SEHS4653
Control System Analysis

Unit 7
State Space Analysis
(Reference: [1] chapter 2.4-5)

Content

- Modeling in State Space
- Relationship with Transfer Function
  - State Transition Matrix

Modeling in State Space

State

- Smallest set of variables (called state variables) in a dynamic system such that knowledge of these variables at $t = t_0$, together with knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$

State Variables

- Variables making up the smallest set of variables that determine the state of the dynamic system
- If at least *n* variables $x_1, x_2, \cdots, x_n$ are needed to completely describe the behavior of a dynamic system, then such *n* variables are a set of state variables
- State variables need not be physically measurable or observable quantities
- Such freedom in choosing state variables is an advantage of the state-space methods
- Practically, however, it is convenient to choose easily measurable quantities for the state variables

Modeling in State Space

State Vector

- If *n* state variables are needed to completely describe the behavior of a given system, then these *n* state variables can be considered the *n* components of a vector **x**, called a state vector

State Space

- The *n*-dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, ⋯, x_n axis, where $x_1, x_2, \cdots, x_n$ are state variables, is called a state space. Any state can be represented by a point in the state space

State-Space Equations

- In state-space analysis we are concerned with three types of variables that are involved in the modeling of dynamic systems: input variables, output variables, and state variables

Modeling in State Space

State-Space Equations

- Assume that a multiple-input, multiple-output system has *r* inputs $u_1(t), u_2(t), \cdots, u_r(t)$, *m* outputs $y_1(t), y_2(t), \cdots, y_m(t)$, and *n* state variables $x_1(t), x_2(t), \cdots, x_n(t)$
- Then, the system may be described by

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1, x_2, \cdots, x_n; u_1, u_{2,} \cdots, u_r; t) \\
\dot{x}_2(t) &= f_2(x_1, x_2, \cdots, x_n; u_1, u_{2,} \cdots, u_r; t) \\
&\vdots \\
\dot{x}_n(t) &= f_n(x_1, x_2, \cdots, x_n; u_1, u_{2,} \cdots, u_r; t)
\end{aligned}
$$

- The outputs $y_1(t), y_2(t), \cdots, y_m(t)$ of the system may be given

$$
\begin{aligned}
y_1(t) &= g_1(x_1, x_2, \cdots, x_n; u_1, u_{2,} \cdots, u_r; t) \\
y_2(t) &= g_2(x_1, x_2, \cdots, x_n; u_1, u_{2,} \cdots, u_r; t) \\
&\vdots \\
y_m(t) &= g_m(x_1, x_2, \cdots, x_n; u_1, u_{2,} \cdots, u_r; t)
\end{aligned}
$$

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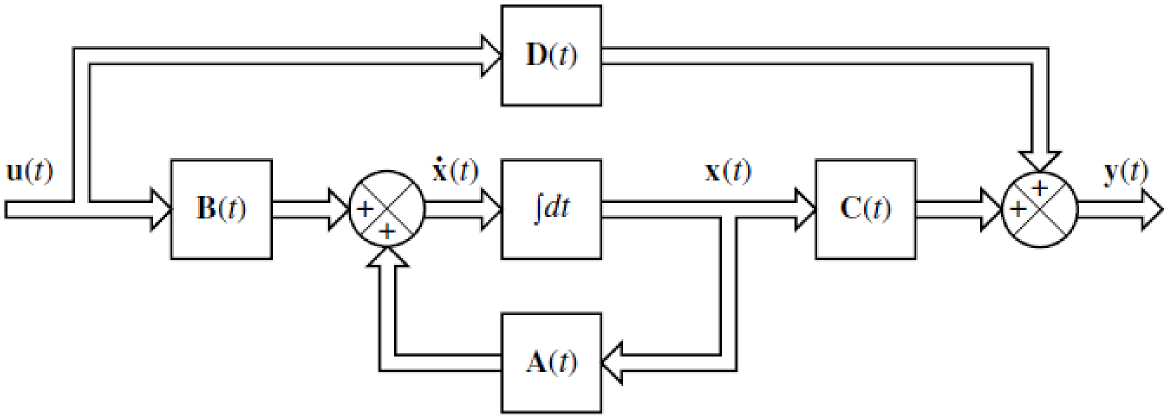
Modeling in State Space

State-Space Equations

- Then ,we have the following linearized state equation and output equation,

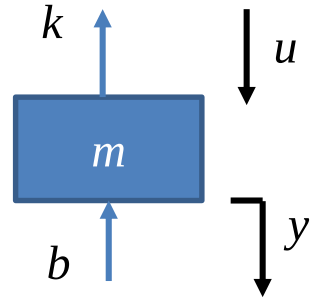
$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

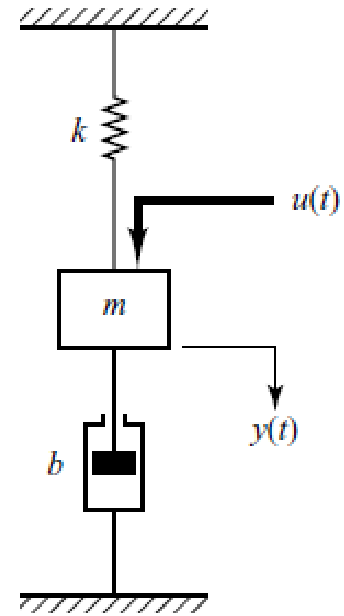
where **A**(t) is called the state matrix, **B**(t) the input matrix, **C**(t) the output matrix, and **D**(t) the direct transmission matrix



Example 1

Consider the mechanical system shown below. We assume that the system is linear. The external force *u*(*t*) is the input to the system, and the displacement *y*(*t*) of the mass is the output. The displacement is measured from the equilibrium position in the absence of the external force. This system is a single-input, single-output system. Represent the system in state-space equations in standard form.





Answer:

The system equation:

Let's define 2 state variables for this 2nd order system,

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

Re-write the equations, we have

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{y} = \frac{1}{m}(-kx_1 - bx_2) + \frac{1}{m}u$$

Hence, the state-space equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \quad \text{and} \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Relationship with Transfer Function

- Consider the state-space equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

- Taking Laplace transform of these equations (with zero initial condition), we have

$$sX(s) = AX(s) + BU(s)$$
$$(sI - A)X(s) = BU(s)$$
$$X(s) = (sI - A)^{-1}BU(s)$$

$$Y(s) = CX(s) + DU(s)$$
$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$
$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

State Transition Matrix

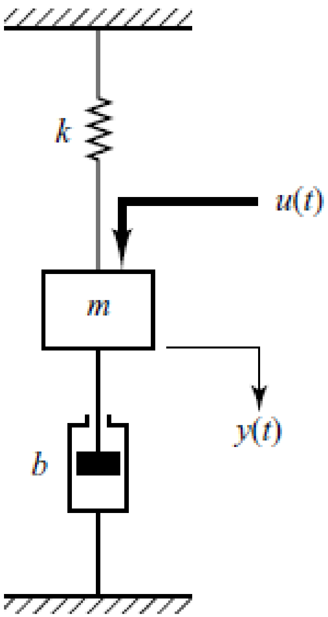
$$\Phi(s) = (sI - A)^{-1}$$

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D = G(s)$$

Transfer Function

Example 2

Use the state-space equations obtained in Example 1 to find the transfer function of the mechanical system.



<u>Answer:</u>

Here are the state-space equations of Example 1,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \qquad \text{and} \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

With $G(s) = C(sI - A)^{-1}B + D$

$$\therefore G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

Example 2

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Answer:

$$\begin{bmatrix} s & -1 \\ \frac{k}{m} & s+\frac{b}{m} \end{bmatrix}^{-1} = \frac{1}{(s)\left(s+\frac{b}{m}\right)-(-1)\left(\frac{k}{m}\right)}\begin{bmatrix} s+\frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} = \frac{1}{s^2+\frac{b}{m}s+\frac{k}{m}}\begin{bmatrix} s+\frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$$

$$\therefore G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix}\frac{1}{s^2+\frac{b}{m}s+\frac{k}{m}}\begin{bmatrix} s+\frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$= \frac{1}{s^2+\frac{b}{m}s+\frac{k}{m}}\begin{bmatrix} s+\frac{b}{m} & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \frac{1}{s^2+\frac{b}{m}s+\frac{k}{m}}\left(\frac{1}{m}\right)$$

$$G(s) = \frac{1}{ms^2+bs+k}$$

Example 3

Compute the state transition matrix in time-domain for the following system.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

<u>Answer:</u>

The equation of state-transition matrix, $\Phi(s) = (sI - A)^{-1}$

$$\Phi(s) = \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{(s)(s+3) - (-1)(2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

Example 3

Answer:

By partial fraction, we have

$$\mathfrak{L}^{-1}\begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{s+2}+\dfrac{2}{s+1} & -\dfrac{1}{s+2}+\dfrac{1}{s+1} \\ \dfrac{2}{s+2}-\dfrac{2}{s+1} & \dfrac{2}{s+2}-\dfrac{1}{s+1} \end{bmatrix}$$

$$\therefore \phi(t) = \begin{bmatrix} -e^{-2t}+2e^{-t} & -e^{-2t}+e^{-t} \\ 2e^{-2t}-2e^{-t} & 2e^{-2t}-e^{-t} \end{bmatrix}$$

$$\frac{?}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$s=-2, B(-1) = \begin{cases} -2+3 \Rightarrow B=-1 \\ 1 \Rightarrow B=-1 \\ -2 \Rightarrow B=2 \\ -2 \Rightarrow B=2 \end{cases}$$

$$A(s+2)+B(s+1) = ?$$

$$s=-1, A = \begin{cases} -1+3 \Rightarrow A=2 \\ 1 \Rightarrow A=1 \\ -2 \Rightarrow A=-2 \\ -1 \Rightarrow A=-1 \end{cases}$$