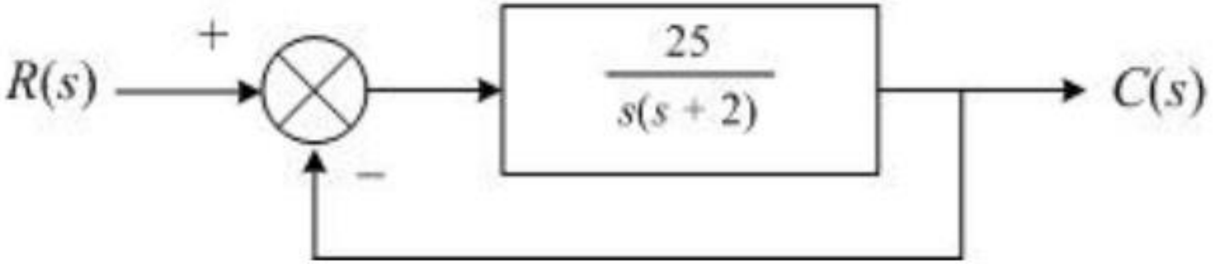
Q1

Given the following system,



(a) Determine the damping ratio, undamped natural frequency and damped natural frequency.
(b) Obtain the unit-step response $c(t)$ of this system when all initial conditions are zero.
(c) Determine the rise time, t_r , the peak time t_p , the 2% settling time t_s and the percentage of overshoot

Ans: (a) $\zeta = 0.2$, $\omega_n = 5$ rad/s, $\omega_d = 4.899$ rad/s; (b) $c(t) = 1 - 1.021e^{-t}\sin(4.899t + 1.369)$; (c) $t_r = 0.362$ sec, $t_p = 0.641$ sec, $t_s = 4$ sec, $M_p = 52.66\%$

Solution

(a) The closed-loop transfer function,

$$\frac{C(s)}{R(s)}=\frac{\frac{25}{s(s+2)}}{1+\frac{25}{s(s+2)}}=\frac{25}{s(s+2)+25}=\frac{25}{s^2+2s+25}$$

Compared the above transfer function with the 2nd order system, we have

$$\frac{C(s)}{R(s)}=\frac{25}{s^2+2s+25}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Equating terms, we have $2\zeta\omega_n=2,\quad \omega_n^2=25$

$\therefore\ \omega_n=5$ rad/s, $(2\zeta)(5)=2\Rightarrow\zeta=0.2$

Since $\omega_d=\omega_n\sqrt{1-\zeta^2}=(5)\left(\sqrt{1-0.2^2}\right)=4.899$ rad/s

(b) The unit-step response,

$$C(s)=\frac{25}{s^2+2s+25}\frac{1}{s}$$

Taking inverse Laplace transform, we have

$$c(t)=1-\frac{1}{\sqrt{1-0.2^2}}e^{-(0.2)(5)t}\sin\left(5\sqrt{1-0.2^2}t+\phi\right),\phi=\cos^{-1}0.2$$

$$\therefore\ c(t)=1-1.021e^{-t}\sin(4.899t+1.369)$$

(c) Rise time: $t_r=\frac{\pi-\beta}{\omega_d}=\frac{\pi-1.3694}{4.899}=0.362\text{ sec}$

$\beta=\tan^{-1}\frac{\omega_d}{\zeta\omega_n}=\tan^{-1}\frac{4.899}{(0.2)(5)}=1.3694\text{ rad}$

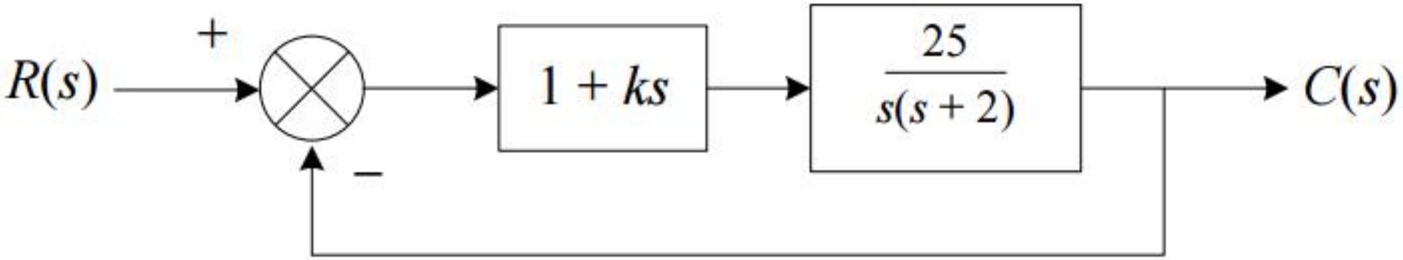
Peak time: $t_p=\frac{\pi}{\omega_d}=\frac{\pi}{4.899}=0.641\text{ sec}$

2% Settling time: $t_s=\frac{4}{\zeta\omega_n}=\frac{4}{(0.2)(5)}=4\text{ sec}$

% of overshoot: $M_p=e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}=e^{-\frac{0.2}{\sqrt{1-0.2^2}}\pi}=0.5266\text{ (or 52.66\%)}$

Q2

To improve the transient behavior of the system in Question 1, a proportional and derivative controller is added as shown below.



(a) Determine the value of k such that the resulting system will have a damping ratio of 0.5.

(b) Obtain the unit-step response $c(t)$ of this system when all initial conditions are zero.

Ans: (a) $k = 0.12$; (b) $c(t) = 1 + 0.693e^{-2.5t}\sin 4.33t - 1.15e^{-2.5t}\sin(4.33t + 1.047)$

Solution

(a) The closed-loop transfer function of the system,

$$\frac{C(s)}{R(s)} = \frac{(1+ks)\frac{25}{s(s+2)}}{1+(1+ks)\frac{25}{s(s+2)}} = \frac{25(1+ks)}{s(s+2)+25(1+ks)} = \frac{25(1+ks)}{s^2+(2+25k)s+25}$$

Equating the terms with the 2nd order equation, we have

$$\omega_n^2 = 25 \Rightarrow \omega_n = 5 \text{ rad/s}$$

$$2\zeta\omega_n = 2+25k \Rightarrow (2)(0.5)(5) = 2+25k \Rightarrow k = 0.12$$

(b) The unit-step response of the system,

$$C(s) = \left(\frac{25(1+0.12s)}{s^2+(2+25(0.12))s+25}\right)\left(\frac{1}{s}\right) = \left(\frac{3s+25}{s^2+5s+25}\right)\left(\frac{1}{s}\right)$$

Rearrange the above equation, we have

$$C(s) = \frac{3s+25}{s(s^2+5s+25)} = \frac{3s}{s(s^2+5s+25)} + \frac{25}{s(s^2+5s+25)}$$

$$= \frac{3}{25}\frac{25}{s^2+5s+25} + \frac{25}{s(s^2+5s+25)}$$

Taking inverse Laplace transform, we have

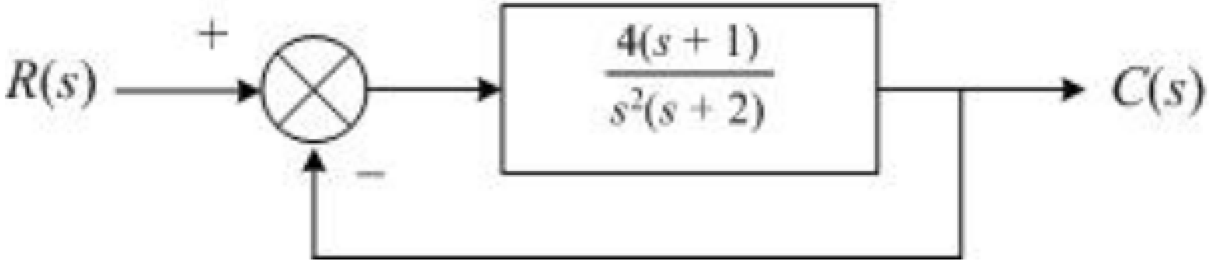
$$c(t) = \left(\frac{3}{25}\right)\left(\frac{5}{\sqrt{1-0.5^2}}e^{-(0.5)(5)t}\sin\left(5\sqrt{1-0.5^2}t\right)\right) + 1$$

$$- \frac{1}{\sqrt{1-0.5^2}}e^{-(0.5)(5)t}\sin\left(5\sqrt{1-0.5^2}t + \phi\right), \phi = \cos^{-1} 0.5$$

$$c(t) = 1 + 0.693e^{-2.5t}\sin 4.33t - 1.15e^{-2.5t}\sin(4.33t + 1.047)$$

Q3

Given the following unity feedback system.



(a) Find the position, velocity, and acceleration error constants.

(b) Determine the steady-state error when the input is $R(s) = \frac{3}{s} - \frac{1}{s^2} + \frac{1}{2s^3}$.

Ans: (a) $K_p = \infty, \quad K_v = \infty, \quad K_a = 2$; (b) $e_\infty = 0.25$

Solution

(a)

Position error constant: $$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{4(s+1)}{s^2(s+2)} = \frac{4}{0} = \infty$$

Velocity error constant: $$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} s\frac{4(s+1)}{s^2(s+2)} = \frac{4}{0} = \infty$$

Acceleration error constant: $$K_a = \lim_{s\to 0} s^2G(s)H(s) = \lim_{s\to 0} s^2\frac{4(s+1)}{s^2(s+2)} = \frac{4}{2} = 2$$

(b)

State-error (Position): $$e_{p,ss}(\infty) = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

State-error (Velocity): $$e_{v,ss}(\infty) = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

State-error (Acceleration): $$e_{a,ss}(\infty) = \frac{1}{K_a} = \frac{1}{2} = 0.5$$

The input signal, $R(s)$, consists of 3 elements, step $\left(\frac{3}{s}\right)$+ ramp $\left(\frac{1}{s^2}\right)$ + parabolic $\left(\frac{1}{2s^3}\right)$inputs, because the system is LTI, the principle of superposition holds, so the steady-state error will be equal to,

$$e_{ss}(\infty) = 3e_{p,ss}(\infty) - e_{v,ss}(\infty) + \frac{1}{2}e_{a,ss}(\infty) = 3(0) - 0 + \frac{1}{2}(0.5) = 0.25$$