Q1 question

Solve the following differential equations using Laplace transform method.

(a) $\dot{y}(t) + y(t) = 2e^{t}$, $y(0) = 4$ Ans: $y(t) = e^{t} + 3e^{-t}$

(b) $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 1$, $y(0) = -1$, $\dot{y}(0) = 0$ Ans: $y(t) = \frac{1}{2} - 3e^{-t} + \frac{3}{2}e^{-2t}$

Q1 solution

(a) Taking Laplace transform,

$$sY(s) - y(0) + Y(s) = (2)\left(\frac{1}{s-1}\right)$$

Putting initial values,

$$(s+1)Y(s) - 4 = \frac{2}{s-1}$$

$$Y(s) = \frac{2}{(s+1)(s-1)} + \frac{4}{s+1}$$

Partial fraction expansion, we have

$$\frac{2}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1}$$

Equating the terms in the numerator, $A(s-1) + B(s+1) = 2$

Put $s = 1$, $A(1-1) + B(1+1) = 2 \Rightarrow 2B = 2 \Rightarrow B = 1$
Put $s = -1$, $A(-1-1) + B(+1) = 2 \Rightarrow -2A = 2 \Rightarrow A = -1$

Hence,

$$Y(s) = \frac{-1}{s+1} + \frac{1}{s-1} + \frac{4}{s+1} = \frac{3}{s+1} + \frac{1}{s-1}$$

Taking inverse Laplace transform, $y(t) = e^{t} + 3e^{-t}$

Q1 solution

(b) Taking Laplace transform,

$$[s^2Y(s) - sy(0) - \dot{y}(0)] + 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s}$$

Putting initial values,

$$[s^2Y(s) - s(-1) - 0] + 3[sY(s) - (-1)] + 2Y(s) = \frac{1}{s}$$

$$Y(s)[s^2 + 3s + 2] + s + 3 = \frac{1}{s}$$

$$Y(s) = \frac{\frac{1}{s} - s - 3}{s^2 + 3s + 2} = \frac{1 - 3s - s^2}{s(s^2 + 3s + 2)}$$

Partial fraction expansion, we have

$$\frac{1 - 3s - s^2}{s(s^2 + 3s + 2)} = \frac{1 - 3s - s^2}{s(s + 1)(s + 2)} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2}$$

Equating the terms in the numerator,

$$A(s + 1)(s + 2) + Bs(s + 2) + Cs(s + 1) = 1 - 3s - s^2$$

Put $s = 0$, we have

$$A(0 + 1)(0 + 2) + B(0)(0 + 2) + C(0)(0 + 1) = 1 - 3(0) - (0)^2 \Rightarrow A = \frac{1}{2}$$

Put $s = -1$, we have

$$A(-1 + 1)(-1 + 2) + B(-1)(-1 + 2) + C(-1)(-1 + 1) = 1 - 3(-1) - (-1)^2 \Rightarrow B = -3$$

Put $s = -2$, we have

$$A(-2 + 1)(-2 + 2) + B(-2)(-2 + 2) + C(-2)(-2 + 1) = 1 - 3(-2) - (-2)^2 \Rightarrow C = \frac{3}{2}$$

Hence,

$$Y(s) = \frac{1}{2}\frac{1}{s} + (-3)\frac{1}{s + 1} + \frac{3}{2}\frac{1}{s + 2}$$

Taking inverse Laplace transform,

$$y(t) = \frac{1}{2} - 3e^{-t} + \frac{3}{2}e^{-2t}$$

Q2 question

Find the unit-step and unit-impulse response of the following systems in time domain.

(a) $\ddot{y}(t) + 2\dot{y}(t) + y(t) = r(t)$ Ans: $u(t) = 1 - te^{-t} - e^{-t}$, $h(t) = te^{-t}$

(b) $\ddot{y}(t) + \dot{y}(t) = r(t)$ Ans: $u(t) = t - 1 + e^{-t}$, $h(t) = 1 - e^{-t}$

(c) $\ddot{y}(t) + 2\dot{y}(t) + 2y(t) = r(t)$ Ans: $u(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t$, $h(t) = e^{-t}\sin t$

Q2 solution

(a) Taking Laplace Transform with zero initial condition,

$$s^2Y(s) + 2sY(s) + Y(s) = R(s)$$
$$Y(s) = \frac{1}{s^2 + 2s + 1}R(s)$$

Unit-impulse response

$$Y(s) = \frac{1}{s^2 + 2s + 1}(1) = \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform,

$$y(t) = h(t) = te^{-t}$$

Unit-step response

$$Y(s) = \frac{1}{s^2 + 2s + 1}\left(\frac{1}{s}\right) = \frac{1}{s}\frac{1}{(s+1)^2}$$

Partial fraction expansion, we have

$$\frac{1}{s}\frac{1}{(s+1)^2} = \frac{A}{s} + \frac{B}{(s+1)^2} + \frac{C}{s+1}$$

Equating the terms in the numerator, $A(s+1)^2 + Bs + Cs(s+1) = 1$

Put $s = 0$, we have

$$A(0+1)^2 + B(0) + C(0)(0+1) = 1 \Rightarrow A = 1$$

Put $s = -1$, we have

$$A(-1+1)^2 + B(-1) + C(-1)(-1+1) = 1 \Rightarrow B = -1$$

Put $s = 1$, we have

$$A(1+1)^2 + B(1) + C(1)(1+1) = 1 \Rightarrow C = -1$$

Hence,

$$Y(s) = \frac{1}{s} + \frac{-1}{(s+1)^2} + \frac{-1}{s+1}$$

Taking inverse Laplace transform,

$$y(t) = u(t) = 1 - te^{-t} - e^{-t}$$

Q2 solution

(b) Taking Laplace Transform with zero initial condition,

$$s^2Y(s) + sY(s) = R(s)$$
$$Y(s) = \frac{1}{s(s+1)}R(s)$$

Unit-impulse response

$$Y(s) = \frac{1}{s(s+1)}(1)$$

Partial fraction expansion, we have

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

Equating the terms in the numerator, $A(s+1) + Bs = 1$

Put $s = 0$, we have $A(0+1) + B(0) = 1 \Rightarrow A = 1$
Put $s = -1$, we have $A(-1+1) + B(-1) = 1 \Rightarrow B = -1$

Hence,

$$Y(s) = \frac{1}{s} + \frac{-1}{s+1}$$

Taking inverse Laplace transform,

$$y(t) = h(t) = 1 - e^{-t}$$

Unit-step response

$$Y(s) = \frac{1}{s(s+1)}\left(\frac{1}{s}\right) = \frac{1}{s^2}\frac{1}{s+1}$$

Partial fraction expansion, we have

$$\frac{1}{s^2}\frac{1}{s+1} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

Equating the terms in the numerator, $A(s+1) + Bs(s+1) + Cs^2 = 1$

Put $s = 0$, we have

$$A(0+1) + B(0)(0+1) + C(0)^2 = 1 \Rightarrow A = 1$$

Put $s = -1$, we have

$$A(-1+1) + B(-1)(-1+1) + C(-1)^2 = 1 \Rightarrow C = 1$$

Put $s = 1$, we have

$$A(1+1) + B(1)(1+1) + C(1)^2 = 1 \Rightarrow B = -1$$

Hence,

$$Y(s) = \frac{1}{s^2} + \frac{-1}{s} + \frac{1}{s+1}$$

Taking inverse Laplace transform,

$$y(t) = u(t) = t - 1 + e^{-t}$$

Q2 solution

Taking Laplace Transform with zero initial condition,

$$s^2Y(s) + 2sY(s) + 2Y(s) = R(s)$$
$$Y(s) = \frac{1}{s^2 + 2s + 2}R(s)$$

Unit-impulse response

$$Y(s) = \frac{1}{s^2 + 2s + 2}(1) = \frac{1}{(s+1)^2 + 1^2}$$

(The above method is called "completing square")

Taking inverse Laplace transform,

$$y(t) = h(t) = e^{-t}\sin t$$

Unit-step response

$$Y(s) = \frac{1}{s^2 + 2s + 2}\left(\frac{1}{s}\right)$$

Partial fraction expansion, we have

$$\frac{1}{s^2 + 2s + 2}\left(\frac{1}{s}\right) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}$$

Equating the terms in the numerator, $A(s^2 + 2s + 2) + (Bs + C)(s) = 1$

Put $s = 0$, we have

$$A(0^2 + 2(0) + 2) + (B(0) + C)(0) = 1 \Rightarrow A = 1/2$$

Put $s = -1$, we have

$$A((-1)^2 + 2(-1) + 2) + (B(-1) + C)(-1) = 1 \Rightarrow B - C = 1/2$$

Put $s = 1$, we have

$$A(1^2 + 2(1) + 2) + (B(1) + C)(1) = 1 \Rightarrow B + C = -3/2$$

Solving the above two equations, we have

$$\therefore B = -1/2 \quad C = -1$$

Hence,

$$Y(s) = \frac{1/2}{s} + \frac{-\frac{1}{2s} - 1}{s^2 + 2s + 2} = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\left[\frac{s+2}{(s+1)^2 + 1^2}\right] = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\left[\frac{s+1+1}{(s+1)^2 + 1^2}\right]$$

$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\left[\frac{s+1}{(s+1)^2 + 1^2}\right] - \frac{1}{2}\left[\frac{1}{(s+1)^2 + 1^2}\right]$$

Taking inverse Laplace transform,

$$y(t) = u(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t$$