{x}_1(t) = -4x_1(t) + x_2(t)$ and $\dot{x}_2(t) = -5x_1(t) - 2x_2(t) + 5v(t)$

Hence, the state space equations are,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} v(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The state transmition matrix in s-domain, $\Phi(s) = (sI - A)^{-1}$

$$\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 1 \\ -5 & -2 \end{bmatrix}\right)^{-1} = \begin{bmatrix} s+4 & -1 \\ 5 & s+2 \end{bmatrix}^{-1} = \frac{1}{(s+2)(s+4) - (5)(-1)} \begin{bmatrix} s+2 & 1 \\ -5 & s+4 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+2}{s^2+6s+13} & \frac{1}{s^2+6s+13} \\ \frac{-5}{s^2+6s+13} & \frac{s+4}{s^2+6s+13} \end{bmatrix} = \begin{bmatrix} \frac{(s+3)-1}{(s+3)^2+2^2} & \frac{1}{(s+3)^2+2^2} \\ \frac{-5}{(s+3)^2+2^2} & \frac{(s+3)+1}{(s+3)^2+2^2} \end{bmatrix}$$

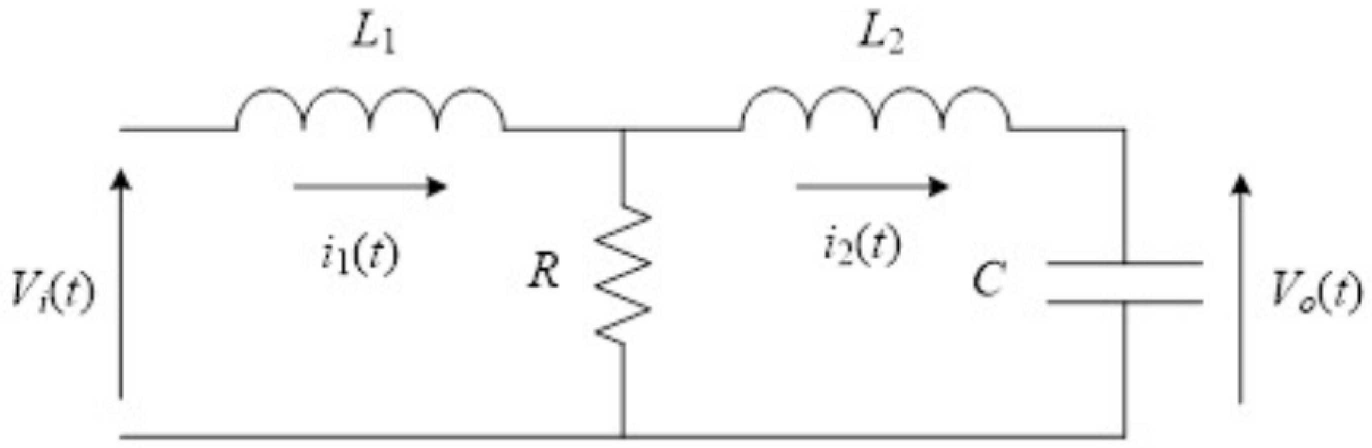
$$= \begin{bmatrix} \frac{s+3}{(s+3)^2+2^2} - \frac{1}{2}\frac{2}{(s+3)^2+2^2} & \frac{1}{2}\frac{2}{(s+3)^2+2^2} \\ -\frac{5}{2}\frac{2}{(s+3)^2+2^2} & \frac{s+3}{(s+3)^2+2^2} + \frac{1}{2}\frac{2}{(s+3)^2+2^2} \end{bmatrix}$$

Then, taking inverse Laplace transform (rules 15 and 16), we have

$$\phi(t) = \begin{bmatrix} e^{-3t}\cos(2t) - \frac{1}{2}e^{-3t}\sin(2t) & \frac{1}{2}e^{-3t}\sin(2t) \\ -\frac{5}{2}e^{-3t}\sin(2t) & e^{-3t}\cos(2t) + \frac{1}{2}e^{-3t}\sin(2t) \end{bmatrix}$$

Question

9. Obtaint the state space equations for the following circuit.



(Ans: $\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}0 & 0 & \frac{1}{C}\\ 0 & -\frac{R}{L_1} & \frac{R}{L_1}\\ -\frac{1}{L_2} & \frac{R}{L_2} & -\frac{R}{L_2}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{L_1}\\ 0\end{bmatrix}u(t),\ y(t)=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$)

End of Tutorial Questions (Part 4)

Solution

9. We first write out the differential equations of the system,

$$V_i(t) = L_1 \frac{d}{dt} i_1(t) + [i_1(t) - i_2(t)]R$$

$$[i_1(t) - i_2(t)]R = L_2 \frac{d}{dt} i_2(t) + \frac{1}{C}\int i_2(t)\,dt$$

$$V_o(t) = \frac{1}{C}\int i_2(t)\,dt$$

Since state space equations related differntial functions instead of integrals, hence, we need to transform integral into differentiation as follow,

$$V_o(t) = \frac{1}{C}\int i_2(t)\,dt \;\Rightarrow i_2(t) = C\frac{d}{dt}V_o(t)$$

Let $x_1(t) = V_o(t) = y(t)$, $x_2(t) = i_1(t)$, $x_3(t) = i_2(t)$ and $u(t) = V_i(t)$, then we have,

$$u(t) = L_1\dot{x}_2(t) + x_2(t)R - x_3(t)R$$

$$x_2(t)R - x_3(t)R = L_2\dot{x}_3(t) + x_1(t)$$

$$x_3(t) = C\dot{x}_1(t)$$

Rearranging the terms,

$$\dot{x}_2(t) = -\frac{R}{L_1}x_2(t) + \frac{R}{L_1}x_3(t) + \frac{1}{L_1}u(t)$$

$$\dot{x}_3(t) = -\frac{1}{L_2}x_1(t) + \frac{R}{L_2}x_2(t) - \frac{R}{L_2}x_3(t)$$

$$\dot{x}_1(t) = \frac{1}{C}x_3(t)$$

Hence, the state space equations of the sytsem is,

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{C} \\ 0 & -\frac{R}{L_1} & \frac{R}{L_1} \\ -\frac{1}{L_2} & \frac{R}{L_2} & -\frac{R}{L_2}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{L_1}\\ 0\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

End of Tutorial Questions (Part 4) Solution