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SEHS4653
Control System Analysis

Unit 5

Frequency Response Analysis

(Reference: [1] chapter 7.1 to 7.7)

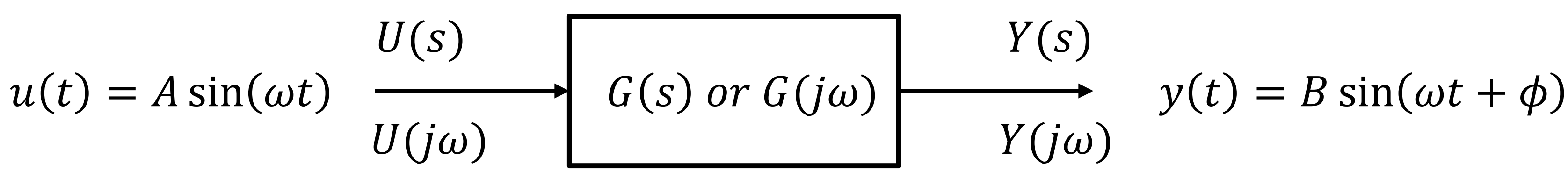
Content

- Introduction
- Bode Diagram (Exact vs Asymptotic)
- Polar (or Nyquist) Plot
- Log-Magnitude-versus-Phase Plot (Nichols Plot)
- Nyquist Stability Criterion
- Relative Stability Analysis
  - Gain Margin and Phase Margin

Introduction

Frequency Response Approach

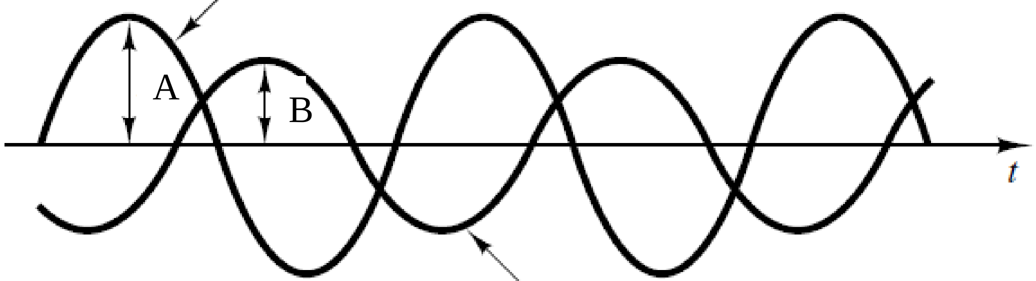
- Steady-state response of a system to a sinusoidal input
- Varying the frequency of the input signal over a certain range and study the resulting response
- Use the data obtained from measurements (experimentally) on the physical system without deriving its mathematical model (without the transfer function of the control system)
- Replacing s in the transfer function $G(s)$ by $j\omega$, where ω is the frequency
- Graphical forms: Bode Diagram, Nyquist (Polar) Plot and Nichols Plot



ϕ: Phase Shift

Introduction

Frequency Response Approach



$$\frac{Y(j\omega)}{U(j\omega)} = G(j\omega)$$

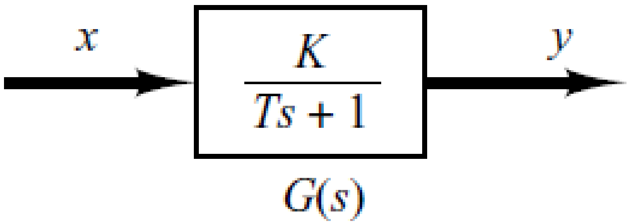
- The function $G(j\omega)$ is called the sinusoidal transfer function, which is a complex quantity
- It can be represented by the magnitude and phase angle with frequency as a parameter

$$|G(j\omega)| = \left|\frac{Y(j\omega)}{U(j\omega)}\right| =$$ amplitude ratio of the output sinusoid to the input sinusoid

$$\angle G(j\omega) = \angle\frac{Y(j\omega)}{U(j\omega)} =$$ phase shift of the output sinusoid with respect to the input sinusoid

Example 1

Consider the system shown below, the transfer function $G(s)$ is



Replacing s in the transfer function $G(s)$ by $j\omega$,

$$G(j\omega) = \frac{K}{j\omega T + 1}$$

The amplitude ratio of the output to the input is,

$$|G(j\omega)| = \frac{K}{\sqrt{(\omega T)^2 + 1^2}} = \frac{K}{\sqrt{1 + \omega^2 T^2}}$$

While the phase angle ϕ is,

$$\angle G(j\omega) = \tan^{-1}\frac{0}{K} - \tan^{-1}\frac{\omega T}{1} = -\tan^{-1}\omega T$$

Bode Diagram

Overview

- Consists of 2 graphs: logarithm of the magnitude of a sinusoidal transfer function and phase angle
- Both are plotted against the frequency on a logarithmic scale
- The logarithmic magnitude of $G(j\omega)$ is $20\log_{10}|G(j\omega)|$ dB (decibels)
- The phase angle (or phase shift) is in degrees or radians
- The curves are drawn on semilog paper, using the log scale for frequency and the linear scale for either magnitude or phase angle

Exact Bode Diagram

- Substitute different values of ω (rad/s) into the magnitude and phase angle equations for plotting

Asymptotic Bode Diagram

- Identify basic factors of $G(j\omega)H(j\omega)$ for plotting

Bode Diagram

Basic Factors of $G(j\omega)H(j\omega)$ for plotting Asymptotic Bode Diagram

- Gain K
- Integral $\left(\frac{1}{j\omega}\right)$ and derivative $(j\omega)$ factors
- First-order factors, e.g. $(1 + j\omega T)$ and $\left(\frac{1}{1+j\omega T}\right)$
- Quadratic factors, e.g. $\left((j\omega)^2 + 2j\omega\zeta\omega_n + \omega_n^2\right)$ and $\left(\frac{1}{(j\omega)^2+2j\omega\zeta\omega_n+\omega_n^2}\right)$

Bode Diagram (Asymptotic)

The Gain K

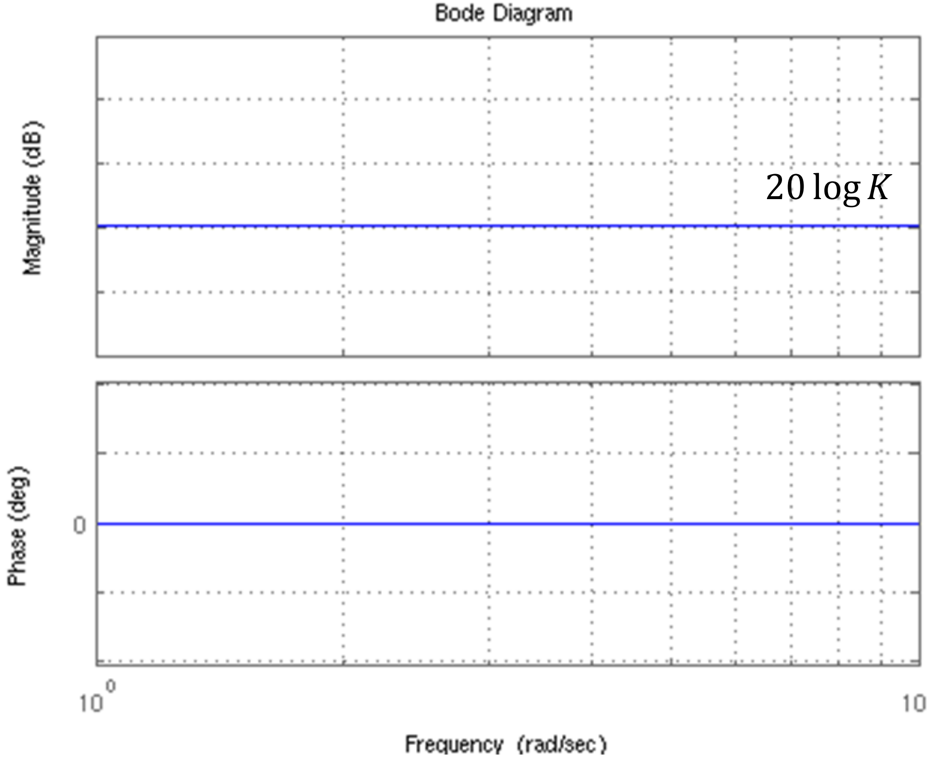
- A number greater than unity has a positive value in decibels, while a number smaller than unity has a negative value
- Consider $G(j\omega) = K$, K = constant

$G(j\omega) = K$

Magnitude: $20\log|G(j\omega)| = 20\log K$

Phase Angle: $\angle G(j\omega) = \tan^{-1}\frac{0}{K} = 0°$

- The effect of varying the gain K in the transfer function is that it raises or lowers the log-magnitude curve of the transfer function by the corresponding constant amount, but it has no effect on the phase curve.



Bode Diagram (Asymptotic)

Integral Factor (Pole)

$$G(j\omega) = \frac{1}{j\omega}$$

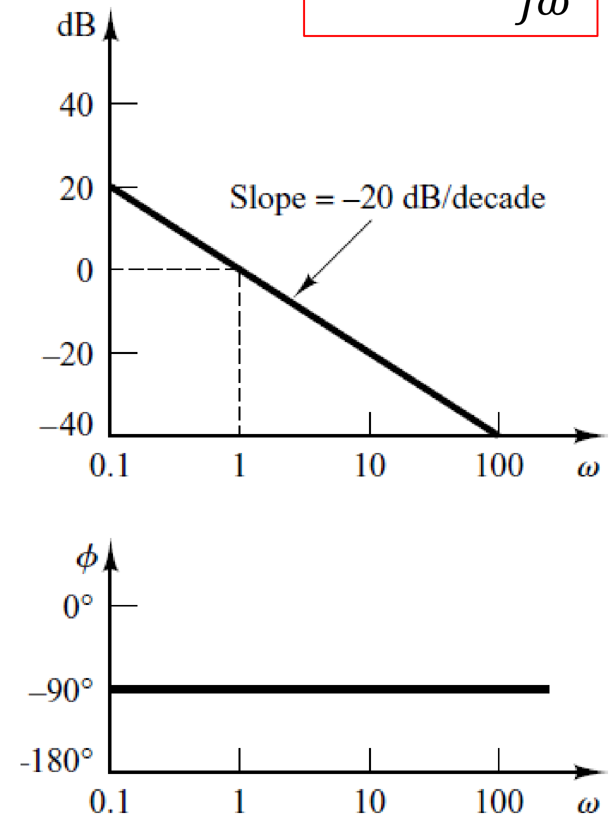
Consider $G(j\omega) = \frac{1}{j\omega}$

Magnitude: $20\log\left|\frac{1}{j\omega}\right| = 20\log\left(\frac{1}{\omega}\right)$

$$= 20\log\omega^{-1} = -20\log\omega \quad \text{(dB)}$$

- The slope of the line is –20 dB / decade

Phase angle: $\angle G(j\omega) = -\tan^{-1}\frac{\omega}{0} = -90°$



Bode diagram of $G(j\omega) = 1/j\omega$

Bode Diagram (Asymptotic)

Derivative Factor (Zero)

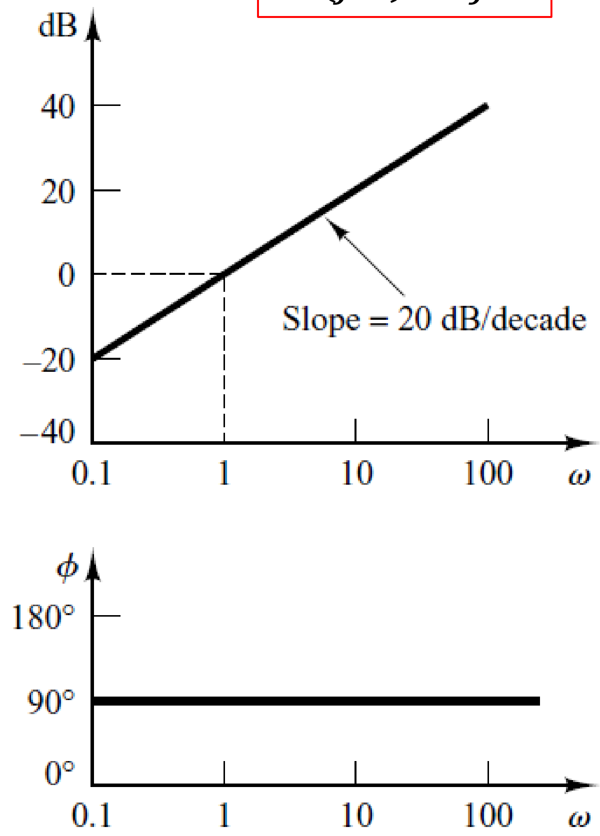
Consider $G(j\omega) = j\omega$

Magnitude: $20\log|j\omega| = 20\log\omega$ (dB)

- The slope of the line is +20 dB / decade

Phase angle: $\angle G(j\omega) = \tan^{-1}\frac{\omega}{0} = +90°$

$G(j\omega) = j\omega$



Bode diagram of $G(j\omega) = j\omega$

Bode Diagram (Asymptotic)

First-Order Factors (Pole)

$$G(j\omega) = \frac{1}{1 + j\omega T}$$

Consider $G(j\omega) = \frac{1}{1+j\omega T}$

Magnitude: $20\log|G(j\omega)| = 20\log\left|\frac{1}{1+j\omega T}\right|$

$$= 20\log\frac{1}{\sqrt{1^2 + (\omega T)^2}} = -20\log\sqrt{1 + \omega^2 T^2}\ \text{(dB)}$$

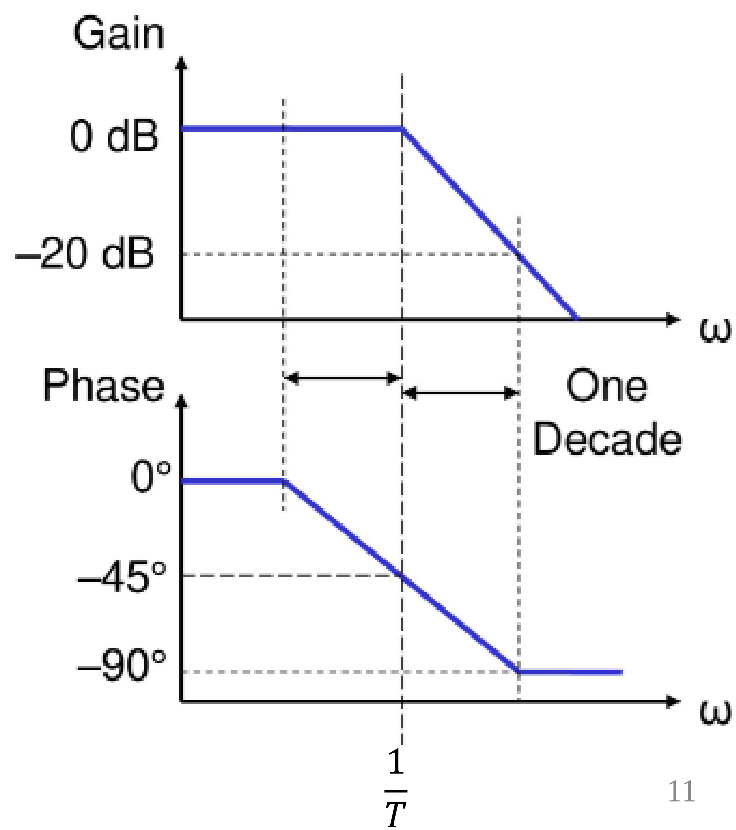
Phase angle: $\angle G(j\omega) = \phi = -\tan^{-1}\omega T$

- At low frequencies, $\omega \ll 1/T$,

$$-20\log\sqrt{1 + \omega^2 T^2} \approx -20\log 1 = 0\ \text{dB}$$

- At high frequencies, $\omega \gg 1/T$,

$$-20\log\sqrt{1 + \omega^2 T^2} \approx -20\log\omega T\ \ \text{dB}$$



Bode Diagram (Asymptotic)

First-Order Factors (Pole)

Magnitude: $-20\log\sqrt{1+\omega^2T^2}$ dB

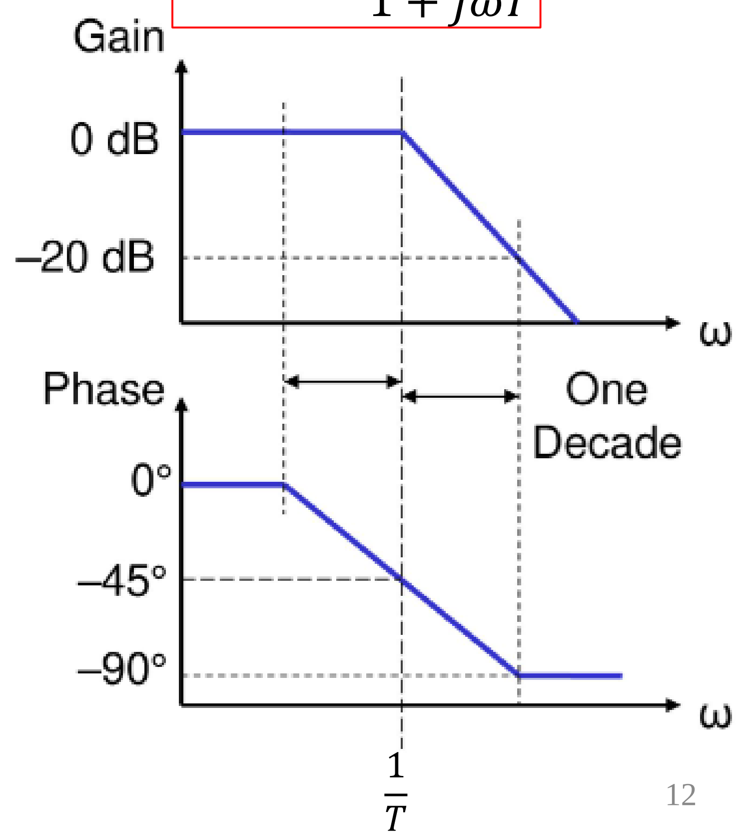
Phase: $\angle G(j\omega) = \phi = -\tan^{-1}\omega T$

$$G(j\omega) = \frac{1}{1+j\omega T}$$

The frequency at which the two asymptotes meet is called the corner frequency or break frequency

- Corner frequency for this example: $\omega = 1/T$
- At low frequencies, $\approx -20\log 1 = 0$ dB
- At high frequencies, $\approx -20\log 10 = -20$ dB

- At $\omega = 0$ rad/s, $\phi = 0°$
- At $\omega \to \infty$ rad/s, $\phi = -90°$
- At $\omega = 1/T$ rad/s, $\phi = -\tan^{-1}\left(\frac{1}{T}\right)T = -45°$



The error in the magnitude curve caused by the use of asymptotes at corner frequency is,

$$-20\log\sqrt{1+\left(\frac{1}{T}\right)^2T^2} = -20\log\sqrt{2} = -3.01 \text{ dB}$$

Bode Diagram (Asymptotic)

First-Order Factors (Zero)

Consider $G(j\omega) = 1 + j\omega T$

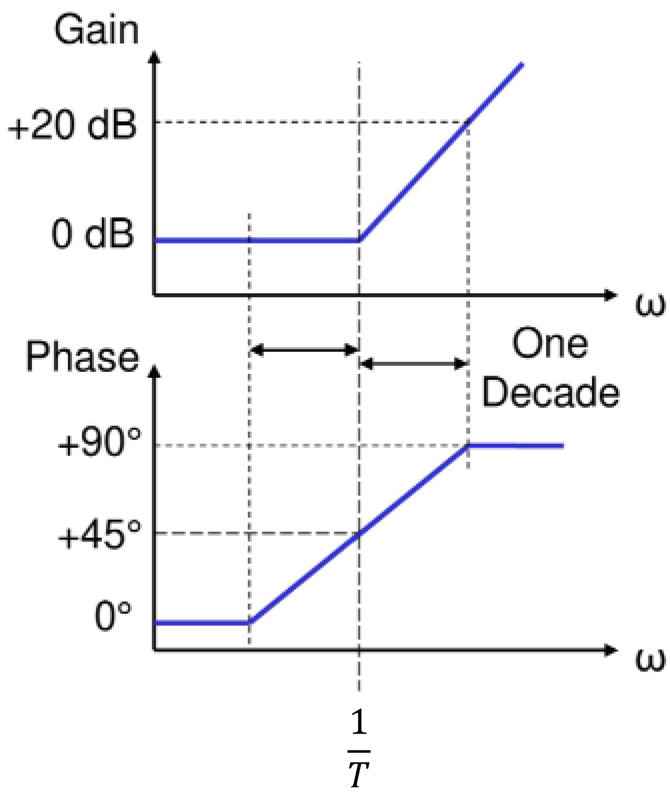
The log-magnitude and the phase-angle curves need only be changed in sign of the previous case

Magnitude: $20\log|G(j\omega)| = 20\log\sqrt{1^2 + \omega^2 T^2}$ (dB)

Phase angle: $\angle G(j\omega) = \phi = \tan^{-1}\omega T$

The corner frequency is the same

$G(j\omega) = 1 + j\omega T$



Bode Diagram (Asymptotic)

Quadratic Factors (Pole)

Consider $G(j\omega) = \frac{1}{(j\omega)^2 + 2j\omega\zeta\omega_n + \omega_n^2}$

$$\therefore G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$$

- If $\zeta = 1$, this quadratic factor can be expressed as a product of two first-order factors with real poles
- If $0 < \zeta < 1$, this quadratic factor is the product of two complex conjugate factors
- Asymptotic approximations to the frequency-response curves are not accurate for a factor with low values of ζ

Magnitude: $$20\log\left|\frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}\right| = -20\log\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

At low frequencies such that $\omega \ll \omega_n$, the log-magnitude becomes, $-20\log 1 = 0$ dB

At high frequencies such that $\omega \gg \omega_n$, the log-magnitude becomes,

$$-20\log\frac{\omega^2}{\omega_n^2} = -40\log\frac{\omega}{\omega_n}\ \text{dB}$$

Bode Diagram

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$$

Quadratic Factors (Pole)

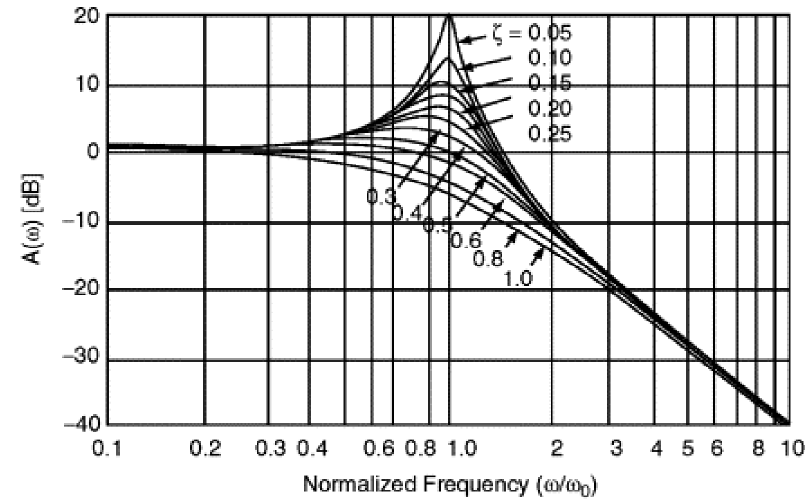
Phase: $$\phi = -\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

At $\omega = 0$, the phase angle equals 0°

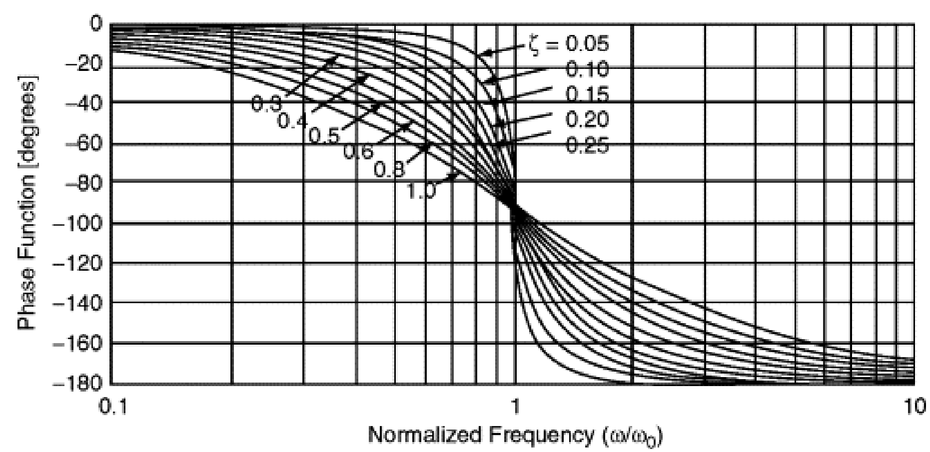
At the corner frequency $\omega = \omega_n$, the phase angle is –90° regardless of ζ,

$$\phi = -\tan^{-1}\left(\frac{2\zeta}{0}\right) = -\tan^{-1}\infty = -90°$$

At $\omega = \infty$, the phase angle becomes –180°



(A) Magnitude Plot



(B) Phase Plot

Bode Diagram (Asymptotic)

Quadratic Factors (Zero)

Consider $G(j\omega) = (j\omega)^2 + 2j\omega\zeta\omega_n + \omega_n^2 = \boxed{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$

- Similar to the first order factor, merely reversing the sign of the log magnitude and that of the phase angle of the factor

Magnitude: $$= 20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

Phase: $$\phi = \tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

Bode Diagram (Asymptotic)

General Procedure for Plotting Bode Diagrams

- Rewrite the transfer function, $G(j\omega)H(j\omega)$, as a product of the basic factors as discussed before
- Identify the corner frequencies associated with these basic factors
- Draw the asymptotic log-magnitude curves with proper slopes between the corner frequencies
- The phase-angle curve can be drawn by adding the phase-angle curves of individual factors

Advantages

- Much less time than other methods that may be used for computing the frequency response of a transfer function
- The ease of plotting the frequency-response curves for a given transfer function and the ease of modification as compensation is added are the main reasons

Example 2

Plot the Bode diagram for the transfer function,

$$G(s) = \frac{10}{s + 20}$$

<u>Answer:</u>

Step 1: Replacing s by $j\omega$ and rewrite the transfer function as a product of basic factors

$$G(j\omega) = \frac{10}{j\omega + 20} = \frac{\frac{10}{20}}{\frac{j\omega + 20}{20}} = \frac{1/2}{\frac{j\omega}{20} + 1}$$

Magnitude of the Gain constant $= 20\log\left(\frac{1}{2}\right) = -6.01\ \text{dB}$

Step 2: Identify the corner frequencies

Since there is only one first order factor (Pole), the corner frequency is, $\omega = 20$ rad/s

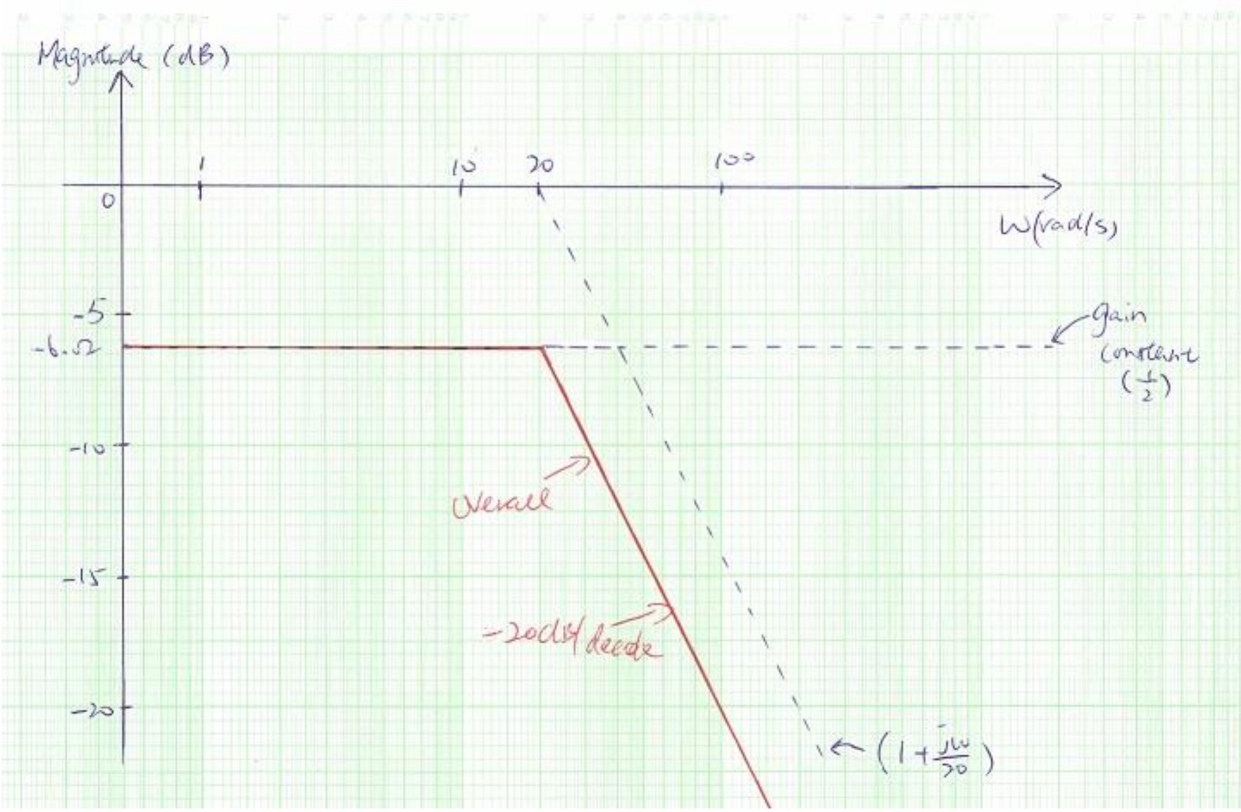
Example 2

$$G(j\omega) = \frac{1/2}{\frac{j\omega}{20} + 1}$$

Answer:

Step 3: Draw the asymptotic log-magnitude and phase-angle curves for individual basic factors

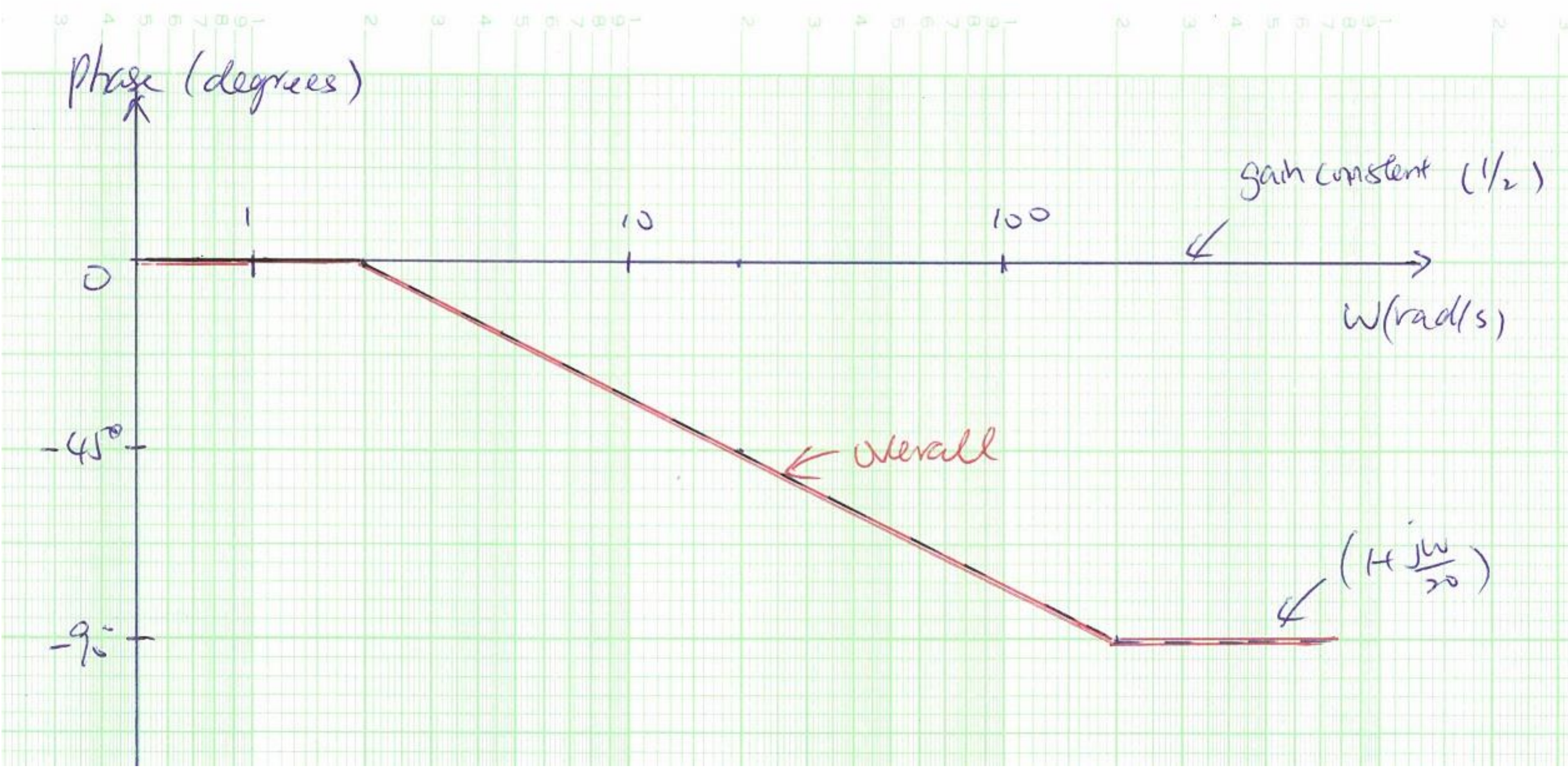
Step 4: Combine the log-magnitude and phase-angle curves



Example 2

$$G(j\omega) = \frac{1/2}{\frac{j\omega}{20} + 1}$$

Answer:



Example 2

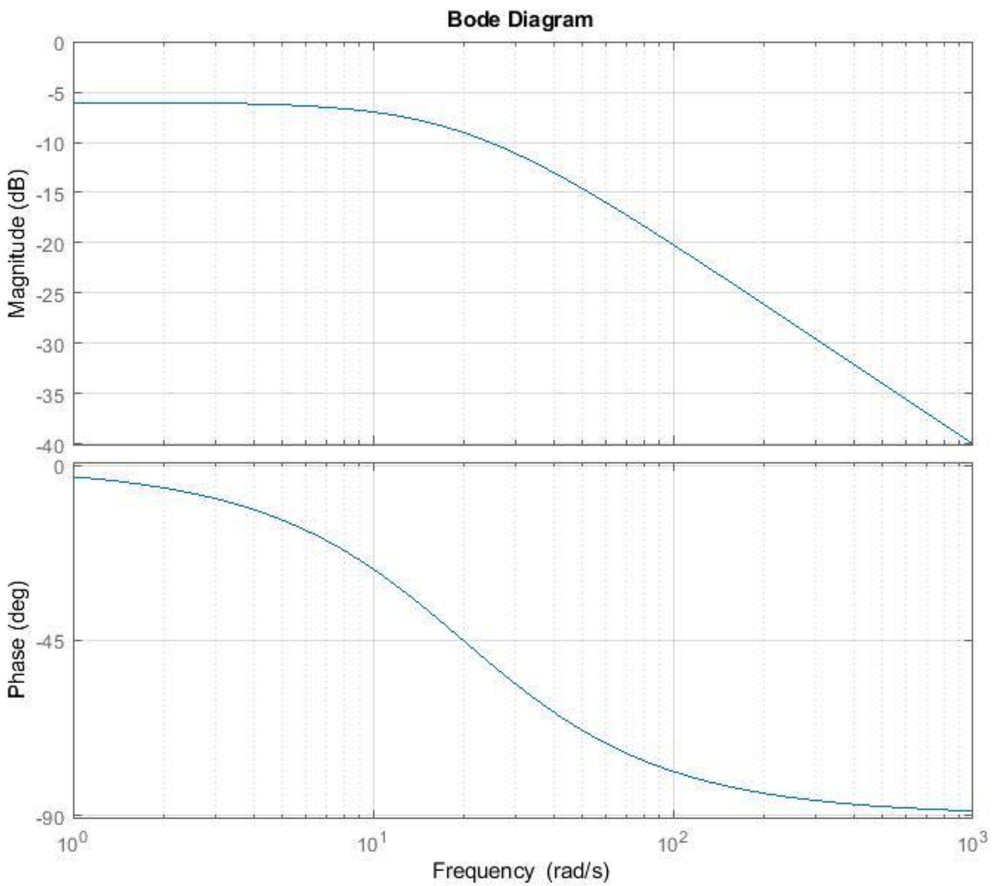
$$G(j\omega) = \frac{10}{j\omega + 20}$$

$$|G(j\omega)| = 20\log\frac{10}{\sqrt{\omega^2 + 20^2}}$$

$$\angle G(j\omega) = -\tan^{-1}\frac{\omega}{20}$$

Answer:

Exact Bode Diagram from Matlab



ω (rad/s)	$\lvert G(j\omega)\rvert$ (dB)	$\angle G(j\omega)$ (°)
1	-6.03	-2.86
3	-6.12	-8.53
5	-6.28	-14.04
10	-6.99	-26.57
20	-9.03	-45
50	-14.62	-68.2
70	-17.24	-74.05
100	-20.17	-78.69
200	-26.06	-84.29
600	-35.57	-88.09
800	-38.06	-88.57
1000	-40	-88.85

Example 3

Plot the Bode diagram for the transfer function,

$$G(s) = \frac{100(s+1)}{(s+5)(s+10)}$$

Answer:

Step 1: Replacing s by $j\omega$ and rewrite the transfer function as a product of basic factors

$$G(j\omega) = \frac{100(j\omega+1)}{(j\omega+5)(j\omega+10)} = \frac{100\left(\frac{j\omega+1}{1}\right)}{\left(\frac{j\omega+5}{5}\right)\left(\frac{j\omega+10}{10}\right)}\left(\frac{1}{(5)(10)}\right) = \frac{(2)(1+j\omega)}{\left(1+\frac{j\omega}{5}\right)\left(1+\frac{j\omega}{10}\right)}$$

Step 2: Identify the corner frequencies

Corner frequency is, $\omega = 5$ rad/s for the pole $\left(1+\frac{j\omega}{5}\right)$

Corner frequency is, $\omega =$10 rad/s for the pole $\left(1+\frac{j\omega}{10}\right)$

Corner frequency is, $\omega = 1$ rad/s for the zero $(1+j\omega)$

Magnitude of the Gain constant

$$= 20\log(2) = 6.01 \text{ dB}$$

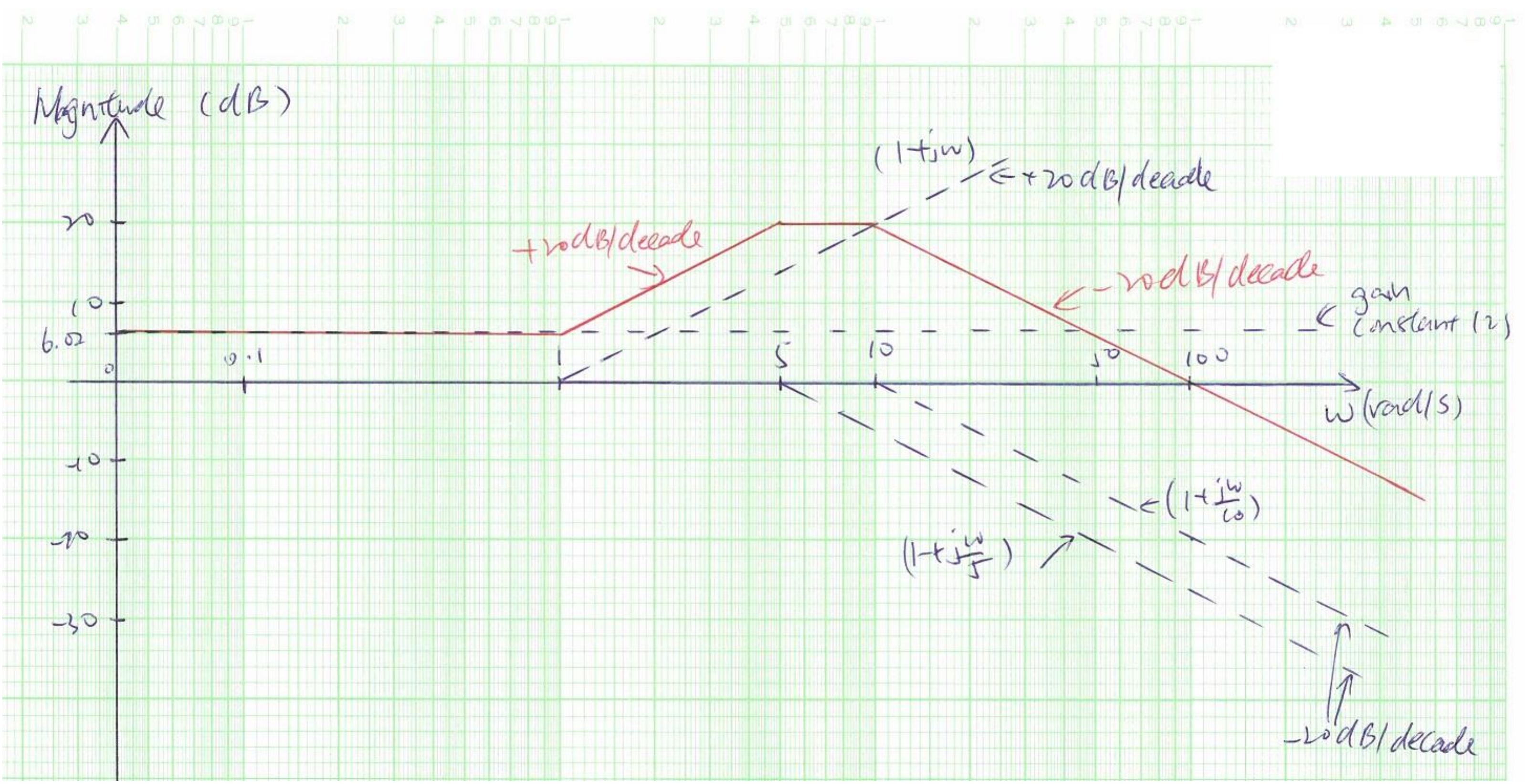
Example 3

$$G(j\omega) = \frac{(2)(1 + j\omega)}{\left(1 + \frac{j\omega}{5}\right)\left(1 + \frac{j\omega}{10}\right)}$$

<u>Answer:</u>

Step 3: Draw the asymptotic log-magnitude and phase-angle curves for individual basic factors

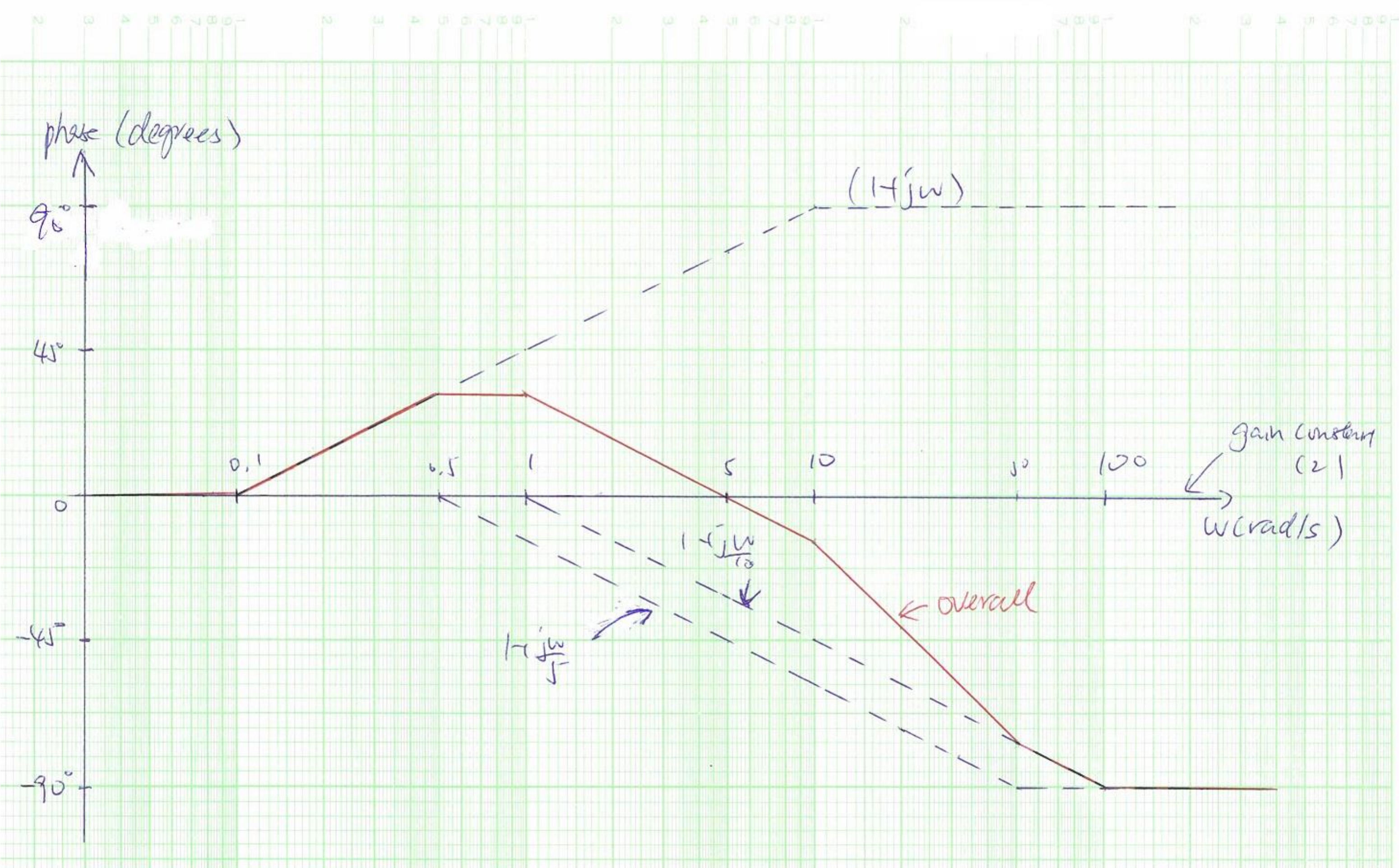
Step 4: Combine the log-magnitude and phase-angle curves



Example 3

$$G(j\omega) = \frac{(2)(1 + j\omega)}{\left(1 + \frac{j\omega}{5}\right)\left(1 + \frac{j\omega}{10}\right)}$$

Answer:



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Example 3

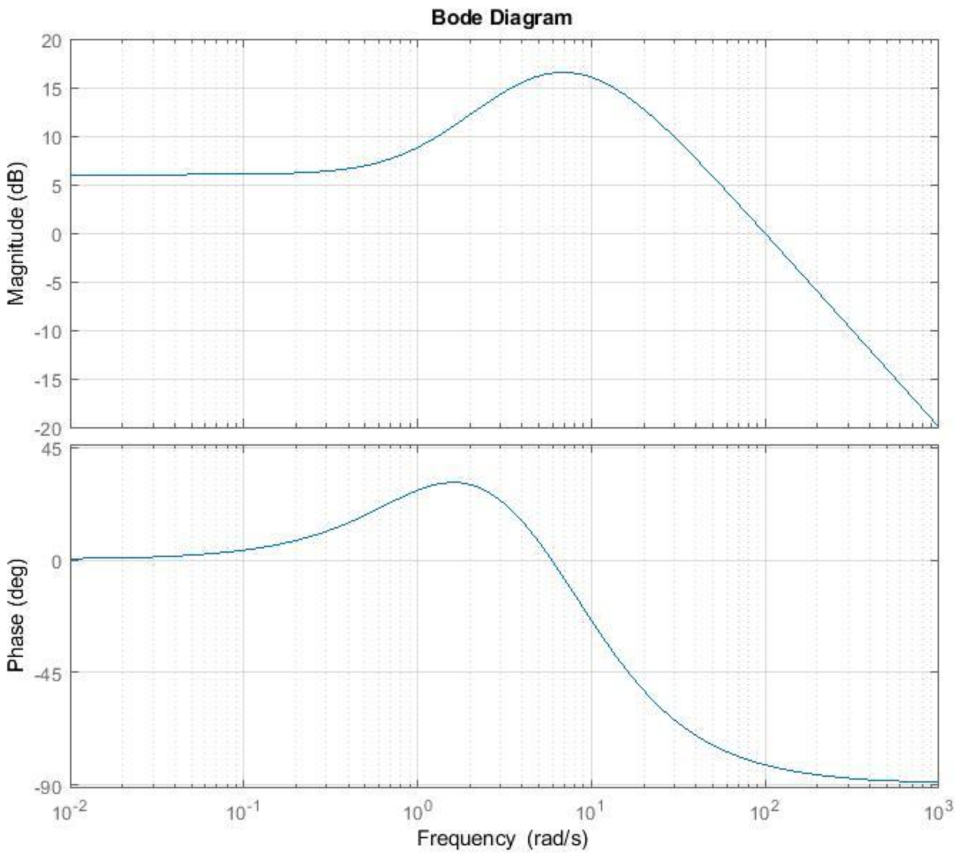
$$G(j\omega) = \frac{100(j\omega+1)}{(j\omega+5)(j\omega+10)}$$

$$|G(j\omega)| = 20\log\frac{100\sqrt{\omega^2+1^2}}{\sqrt{\omega^2+5^2}\sqrt{\omega^2+10^2}}$$

$$\angle G(j\omega) = \tan^{-1}\omega - \tan^{-1}\frac{\omega}{5} - \tan^{-1}\frac{\omega}{10}$$

Answer:

Exact Bode Diagram from Matlab

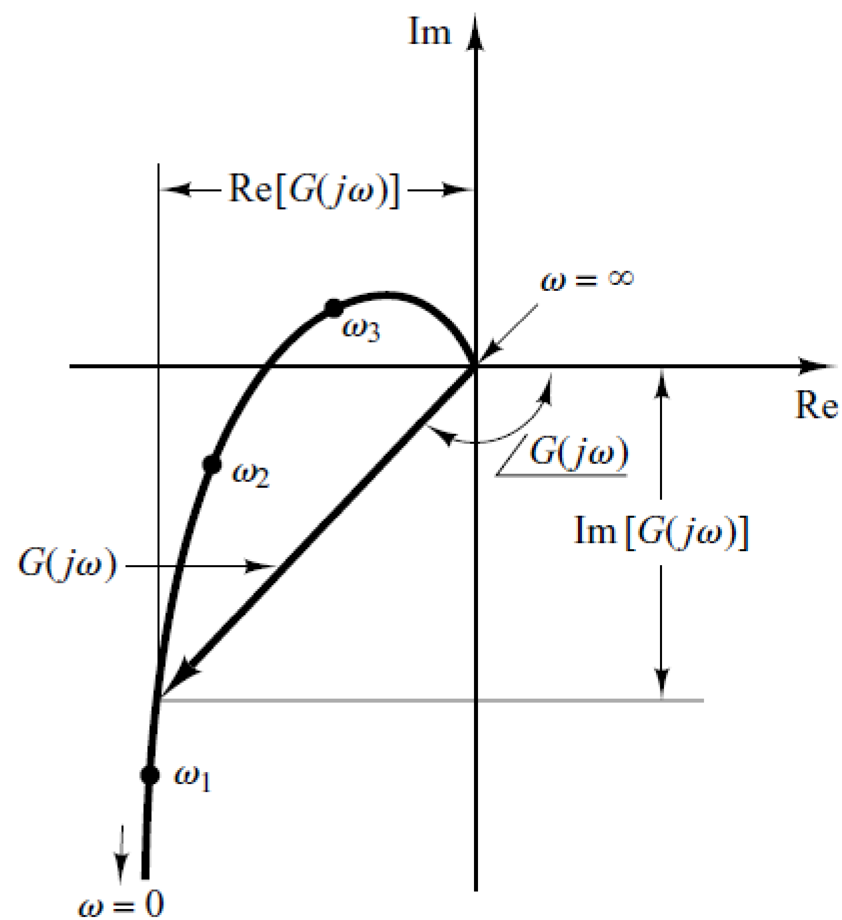


ω (rad/s)	$\lvert G(j\omega)\rvert$ (dB)	$\angle G(j\omega)$ (°)
0.1	6.06	3.99
0.5	6.94	17.99
0.8	8.03	25
1	8.82	27.98
3	14.31	23.9
5	16.19	7.13
10	16.06	-24.15
30	9.89	-64.01
50	5.81	-74.13
100	-0.05	-82
500	-13.98	-88.4
1000	-20	-89.2

Polar (or Nyquist) Plot

Overview

- A plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω is varied from zero to infinity
- Note that in polar plots a positive (negative) phase angle is measured counterclockwise (clockwise) from the positive real axis
- Each point on the polar plot of $G(j\omega)$ represents the terminal point of a vector at a particular value of ω
- It depicts the frequency-response characteristics of a system over the entire frequency range in a single plot



Example 4

The polar plot of the transfer function,

$$G(s) = \frac{10}{s(s+1)}$$

Answer:

Replacing s into $j\omega$, $$G(j\omega) = \frac{10}{j\omega(j\omega+1)}$$

Write the expressions for magnitude and phase of $G(j\omega)$ and varies ω from 0 to ∞.

ω rad/s	M	ϕ
0	∞	–90°
0.5	17.89	–116.57°
1.0	7.071	–135°
2.0	2.236	–153.43°
5.0	0.392	–168.69°
8.0	0.155	–172.87°
10.0	0.995	–174.29°

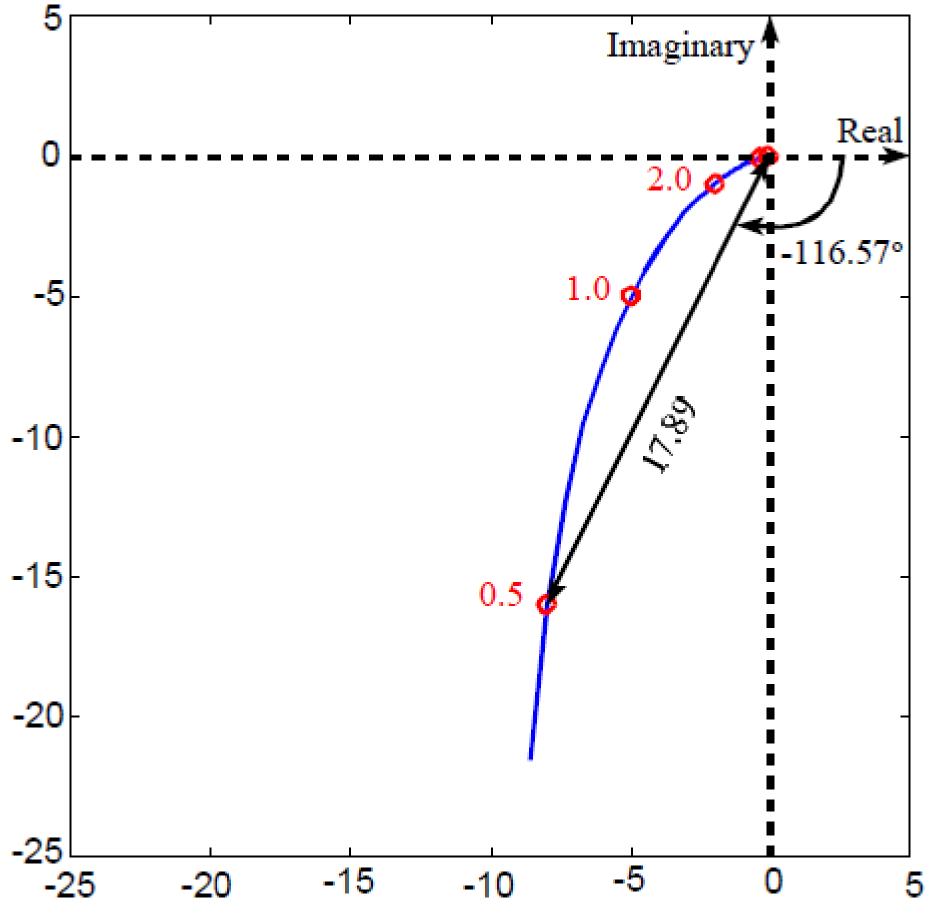
$$M = \frac{10}{\omega\sqrt{\omega^2 + 1^2}}$$

$$\phi = -90^\circ - \tan^{-1}\omega$$

Example 4

Answer:

The polar plot is,



Log-Magnitude-versus-Phase Plot (Nichols Plot)

Overview

- A plot of the logarithmic magnitude in decibels versus the phase angle or phase margin for a frequency range of interest
- The phase margin is the difference between the actual phase angle ϕ and –180°; that is, $\phi - (-180°) = 180° + \phi$
- It combines the 2 curves, log-magnitude curve and the phase-angle curve, in Bode diagrams
- A change in the gain constant of $G(j\omega)$ merely shifts the curve up (for increasing gain) or down (for decreasing gain), but the shape of the curve remains the same
- The relative stability of the closed-loop system can be determined quickly and that compensation can be worked out easily

Example 5

The Nichols plot of the transfer function,

$$G(s) = \frac{10}{s(s+1)}$$

Answer:

Replacing s into $j\omega$, $G(j\omega) = \frac{10}{j\omega(j\omega+1)}$

Write the expressions for magnitude and phase of $G(j\omega)$ and varies ω from 0 to ∞.

ω rad/s	M (dB)	ϕ
0	∞	–90°
0.5	25.05	–116.57°
1.0	16.99	–135°
2.0	6.99	–153.43°
5.0	–8.13	–168.69°
8.0	–16.19	–172.87°
10.0	–20.04	–174.29°

$$M = \frac{10}{\omega\sqrt{\omega^2+1^2}}$$

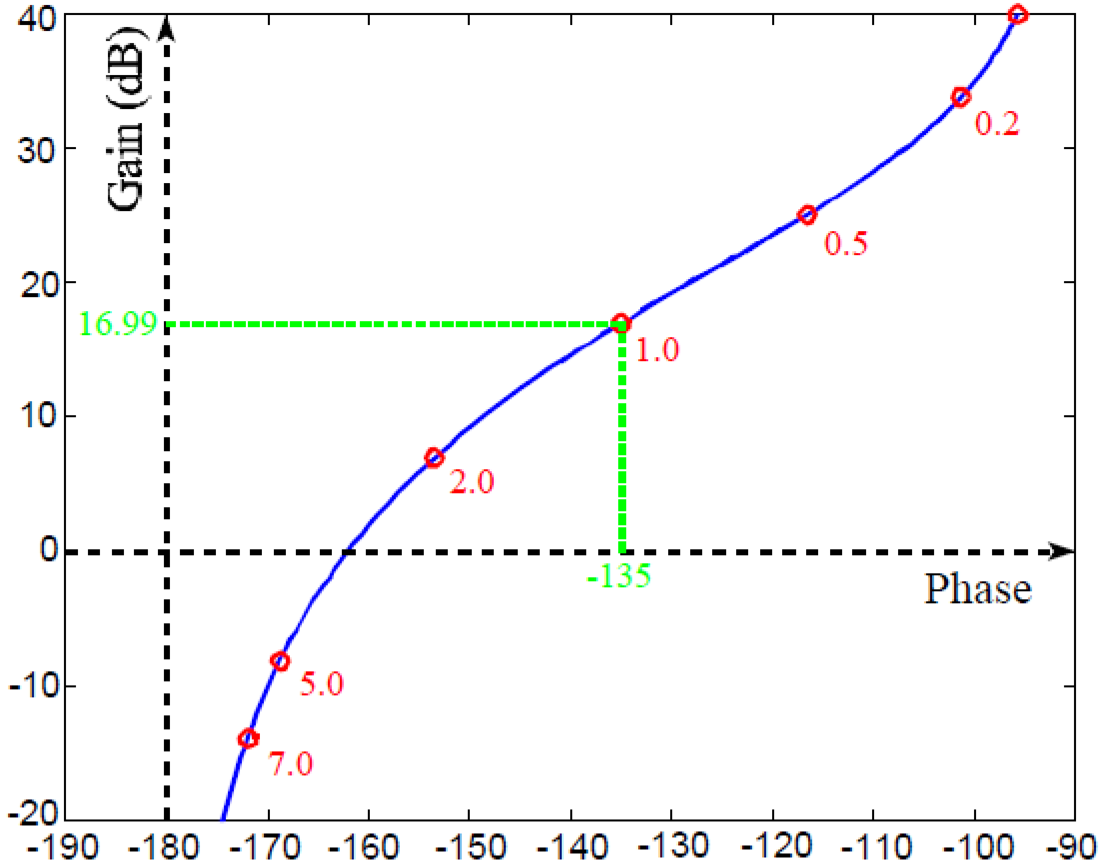
$$= 20\log 10 - 20\log\omega - 20\log\sqrt{\omega^2+1}$$

$$\phi = -90° - \tan^{-1}\omega$$

Example 5

Answer:

The Nichols plot is,



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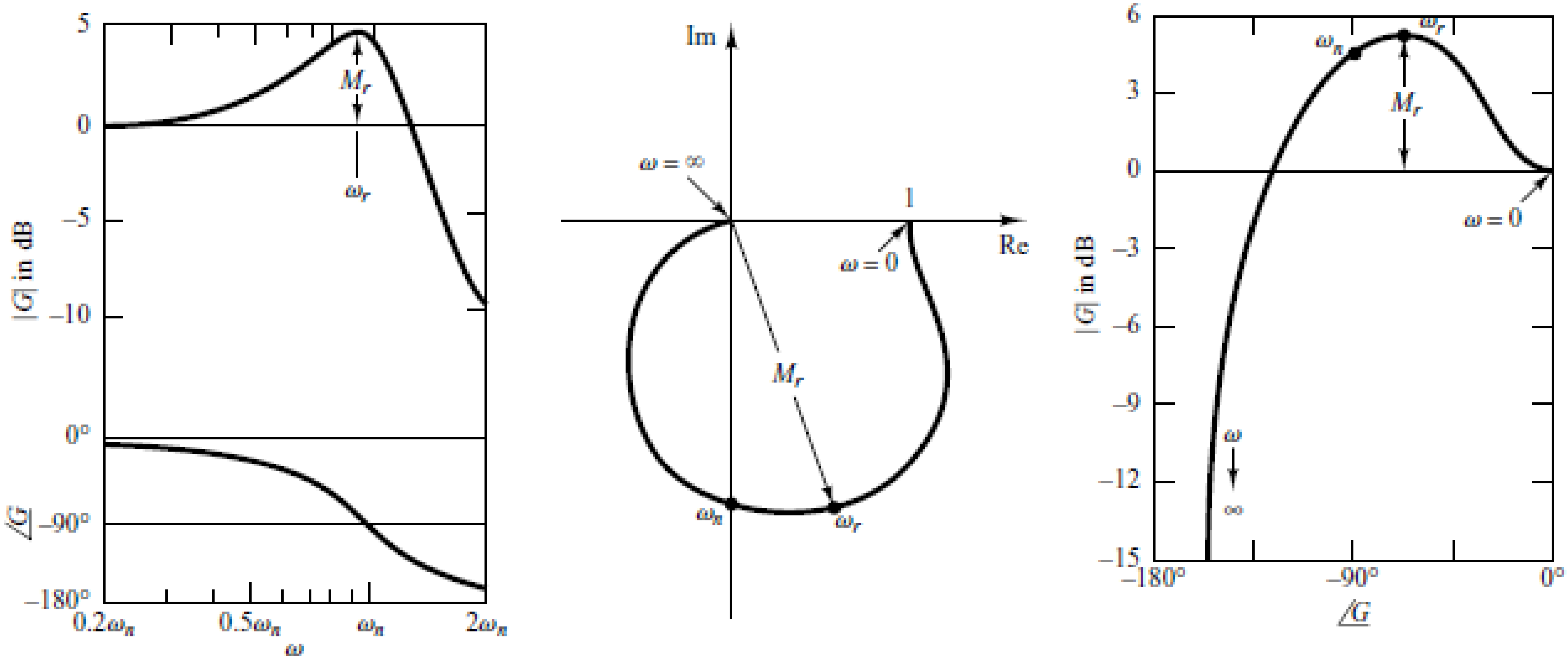
Log-Magnitude-versus-Phase Plot (Nichols Plot)



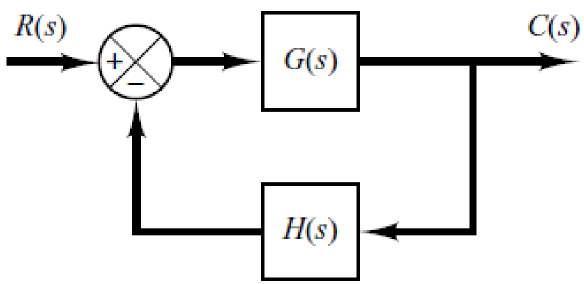
Three representations of the frequency response of $\dfrac{1}{1+2\zeta\left(j\dfrac{\omega}{\omega_n}\right)+\left(j\dfrac{\omega}{\omega_n}\right)^2}$, for $\zeta > 0$.

(a) Bode diagram; (b) polar plot; (c) log-magnitude-versus-phase plot.

Nyquist Stability Criterion

Overview

- The Nyquist stability criterion determines the stability of a closed-loop system from its open-loop frequency response and open-loop poles



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

- For stability, all roots of the characteristic equation, $\Delta(s) = 1 + G(s)H(s) = F(s)$ must lie in the left-half s plane
- The Nyquist stability criterion relates the open-loop frequency response $G(j\omega)H(j\omega)$ to the number of zeros and poles of $\Delta(s)$ of $F(s)$ that lie in the right-half s plane
- The absolute stability of the closed-loop system can be determined graphically from open-loop frequency-response curves

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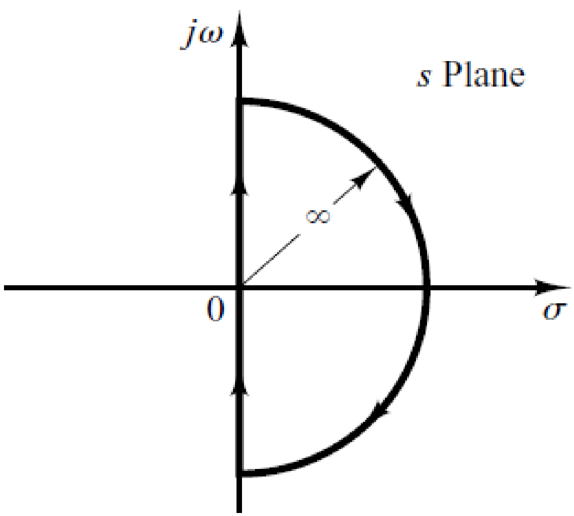
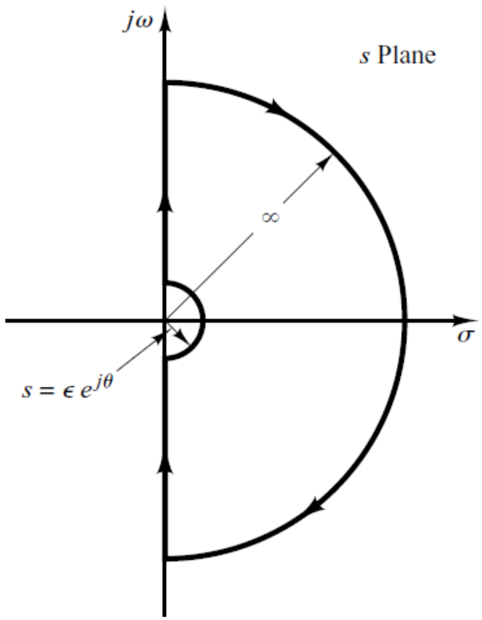
Nyquist Stability Criterion

Stability Analysis of Closed-loop Systems

- Let the closed contour in the *s* plane enclose the entire right-half *s* plane
- This contour consists of the entire $j\omega$ axis from $\omega = -\infty$ to $+\infty$ and a semicircular path of infinite radius in the right-half *s* plane
- The contour encloses all the zeros and poles of $F(s)$ that have positive real parts
- If the function $F(s)$ has poles or zeros at the origin or at some points the $j\omega$ axis, make a detour along an infinitesimal semicircle





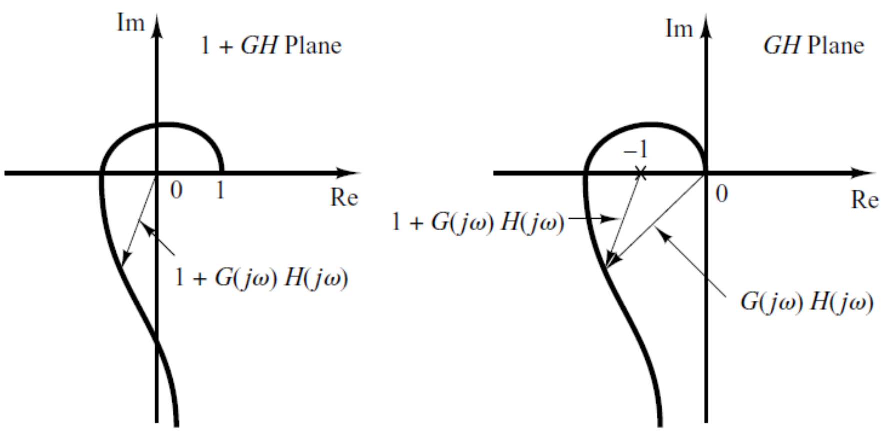
Nyquist Stability Criterion

Stability Analysis of Closed-loop Systems

- If the closed contour in the *s* plane encloses the entire right-half *s* plane, then

$$Z = N + P$$

  - Z = Number of right-half *s* plane zeros of $F(s)$
  - P = Number of right-half *s* plane poles of $G(s)H(s)$
  - N = Number of clockwise encirclement of the origin of the $F(s)$-plane
- A system is stable, we must have $Z = 0$, or $N = -P$ (having P counterclockwise encirclements of the origin)
- The origin of the $F(s)$-plane is the point $(-1 + j0)$ on the $G(j\omega)H(j\omega)$ plane



Hence, feedback control system is stable if and only if, the number of counterclockwise encirclements of the point $(-1 + j0)$ by the map of the Nyquist contour on the *GH*-plane = number of poles of the $G(s)H(s)$ within the Nyquist contour on the *s* plane.

Nyquist Stability Criterion

Practical Approach to Apply the Rule ($Z = N + P$)

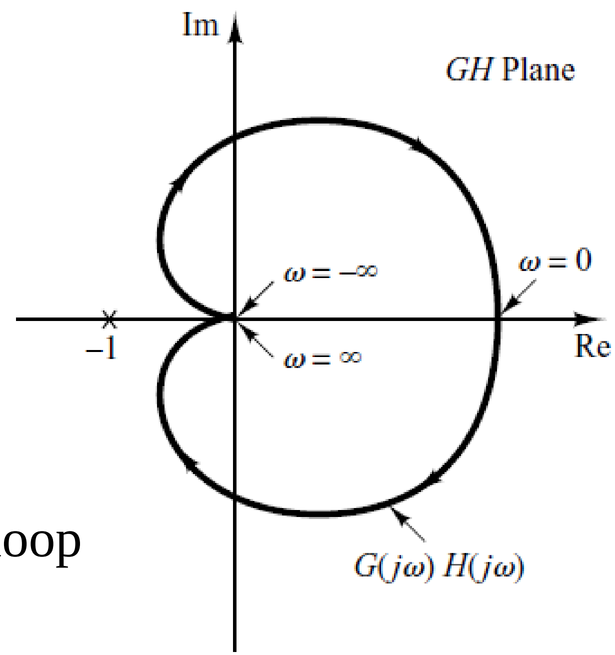
- Determine P by inspecting the denominator of the $G(s)H(s)$
- Determine N:
  - Sketch the open-loop locus (Polar Plot) from $\omega = -\infty$ to $+\infty$
  - Draw a straight line in any direction from $(-1 + j0)$ point
  - Where this line crosses open-loop locus, mark arrow heads in the direction of increasing frequency
  - N = number of clockwise arrows – number of counterclockwise arrows

Example 6

Consider a closed-loop system whose open-loop transfer function is given by

$$G(s)H(s) = \frac{K}{(T_1 s + 1)(T_2 s + 1)}$$

with K, T_1 and T_2 are positive values. Examine the stability of the system with the given polar plot.



Answer:

Nyquist Stability Criterion: $Z = N + P$

$P = 0$

$N = 0$

Hence, $Z = 0$.

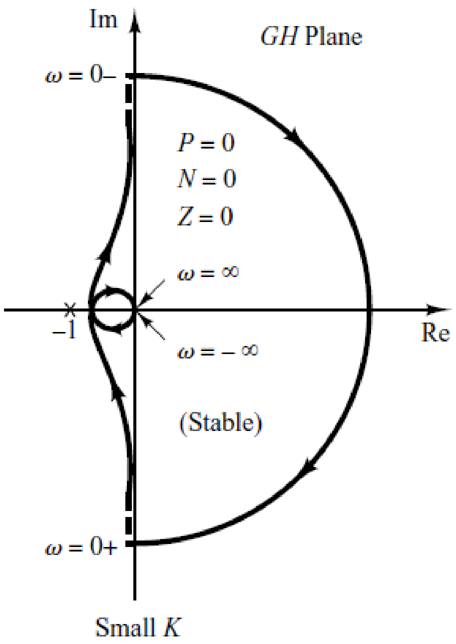
The system is stable since there is no closed-loop poles in the right-half s plane

Example 7

Consider the system with the following open-loop transfer function,

$$G(s)H(s) = \frac{K}{s(T_1 s + 1)(T_2 s + 1)}$$

with K, T_1 and T_2 are positive values. Determine the stability of the system for two cases: (1) the gain K is small and (2) K is large.



Answer:

Nyquist Stability Criterion: $Z = N + P$

$P = 0$

$N = 0$

Hence, $Z = 0$.

The system is stable since there is no closed-loop poles in the right-half s plane

Example 7

<u>Answer:</u>

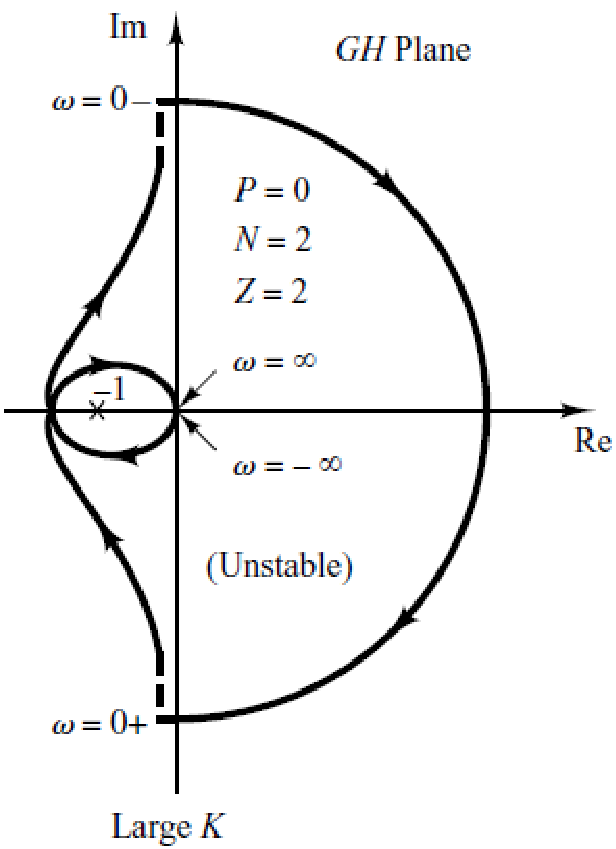
Nyquist Stability Criterion: $Z = N + P$

$P = 0$

$N = 2$ (2 clockwise encirclements of $(-1 + j0)$)

Hence, $Z = 2$.

The system is unstable since there is 2 closed-loop poles in the right-half s plane



Relative Stability Analysis

Relative Stability

- The degree of stability of a stable system, hence we can think of different design strategies to improve the stability of the control systems

- The closer the $G(j\omega)H(j\omega)$ locus comes to encircling the point $(-1 + j0)$, the more oscillatory is the system response

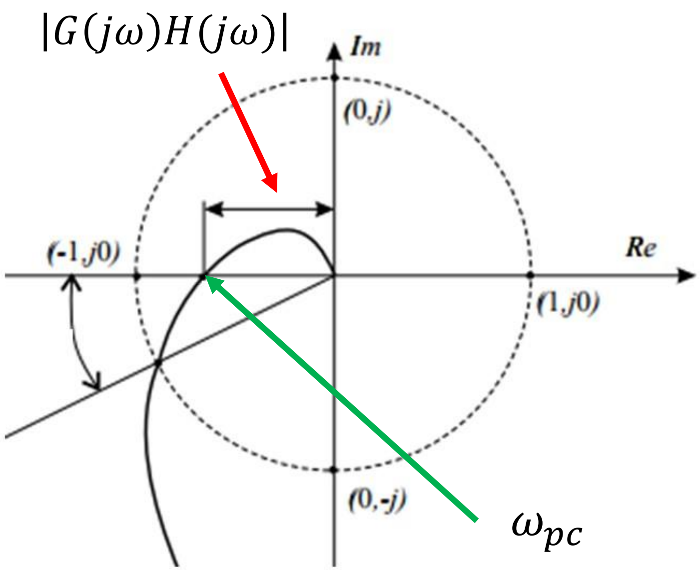
- Hence, the proximity of the open-loop frequency response $(G(j\omega)H(j\omega)$ locus) to the point $(-1 + j0)$ on the *GH*-plane (or $F(s)$-plane) is a measure of the relative stability of a closed-loop system

- It is a common practice to represent the proximity in terms of **phase margin** and **gain margin**

Relative Stability Analysis

Gain Margin

- It is defined as the additional gain required to make the system just unstable
- The amount by which the magnitude of $G(j\omega)H(j\omega)$ must be increased in order to be equal to 1 when $\angle G(j\omega)H(j\omega) = -180°$
- Phase crossover frequency (ω_{pc}) - the frequency at which $\angle G(j\omega)H(j\omega) = -180°$



$$G.M. = \frac{1}{|G(j\omega)H(j\omega)|}$$

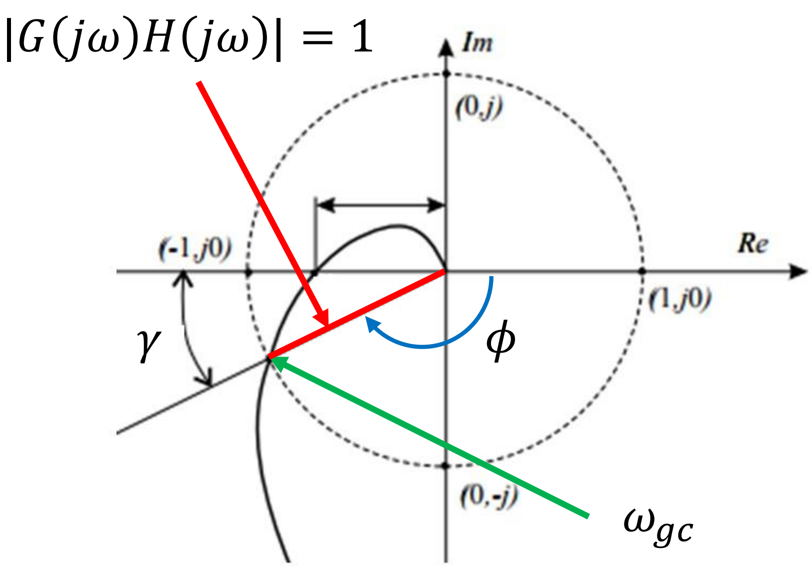
$$G.M.(\text{dB}) = -20\log|G(j\omega)H(j\omega)|$$

Typical Degree Values
$G.M.$ = 1.5 ~ 4.0 (3.5 ~ 12 dB)

Relative Stability Analysis

Phase Margin

- It is defined as the additional phase lag required to make the system just unstable
- The additional phase lag required make $\angle G(j\omega)H(j\omega) = -180°$ at the frequency for which the magnitude of $G(j\omega)H(j\omega)$ is equal to 1
- Gain crossover frequency (ω_{gc}) - the frequency at which $|G(j\omega)H(j\omega)| = 1$



$$P.M. = \gamma = 180° + \angle G(j\omega)H(j\omega)$$

$$\gamma = 180° + \phi$$

Typical Degree Values
$P.M. = \gamma = 30° \sim 60°$

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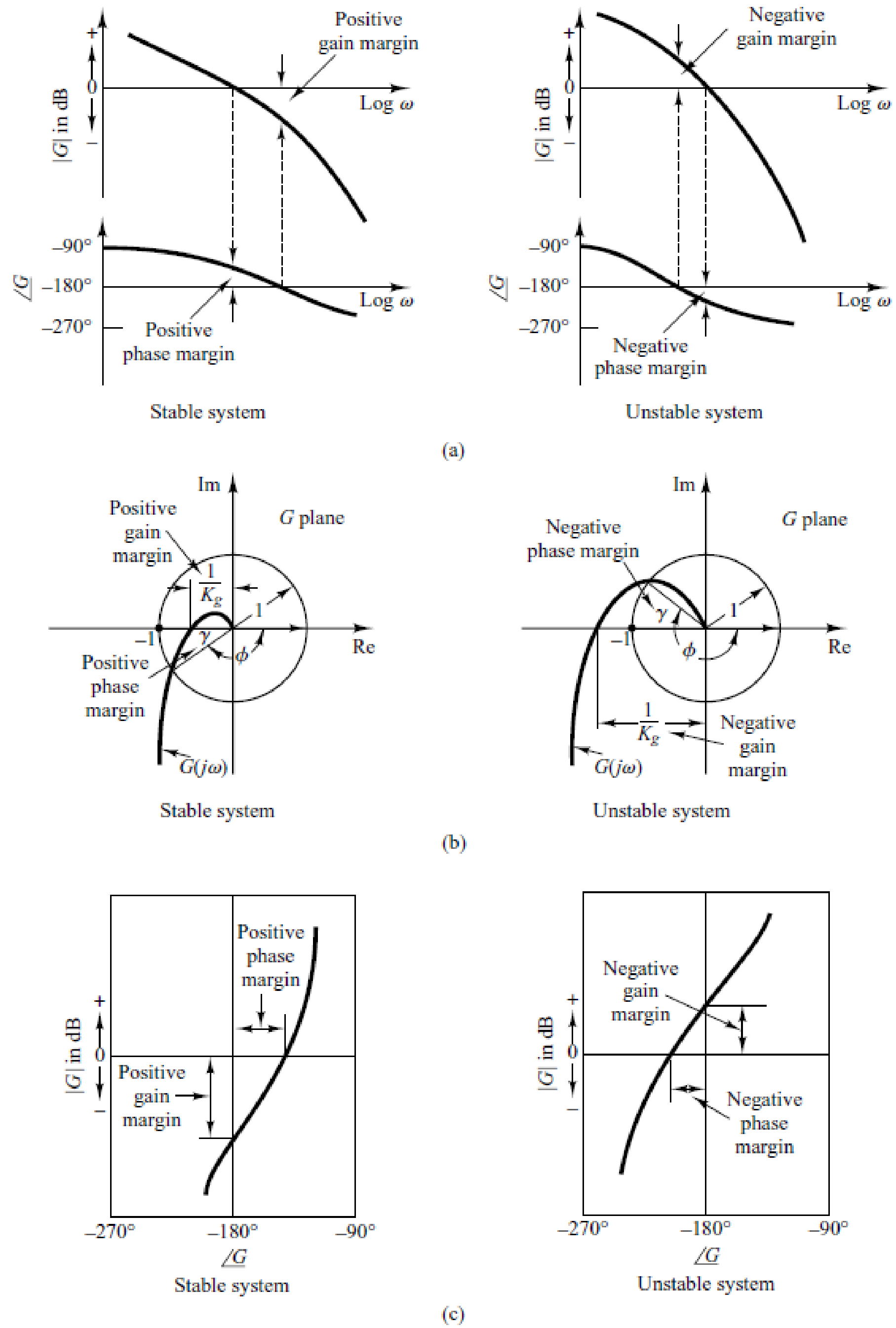
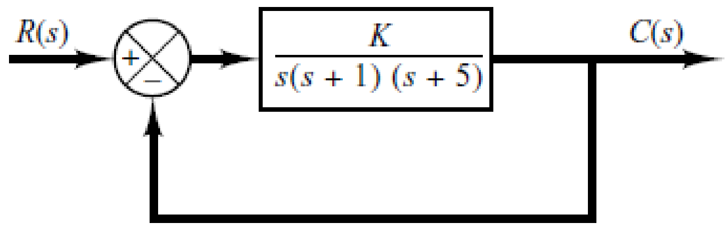
Relative Stability Analysis



(a) Bode diagrams
(b) Polar Plots
(c) Log-magnitude versus-phase plots

Example 8

Obtain the phase and gain margins of the system shown below for the two cases where $K = 10$ and $K = 100$.



Answer:

You can either draw the **Bode diagrams, polar plot or Nichols plot** of the **open-loop** frequency response for determining the G.M. and P.M. with the following magnitude and phase equations.

$$G(j\omega) = \frac{K}{j\omega(j\omega + 1)(j\omega + 5)}$$

$$|G(j\omega)|(\text{dB}) = 20\log K - 20\log\omega - 20\log\sqrt{1+\omega^2} - 20\log\sqrt{5^2+\omega^2}$$

$$\angle G(j\omega)(°) = -90° - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{5}$$

Example 8

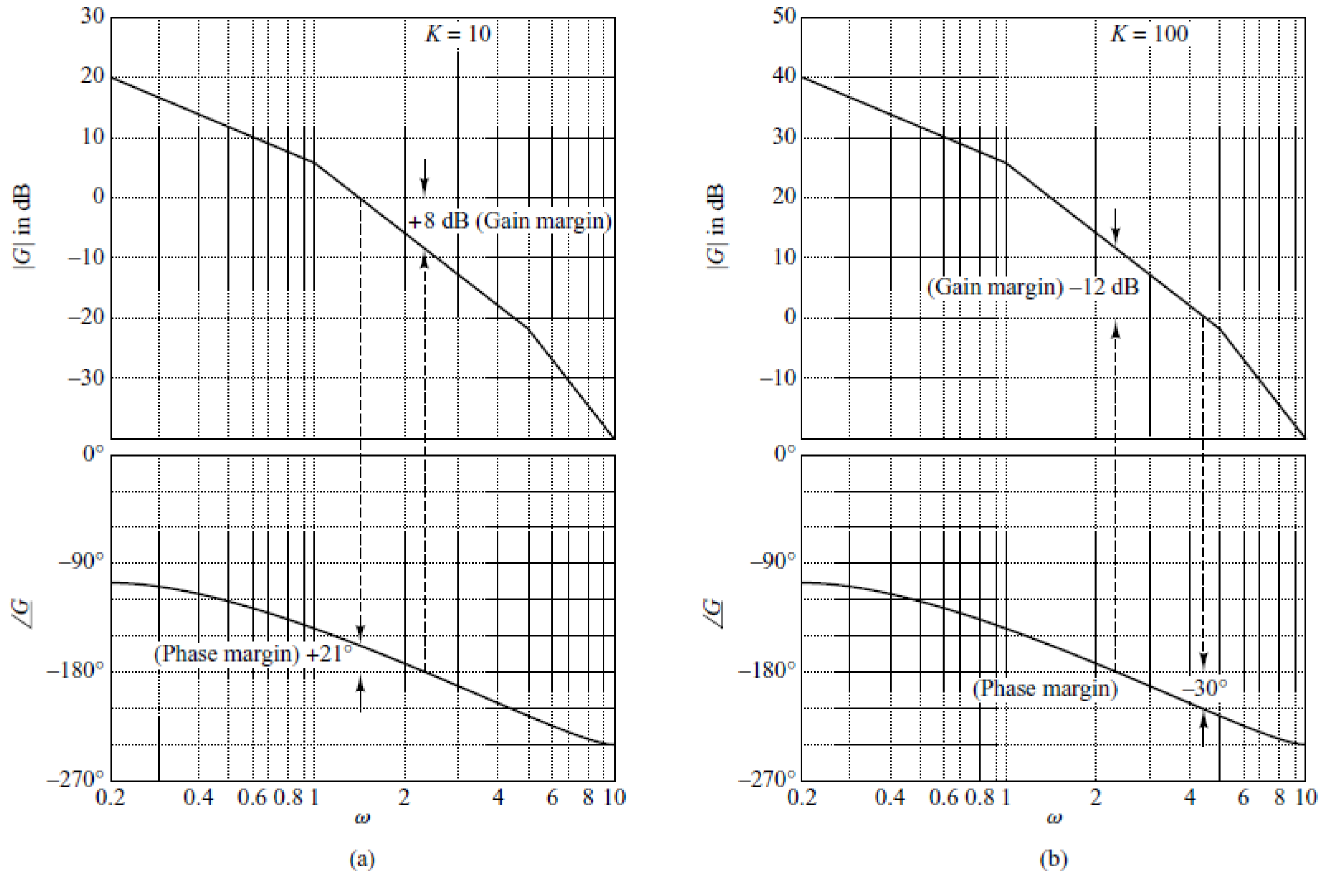
$$|G(j\omega)|(\text{dB}) = 20\log K - 20\log\omega - 20\log\sqrt{1+\omega^2} - 20\log\sqrt{5^2+\omega^2}$$

$$\angle G(j\omega)(^\circ) = -90^\circ - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{5}$$

	K = 10		K = 100	
ω (Rad/s)	Magnitude (dB)	Phase	Magnitude (dB)	Phase
0.2	19.823	–103.6°	39.823	–103.6°
0.5	11.029	–122.28°	31.029	–122.28°
1	2.84	–146.31°	22.84	–146.31°
2	–7.634	–175.24°	12.37	–175.24°
5	–25.119	–213.7°	–5.119	–213.7°
6	–29.1	–220.73°	–9.1	–220.73°
7	–32.584	–226.33°	–12.584	–226.33°
8	–35.685	–230.87°	–15.685	–230.87°
10	–41.01	–237.72°	–21.01	–237.72°

Example 8

Answer:



The phase and gain margins can easily be obtained from the Bode diagram.

The phase and gain margins for K = 10 are P.M. = 21° and G.M. = +8 dB

Therefore, the system gain may be increased by 8 dB before the instability occurs.

The phase and gain margins for K = 100 are P.M. = –30° and G.M. = –12 dB

Thus, the system is stable for K = 10, but unstable for K = 100.

Example 8

Answer:

Beside graphically solved the problem, we can use **analytical method** as below.

$$G(j\omega) = \frac{K}{j\omega(j\omega+1)(j\omega+5)}$$

$$\angle G(j\omega)(°) = -90° - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{5} \qquad |G(j\omega)| = \frac{K}{\omega\sqrt{1+\omega^2}\sqrt{5^2+\omega^2}}$$

Gain Margin

When system phase, $\phi = -180°$, and put it into the phase equation, we can find the phase crossover frequency, ω_{pc},

$$-180° = -90° - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{5}$$

$$90° = \tan^{-1}\omega + \tan^{-1}\frac{\omega}{5}$$

Since $\tan^{-1}X + \tan^{-1}Y = \tan^{-1}\left(\frac{X+Y}{1-XY}\right)$, hence $\infty = \frac{\omega + \frac{\omega}{5}}{1-(\omega)\left(\frac{\omega}{5}\right)}$

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Example 8

Answer:

Gain Margin

The equation is equal to infinity if and only if the denominator is equal to zero, we have

$$1-\frac{\omega^2}{5}=0\Rightarrow\omega^2=5\Rightarrow\omega=2.236\ \text{rad/s}$$

The phase crossover frequency, $\omega_{pc}=2.236$ rad/s. Substitute this into the magnitude equation with K = 10, we have

$$|G(j\omega)|=\frac{10}{2.236\sqrt{1+(2.236)^2}\sqrt{5^2+(2.236)^2}}=0.3333$$

$$|G(j\omega)|=-20\log 0.3333=-9.543\ \text{dB}$$

$\therefore$ G.M. = 0 – (–9.543) = 9.543 dB

Example 8

Answer:

Phase Margin

When system magnitude, $|G(j\omega)|$ = 1 or 0 dB, and put it into the magnitude equation, we can find the gain crossover frequency, ω_{gc}, for K = 10,

$$1 = \frac{10}{\omega\sqrt{1+\omega^2}\sqrt{5^2+\omega^2}} \quad \Rightarrow \quad \sqrt{\omega^2(1+\omega^2)(25+\omega^2)} = 10$$

So, the equation will be, $\omega^6 + 26\omega^4 + 25\omega^2 - 100 = 0$. Substitute $a = \omega^2$ into the equation, we have

$$a^3 + 26a^2 + 25a - 100 = 0$$

By the solving the equation, we have $a = -2.675$ (rejected), $a = -24.83$ (rejected) and $a = 1.506$. Hence, $\omega^2 = 1.506$, $\omega = \omega_{gc} = 1.227$ rad/s

Put $\omega_{gc} = 1.227$ rad/s into $\angle G(j\omega) = -90° - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{5}$,

$$\angle G(j\omega) = -90° - \tan^{-1} 1.227 - \tan^{-1}\frac{1.227}{5} = -154.61°$$

$\therefore$ P.M. = 180° + (–154.61°) = 25.39°

Example 8

Answer:

Repeat the same procedures for K = 100.

Gain Margin

Since the change of gain K will not affect the phase equation and hence the phase crossover frequency, $\omega_{pc} = 2.236$ rad/s. So, the system gain is,

$$|G(j\omega)| = \frac{100}{2.236\sqrt{1+(2.236)^2}\sqrt{5^2+(2.236)^2}} = 3.3335$$

$$|G(j\omega)| = 20\log 3.335 = 10.46 \text{ dB}$$

$\therefore$ G.M. = 0 –10.46 = –10.46 dB

Phase Margin

We need to recalculate the gain crossover frequency, ω_{gc}, for the new gain K.

$$1 = \frac{100}{\omega\sqrt{1+\omega^2}\sqrt{5^2+\omega^2}}$$

Hence, we have $\omega_{gc} = 3.907$ rad/s. Put $\omega_{gc} = 3.907$ rad/s into $\angle G(j\omega)$, we have

$$\angle G(j\omega) = -90° - \tan^{-1} 3.907 - \tan^{-1}\frac{3.907}{5} = -203.65°$$

$\therefore$ P.M. = 180° + (–203.65°) = –23.65°