Question

Gain and Phase Margin

3. Determine the value of K for the open-loop transfer function of a control system,

$$G(s) = \frac{K}{s(s+2)(s+10)}$$

such that the system may have

(a) a gain margin of 6 dB; and (b) a phase margin of 45° **analytically**.

(c) Repeat (a) and (b) **graphically** by using ω = 0.01, 0.04, 0.08, 0.1, 0.5, 0.8, 1, 5, 8, 10 and 20 rad/s.

(Ans: (a) $K = 120.28$; (b) $K = 37.22$; (c) 157.18 and 35.18)

Solution

3. (a)

$$G(s) = \frac{K}{s(s+2)(s+10)}$$

$$G(j\omega) = \frac{K}{j\omega(j\omega+2)(j\omega+10)}$$

$$|G(j\omega)| = 20\log\frac{K}{\omega\sqrt{\omega^2+2^2}\sqrt{\omega^2+10^2}} \quad \text{(dB)}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10} \quad (^\circ)$$

When system phase $\phi = -180^\circ$, and put into the phase equation, we can find the phase crossover frequency, ω_{pc}:

$$-180^\circ = -90^\circ - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10} \quad \Rightarrow \quad \tan^{-1}\frac{\omega}{2} + \tan^{-1}\frac{\omega}{10} = 90^\circ$$

$$\frac{\frac{\omega}{2}+\frac{\omega}{10}}{1-\frac{\omega^2}{20}} = \infty \quad \Rightarrow \quad 1-\frac{\omega^2}{20} = 0$$

$$\therefore \omega = \sqrt{20} \text{ rad/s or } 4.47 \text{ rad/s}$$

Substitute into this phase crossover frequency into the magnitude equation, we have

$$|G(j\omega)| = 20\log\frac{K}{\sqrt{20}\sqrt{\left(\sqrt{20}\right)^2+2^2}\sqrt{\left(\sqrt{20}\right)^2+10^2}} = 20\log\frac{K}{\sqrt{20}\sqrt{24}\sqrt{120}}$$

If gain margin = 6 dB, we have

$$20\log\frac{K}{\sqrt{20}\sqrt{24}\sqrt{120}} = -6$$

$$\therefore K = 120.28$$

3. (b) With the phase margin = 45°, the corresponding open-loop phase = –135°

$$-135° = -90° - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10} \quad \Rightarrow \quad \tan^{-1}\frac{\omega}{2} + \tan^{-1}\frac{\omega}{10} = 45°$$

$$\frac{\frac{\omega}{2}+\frac{\omega}{10}}{1-\frac{\omega^2}{20}} = 1 \quad \Rightarrow \quad 1-\frac{\omega^2}{20} = \frac{\omega}{2}+\frac{\omega}{10} \quad \Rightarrow \quad \frac{\omega^2}{20}+\frac{6\omega}{10}-1=0 \quad \Rightarrow \quad \frac{\omega^2+12\omega-20}{20}=0$$

$$\therefore \omega = 1.48 \text{ rad/s or} -13.48 \text{ rad/s (rejetced)}$$

Noting that the magnitude of the system ($|G(j\omega)|$) must be equal to 0 dB at ω = 1.48 rad/s, we have

$$|G(j\omega)| = 20\log\frac{K}{1.48\sqrt{(1.48)^2+2^2}\sqrt{(1.48)^2+10^2}} = 0$$

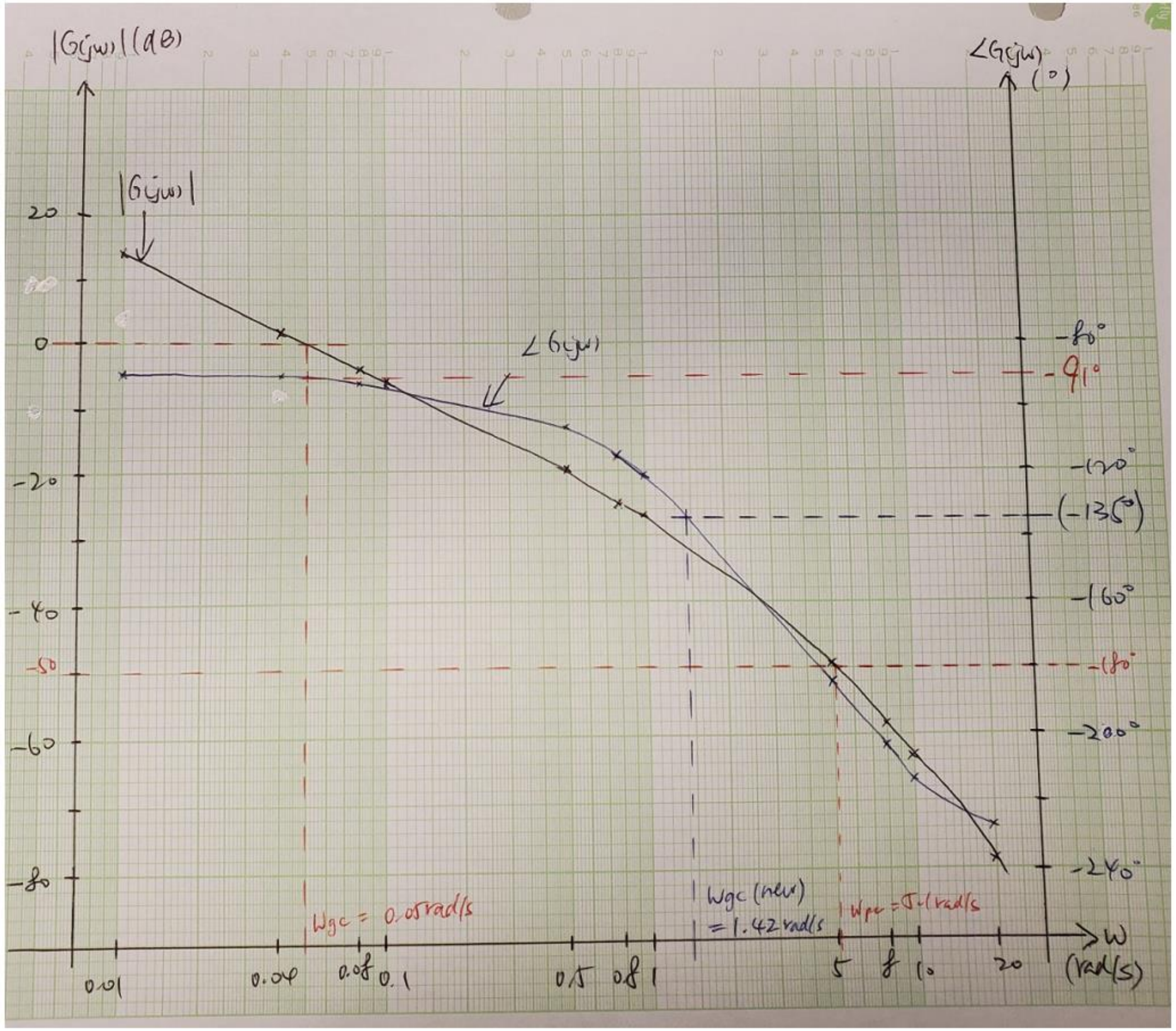
$$\therefore K = 37.22$$

Solution

3. (c) From the system magnitude and phase equations obtained in part (a), we can draw the exact Bode diagrams

ω (rad/s)	$\lvert G(j\omega)\rvert$ (dB)	$\angle G(j\omega)$ (°)
0.01	14.0	–90.3
0.04	1.9	–91.4
0.08	–4.1	–92.7
0.1	–6.0	–93.4
0.5	–20.2	–106.9
0.8	–24.8	–116.4
1	–27.0	–122.3
5	–49.6	–184.8
8	–58.5	–204.6
10	–63.2	–213.7
20	–79.1	–237.7

Solution



From the exact bode diagrams, the gain margin = 50 dB (G.M. = 0 – $|G(j\omega)|$) at 5.1 rad/s. Phase margin = 89° at 0.05 rad/s.

G.M. = 6 = 0 – $|G(j\omega)|$ (phase crossover frequency will not change)

$$\therefore |G(j\omega)| = 20\log\frac{K}{5.1\sqrt{(5.1)^2+2^2}\sqrt{(5.1)^2+10^2}} = -6$$

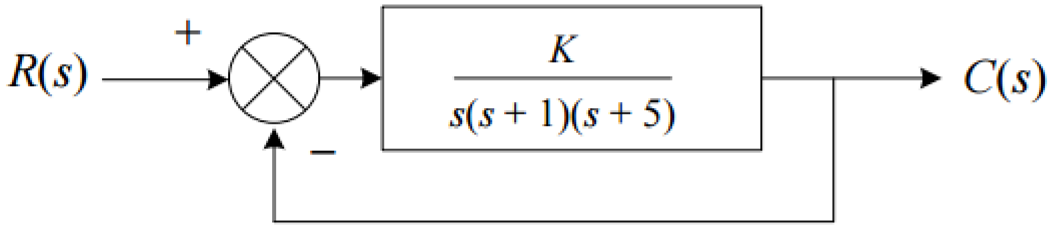
$$\therefore K = 157.18$$

P.M. = $180° + \angle G(j\omega) = 45°,\ \angle G(j\omega) = -135°$

From the exact bode diagrams, the gain crossover frequency at $\angle G(j\omega) = -135°$ is 1.42 rad/s

Question

4. Obtain the phase and gain margins of the system shown below for the two cases: (a) $K = 10$ and (b) $K = 100$ by plotting Bode Diagram. Which system is stable? (Ans: (a) Gain margin = 8.3 dB, phase margin = 26°; (b) Gain margin = –11 dB, phase margin = –19°)



Solution

4. From the block diagram, the open-loop transfer function is,

$$G(s) = \frac{K}{s(s+1)(s+5)}$$

Plot the exact Bode diagrams for finding gain margin and phase margin with $K = 10$ and $K = 100$.

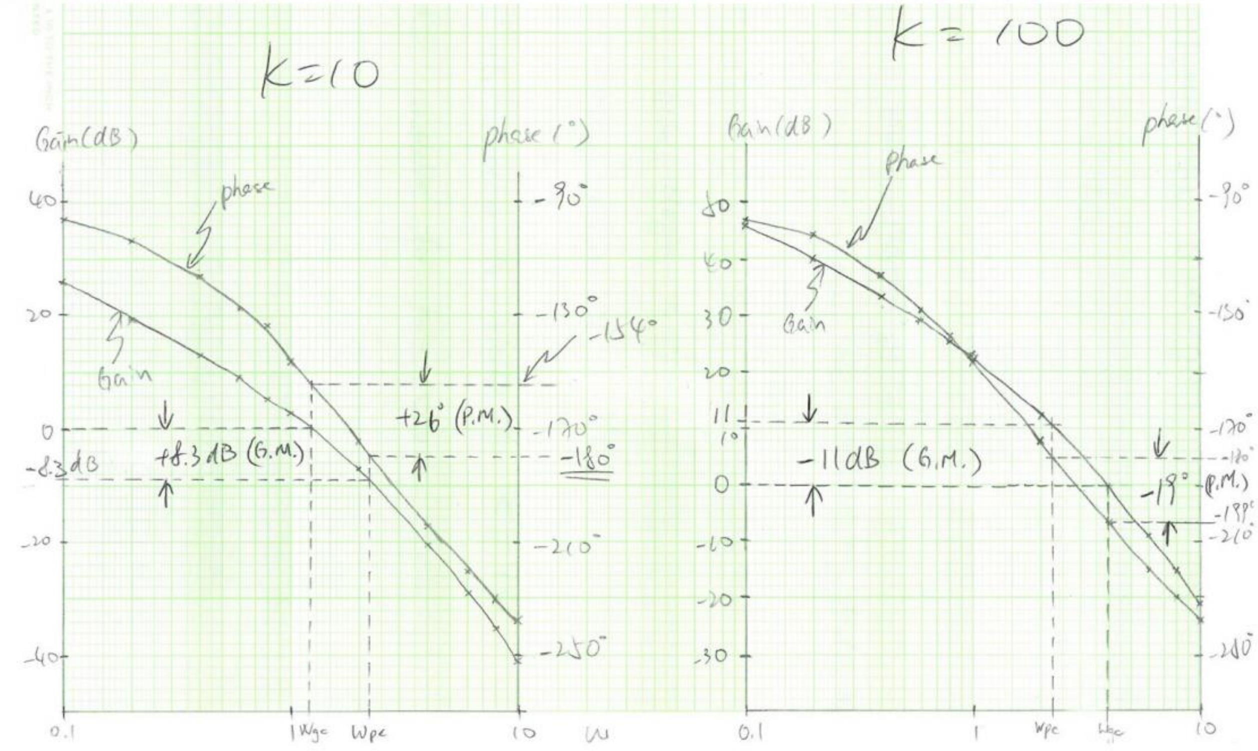
$$G(j\omega) = \frac{K}{j\omega(j\omega+1)(j\omega+5)}$$

$$|G(j\omega)| = 20\log\frac{K}{\omega\sqrt{\omega^2+1^2}\sqrt{\omega^2+5^2}} \quad \text{(dB)}$$

$$\angle G(j\omega) = -90° - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{5} \quad (°)$$

Change of gain K will Not alter the system phase, we have

ω (rad/s)	Magnitude (dB)		Phase (°)
	K = 10	K = 100	
0.1	25.98	45.98	–96.86
0.2	19.82	39.82	–103.13
0.4	13.31	33.31	–116.38
0.6	9.06	29.06	–127.81
0.8	5.70	25.70	–137.75
1	2.84	22.84	–146.31
2	–7.63	12.37	–175.24
4	–20.47	–0.47	–204.62
6	–29.10	–9.1	–220.73
8	–35.68	–15.68	–230.87
10	–41.01	–21.01	–237.72

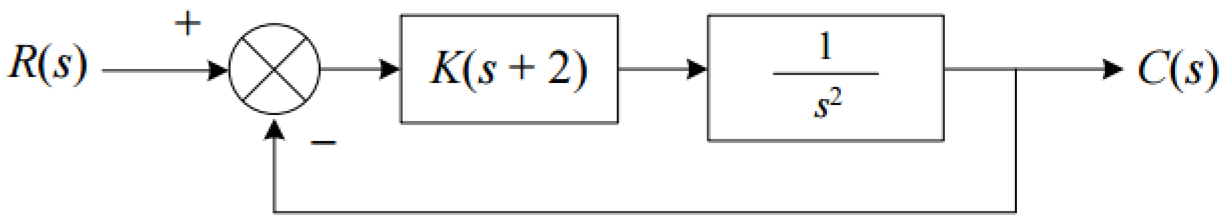


K	Gain margin	Phase margin
10	+8.3 dB	+26°
100	–11 dB	–19°

Hence, the system is stable for $K = 10$, but unstable for $K = 100$.

Question

5. Below shows a block diagram of a space vehicle control system. Determine the gain K such that the phase margin is 50°. (Ans: $K = 1.826$)



End of Tutorial Questions (Part 3)

Solution

5. From the block diagram, the open-loop transfer function, we have

$$G(s) = \frac{K(s+2)}{s^2}$$

$$G(j\omega) = \frac{K(j\omega+2)}{(j\omega)^2}$$

$$|G(j\omega)| = 20\log\frac{K\sqrt{\omega^2+2^2}}{\omega^2} \quad \text{(dB)}$$

$$\angle G(j\omega) = \tan^{-1}\frac{\omega}{2} - 90° - 90° \quad (°)$$

The required phase margin of 50°means that the system phase be equal to −130°. Hence, the gain crossover frequency can be found from the system phase equation as follow,

$$-130° = \tan^{-1}\frac{\omega}{2} - 90° - 90° \quad \Rightarrow \quad \tan^{-1}\frac{\omega}{2} = 50°$$

$$\therefore \omega_{gc} = 2.3835 \text{ rad/s}$$

Noting that the magnitude of the system ($|G(j\omega)|$) must be equal to 0 dB at $\omega = 2.3835$ rad/s, we have

$$20\log\frac{K\sqrt{2.3835^2+2^2}}{2.3835^2} = 0$$

$$\therefore K = 1.826$$