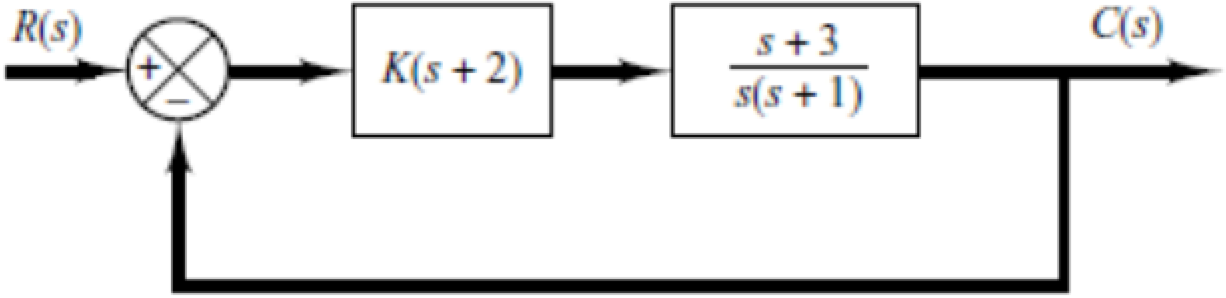
Q6

Sketch the root loci for the systems below, where $K > 0$

(a) $$G(s) = \frac{K}{s(s+3)(s+8)}; \quad H(s) = 1$$

(b)



Solution (a)

The open-loop transfer function is,

$$G(s)H(s) = \frac{K}{s(s+3)(s+8)}$$

1. **Locate the open-loop poles and zeros of $G(s)H(s)$ on the complex plane (or s-plane)**
   Poles: $s = 0, \quad s = -3, \quad s = -8$

2. **Determine the root loci on the real axis**
   Root loci: $(-\infty, -8]$ and $[-3, 0]$

3. **Determine the asymptotes of root loci**

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{n-m} = \frac{\pm 180°(2k+1)}{3-0} = +60°, -60°, \pm 180°$$

   The intersection of the asymptotes and the real axis is found from,

$$s = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{(0)+(-3)+(-8)}{3-0} = -3.667$$

4. **Find the breakaway point and/or break-in points**
   The characteristic equation for the system is,

$$\Delta(s) = s(s+3)(s+8) + K = s^3 + 11s^2 + 24s + K = 0$$

   We have

$$K = -s^3 - 11s^2 - 24s$$

   The breakaway and/or break-in points are found from,

$$\frac{dK}{ds} = -3s^2 - 22s - 24 = 0$$

   from which we get,

$$s = -1.333, \quad -6 \text{ (rejected)}$$

   Only $s = -1.333$ lies on the root loci.

5. **Determine the angle of departure (angle of arrival) of the root locus from a complex pole/zero**
   There are no angle of departure (angle of arrival) since the system has no complex pole/zero.

6. **Find the points where the root loci may cross the imaginary axis**
   The characteristic equation for the system is,

$$\Delta(s) = s(s+3)(s+8) + K = s^3 + 11s^2 + 24s + K = 0$$

Methods 1

The Routh's array,

s^3	1	24
s^2	11	K
s^1	$\frac{(11)(24)-(1)(K)}{11} = 24 - \frac{1}{11}K$	
s^0	$\frac{\left(24-\frac{1}{11}K\right)(K)-(11)(0)}{24-\frac{1}{11}K} = K$	

For stable system, $K > 0, \quad 24 - \frac{1}{11}K > 0 \Rightarrow 24 > \frac{1}{11}K$

$\therefore 0 < K < 264$

Refer to the 2nd row of the Routh's array, we have $11s^2 + 264 = 0$, yielding,

$$s = \pm j4.899$$

Methods 2

Substitute $s = j\omega$ into the characteristic equation to find the points where root-locus branches may cross the imaginary axis, yielding,

$$(j\omega)^3 + 11(j\omega)^2 + 24(j\omega) + K = 0$$

or

$$(K - 11\omega^2) + j\omega(24 - \omega^2) = 0$$

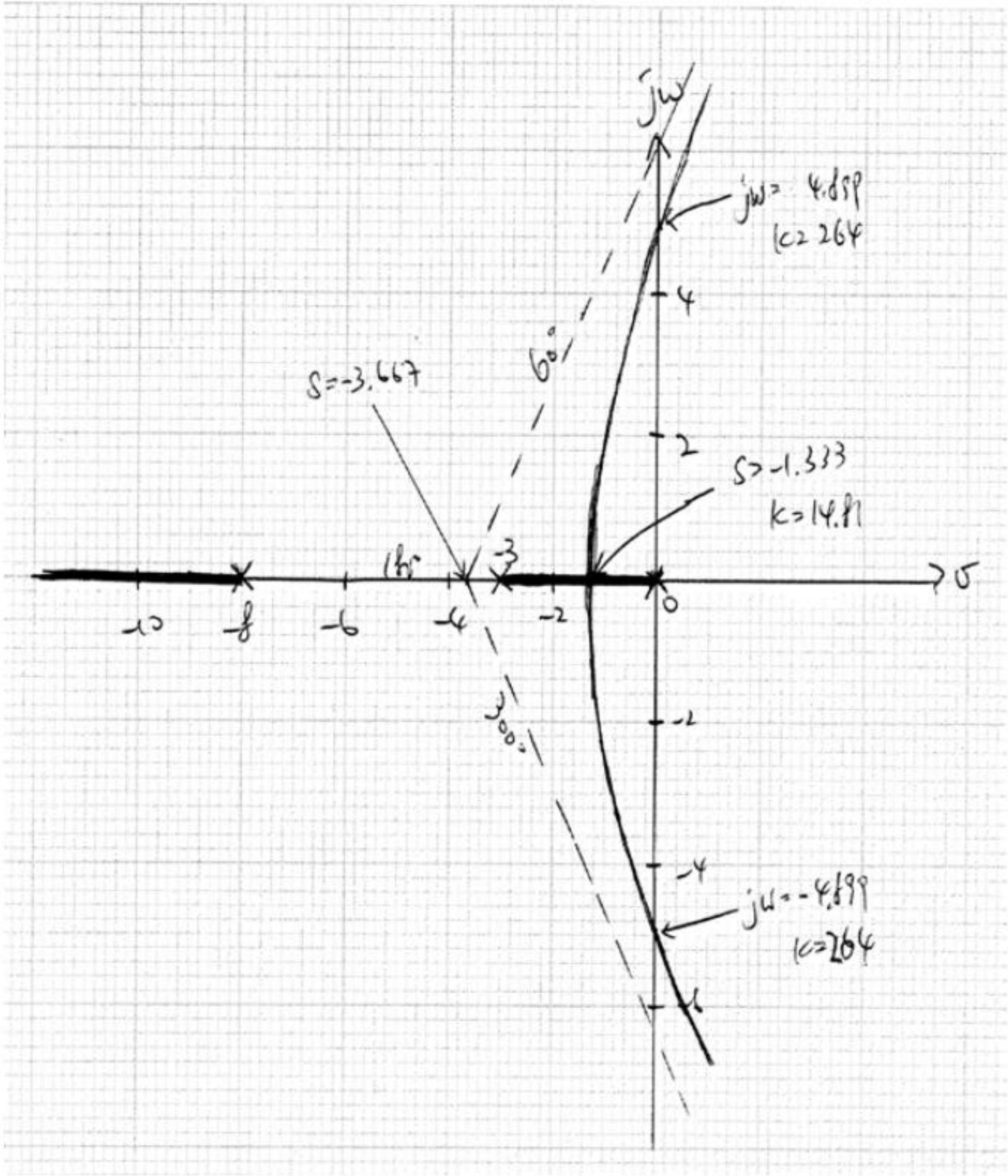
In order to satisfy the equation,

$$K - 11\omega^2 = 0 \text{ and } j\omega(24 - \omega^2) = 0$$

$$\therefore j\omega = \pm\sqrt{24} = \pm j4.899$$

With $\omega = 4.899, K = 11\omega^2 \Rightarrow K = 264$

7. **Draw the root locus on the graph paper**



Solution (b)

(b) The open-loop transfer function is,

$$G(s)H(s) = \frac{K(s+2)(s+3)}{s(s+1)}$$

1. **Locate the open-loop poles and zeros of $G(s)H(s)$ on the complex plane (or s-plane)**
   Poles: $s = 0, \quad s = -1$
   Zeros: $s = -2, \quad s = -3$

2. **Determine the root loci on the real axis**
   Root loci: $[-3, -2]$ and $[-1, 0]$

3. **Determine the asymptotes of root loci**
   Since the number of open-loop poles and zeros are the same. There are NO asymptotes in the complex region of the s-plane.

4. **Find the breakaway point and/or break-in points**
   The characteristic equation for the system is,

$$\Delta(s) = s(s+1) + K(s+2)(s+3) = 1 + \frac{K(s+2)(s+3)}{s(s+1)} = 0$$

or

$$K = -\frac{s(s+1)}{(s+2)(s+3)}$$

The breakaway and/or break-in points are found from,

$$\frac{dK}{ds} = -\frac{(s+2)(s+3)\frac{d}{ds}s(s+1) - s(s+1)\frac{d}{ds}(s+2)(s+3)}{[(s+2)(s+3)]^2}$$

$$\frac{dK}{ds} = -\frac{(s+2)(s+3)(2s+1) - s(s+1)(2s+5)}{[(s+2)(s+3)]^2} = -\frac{4s^2+12s+6}{[(s+2)(s+3)]^2} = 0$$

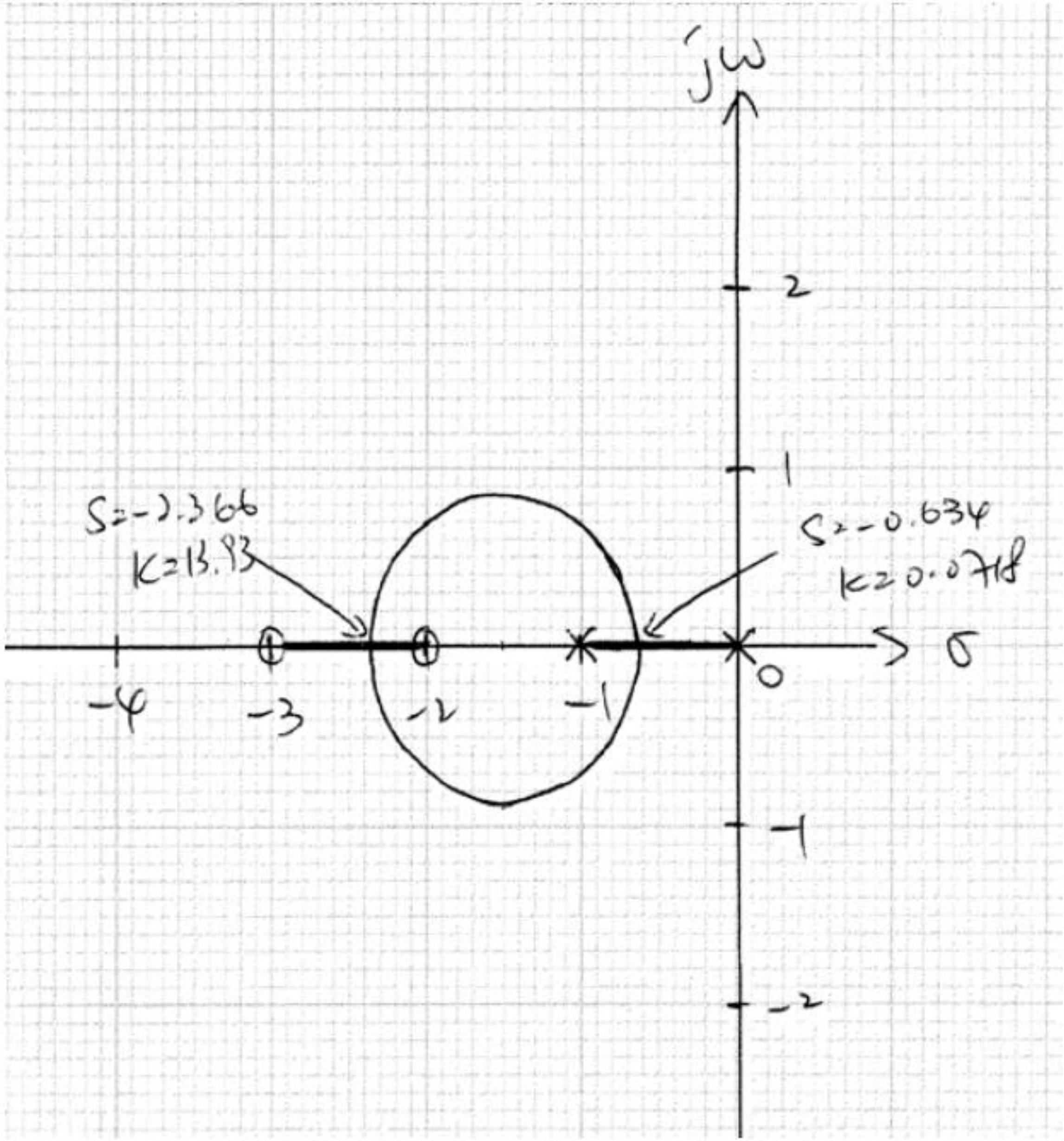
Hence, we have $4s^2 + 12s + 6 = 0$, which yielding,

$$s = -0.634, \quad s = -2.366$$

Both points are on the root loci. Because point lies $s = -0.634$ between two poles, it is a breakaway point, and because point $s = -2.366$ lies between two zeros, it is a break-in point.

5. **Determine the angle of departure (angle of arrival) of the root locus from a complex pole/zero**
   There are no angle of departure (angle of arrival) since the system has no complex pole/zero.

6. **Draw the root locus on the graph paper**
   Determine a sufficient number of points that satisfy the angle condition. (It can be found that the root loci involve a circle with center at –1.5 that passes through the breakaway and break-in points.)


jω
2
1
-1
-2
σ
0
-4
-3
-2
-1
S=-2.366
K=13.93
S=-0.634
K=0.0718

Q7

Sketch the root locus for the open-loop transfer function,

$$G(s)H(s) = \frac{K}{s(s+4)}, \qquad K > 0$$

Comment on the system stability if (a) a pole at $s = -1$ or (b) a zero at $s = -1$ was introduced to the above transfer function.

End of Tutorial Questions (Part 2)

Q7 solution

The open-loop transfer function is,

$$G(s)H(s) = \frac{K}{s(s+4)}$$

1. **Locate the open-loop poles and zeros of $G(s)H(s)$ on the complex plane (or s-plane)**

   Poles: $s = 0, \quad s = -4$

2. **Determine the root loci on the real axis**

   Root loci: $[-4, 0]$

3. **Determine the asymptotes of root loci**

   $$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{n-m} = \frac{\pm 180°(2k+1)}{2-0} = +90°, -90°$$

   The intersection of the asymptotes and the real axis is found from,

   $$s = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{(0)+(-4)}{2-0} = -2$$

4. **Find the breakaway point and/or break-in points**

   The characteristic equation for the system is,

   $$\Delta(s) = s(s+4) + K = s^2 + 4s + K = 0$$

   We have

$$K = -s^2 - 4s$$

The breakaway and/or break-in points are found from,

$$\frac{dK}{ds} = -2s - 4 = 0$$

from which we get,

$$s = -2$$

The point $s = -2$ lies on the root loci.

5. **Determine the angle of departure (angle of arrival) of the root locus from a complex pole/zero**
   There is no angle of departure (angle of arrival) since the system has no complex pole/zero.

6. **Find the points where the root loci may cross the imaginary axis**
   The characteristic equation for the system is,

$$\Delta(s) = s^2 + 4s + K$$

Substitute $s = j\omega$ into the characteristic equation to find the points where root-locus branches may cross the imaginary axis, yielding,

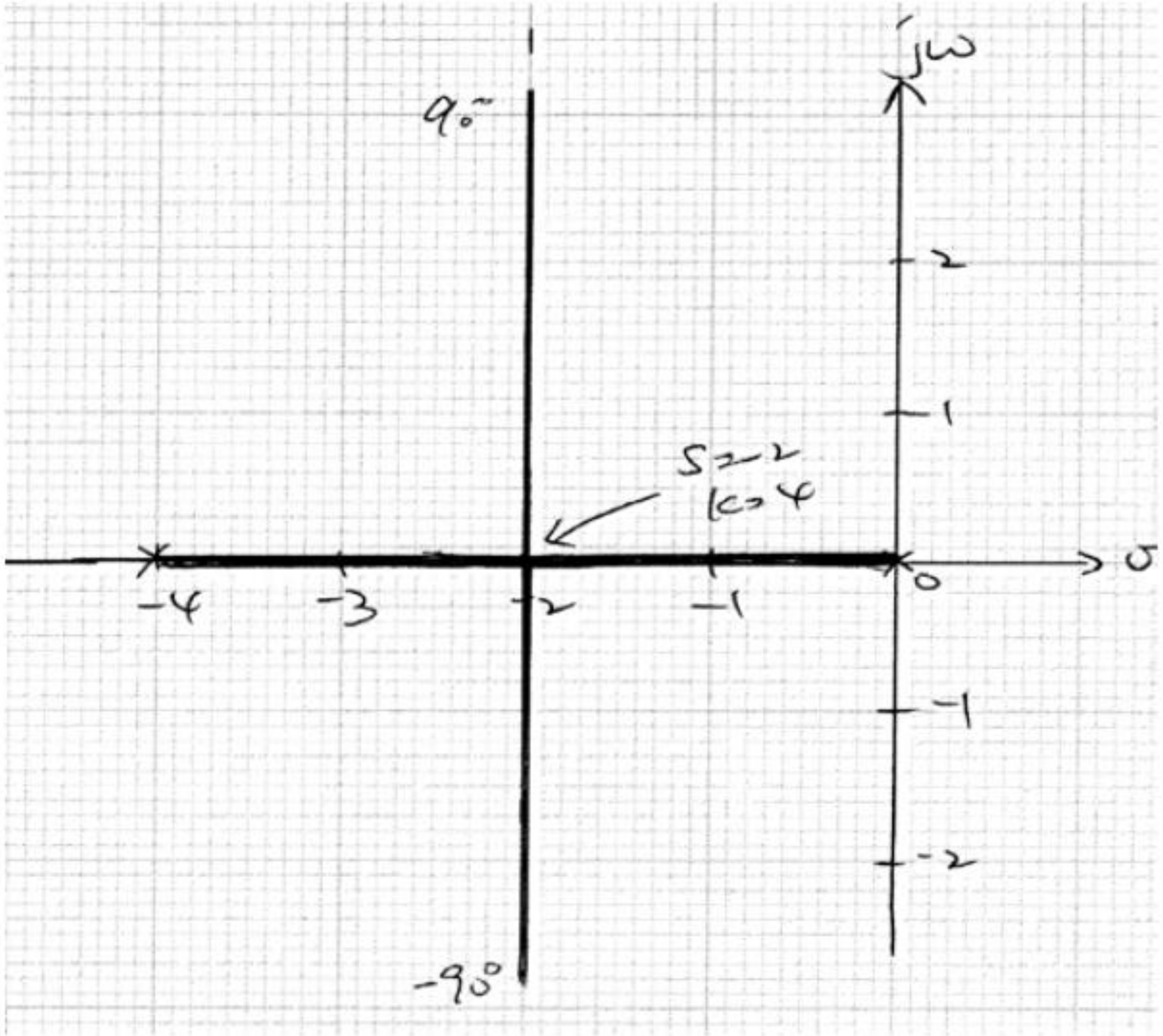
$$(j\omega)^2 + 4(j\omega) + K = 0$$

or

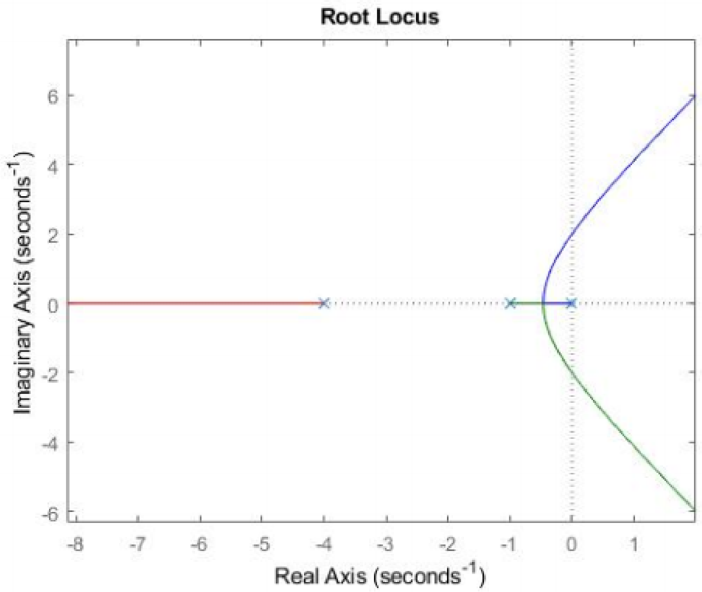
$$(K - \omega^2) + 4j\omega = 0$$

Notice that this equation can be satisfied only if $\omega = 0, \quad K = 0$. The root-locus branches do not cross the $j\omega$ axis.

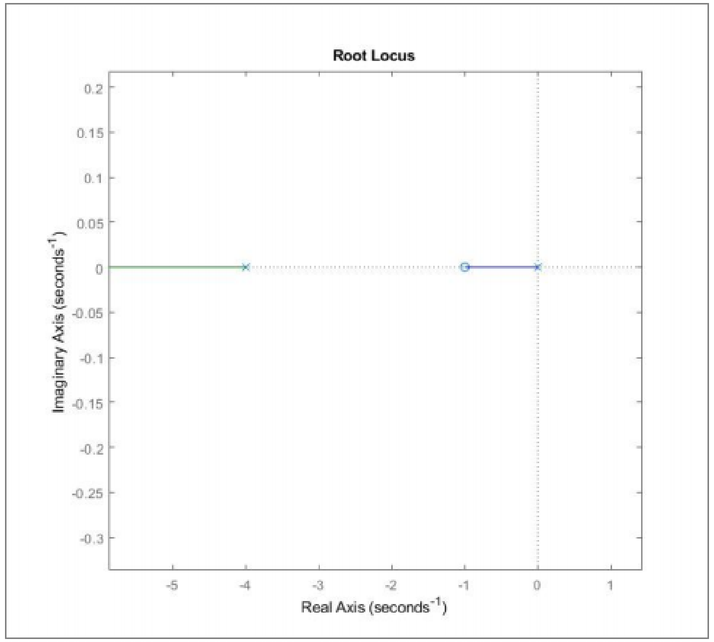
7. **Draw the root locus on the graph paper**



(a) The addition of a pole has the effect of pulling the root locus to the right, tending to lower the system's relative stability. The root locus plot is shown below.



(b) The addition of zeros has the effect of pulling the root locus to the left, tending to make the system more stable. The root locus plot is shown below.



End of Tutorial Questions (Part 2) Solution