THE HONG KONG POLYTECHNIC UNIVERSITY
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School of Professional Education and Executive Development
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SEHS4653
Control System Analysis

Unit 4
System Stability and Root Locus Method
(Reference: [1] chapter 5.6, 6.1-2, 6.4-5)

Content

- Poles and Zeros
- Stability and Pole-Zero Plot
- Routh-Hurwitz Stability Criterion
- Root Locus Method
  - Root Locus Plot Example
  - Summary of General Rules for Constructing Root Loci
  - Root-Locus Approach to Control Systems Design

Poles and Zeros

- Consider the below closed-loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

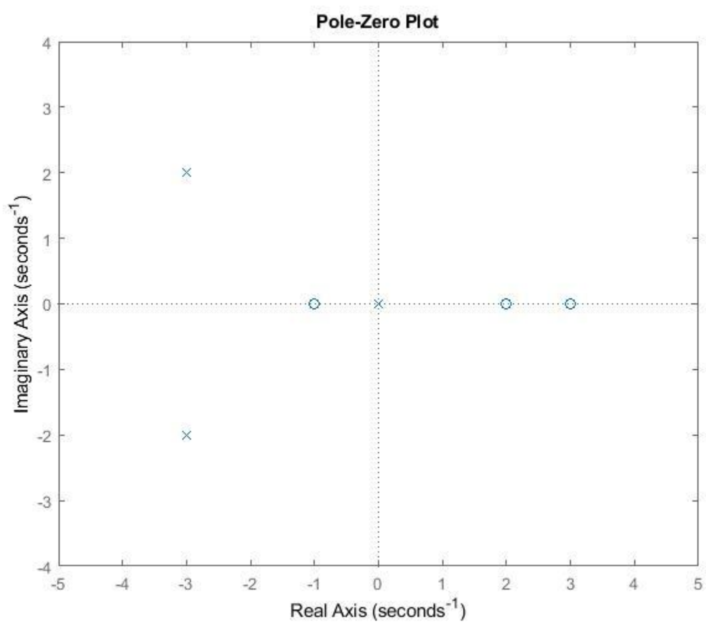
- Rewrite the above equation with roots of the numerator and denominator,

$$\frac{C(s)}{R(s)} = \frac{N(s)}{D(s)} = K\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- Hence,
  - Zeros: roots of numerator, $z_1, z_2, \cdots z_m$
  - Poles: roots of denominator, $p_1, p_2, \cdots p_n$
  - Gain: constant multiplier of the system, $K = \frac{b_m}{a_n}$

Stability and Pole-Zero Plot

- Transfer function is a rational function in the complex variable $s = \sigma + j\omega$
- Poles and zeros may be real or complex and represented graphically by plotting on the complex *s*-plane, known as Pole-Zero Plot
- *x*-axis represents the real part of the poles/zeros; while *y*-axis represents the imaginary part
- Zeros are marked with circle (o) while Poles are marked with cross (x)



Stability and Pole-Zero Plot

Stability

- A system is defined as stable if every bounded input produces bounded output
- Refer to the location of poles on the pole-zero plot

Stable

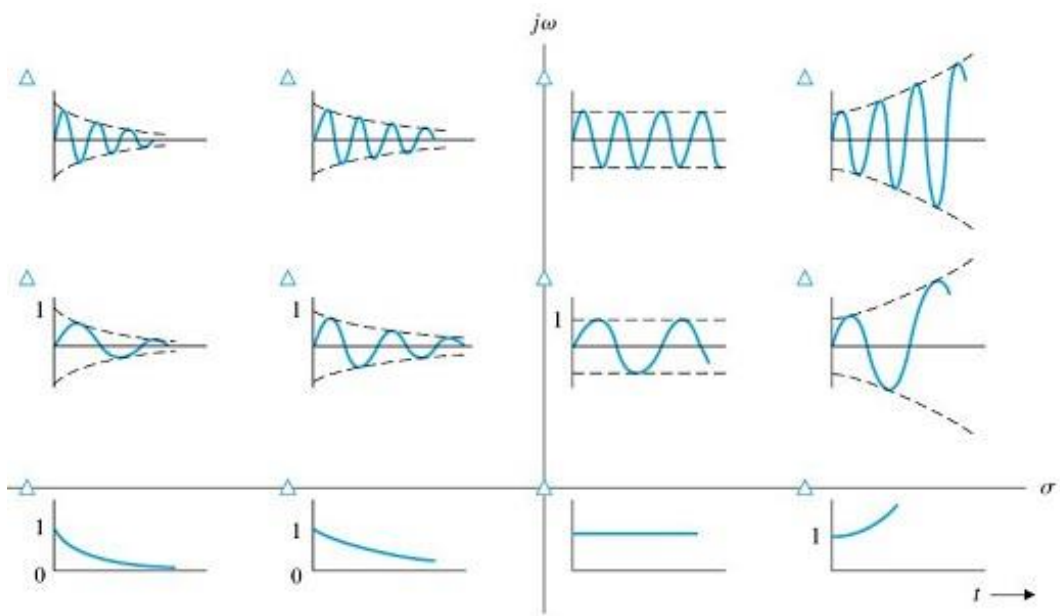
All the poles are in the left-hand side of the pole-zero plot

Marginally Stable

One or more *poles* lie on the vertical axis of the pole-zero plot, i.e. has a zero real value, and no poles lie in the right-hand side

Unstable

At least one *pole* lies in the right-hand side of the pole-zero plot



Impulse response for various root locations in the s-plane.
(The conjugate root is not shown.)

Example 1

Given the pole(s) and zero(s) of a system.

System 1: Poles = –2, – 3

System 2: Poles = – 1.5, +1; Zeros = 0

System 3: Poles = $-1 \pm j2$; Zeros = +2

(a) Write the transfer functions of the system.

(b) Sketch the pole-zero plot of the system.

(c) Determine the stability of the system.

<u>Answer:</u>

(a)

System 1: $G_1(s) = \dfrac{1}{(s+2)(s+3)}$

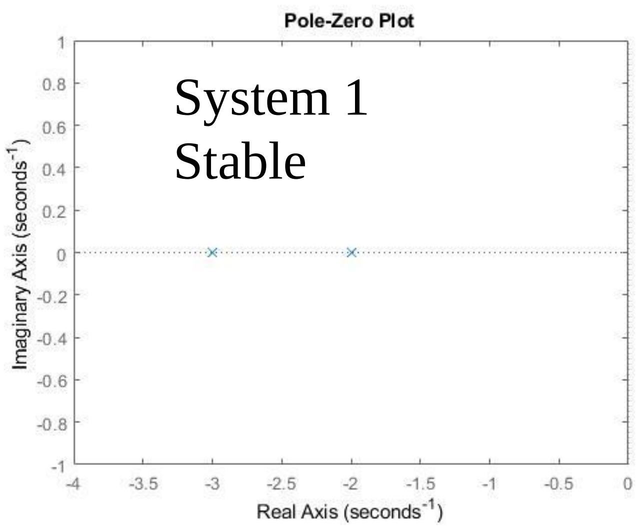
System 2: $G_2(s) = \dfrac{s}{(s+1.5)(s-1)}$

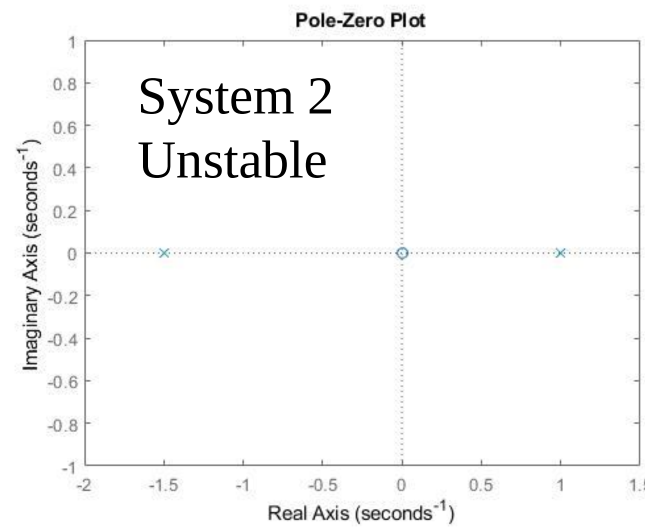
System 3: $G_3(s) = \dfrac{(s-2)}{(s+1+j2)(s+1-j2)} = \dfrac{s-2}{s^2+2s+5}$

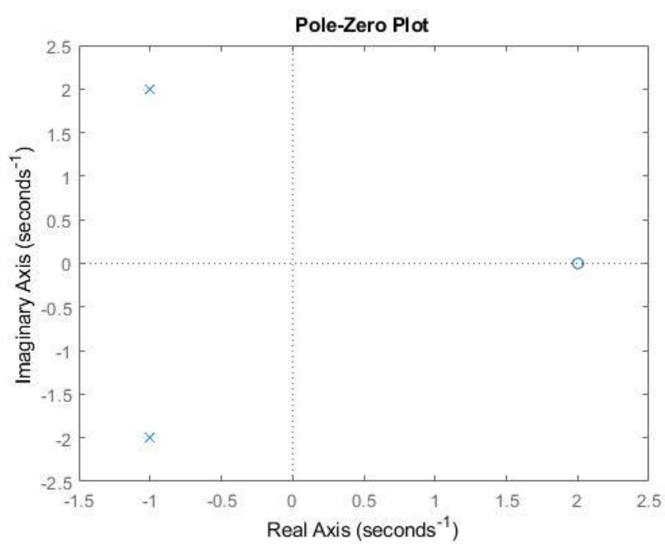
Example 1 (continued)

Answer:

(b) & (c)



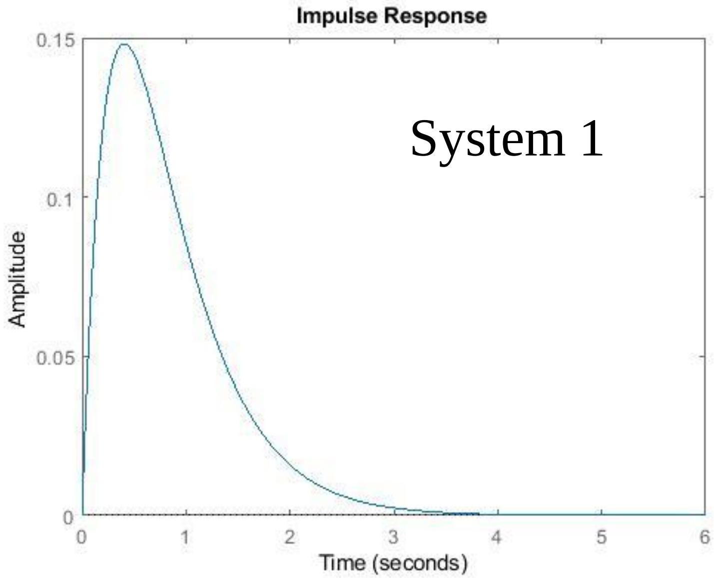


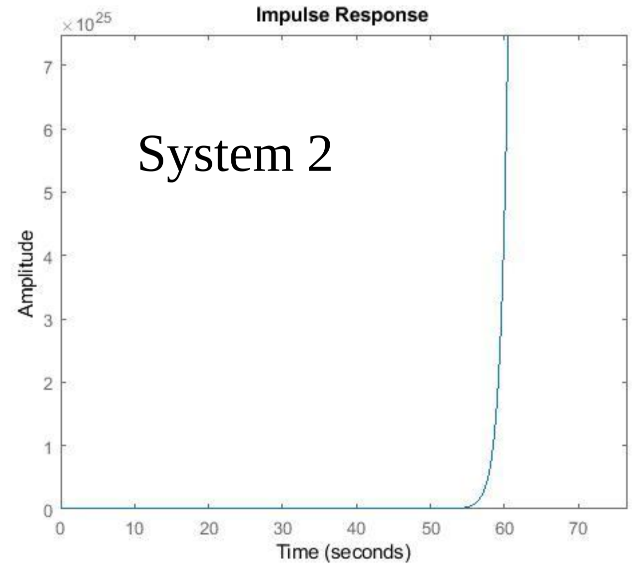


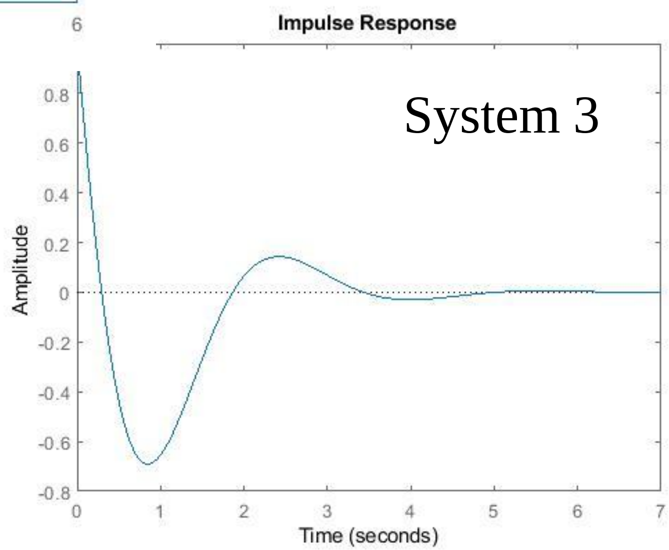
System 3
Stable

Example 1 (continued)

Impulse Response (Time domain)





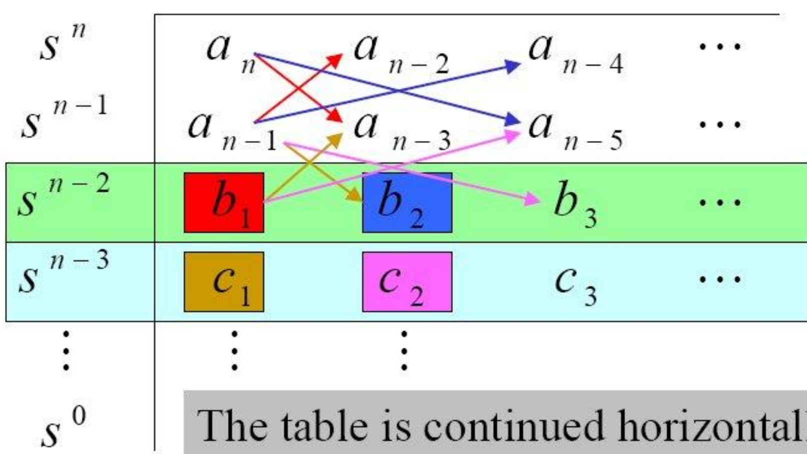


Routh-Hurwitz Stability Criterion

- Method for determining the stability of a system without factoring the denominator
- It tells us whether or not there are unstable roots in a polynomial equation without actually solving
- Information about absolute stability can be obtained directly from the coefficients of the characteristic equation

Routh Array

- Consider an *n*th order characteristic equation,
  $D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$



The table is continued horizontally and vertically until only zero's are obtained.

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$$

$$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$$

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Routh-Hurwitz Stability Criterion

- The Routh-Hurwitz Criterion states that the number of roots of the denominator with positive real parts is equal to the number of changes in the sign in the first column of the Routh array
- Hence, the system is stable if and only if there are no sign changes in the first column of the array
- The number of sign changes in the first column equals the number of poles in the right half *s*-plane

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

The number of roots in the open right half-plane is equal to the number of sign changes in the first column of Routh array.

Example 2

Use the Routh array to determine the stability of a system given the following characteristic equation

$$D(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$$

<u>Answer:</u>

The first 2 rows of the Routh array

s^4	1	3	5
s^3	2	4	

The 3rd row of the array can be calculated as,

$$b_1 = \frac{(2)(3) - (1)(4)}{2} = 1 \qquad b_2 = \frac{(2)(5) - (1)(0)}{2} = 5$$

s^4	1	3	5
s^3	2	4	
s^2	1	5	

The 4th row of the array can be calculated as,

$$c_1 = \frac{(1)(4) - (2)(5)}{1} = -6$$

s^4	1	3	5
s^3	2	4	
s^2	1	5	
s^1	-6		

Example 2 (continued)

Answer:

The last row of the array can be calculated as,

$$d_1 = \frac{(-6)(5) - (1)(0)}{-6} = 5$$

$$\begin{array}{c|ccc} s^4 & 1 & 3 & 5 \\ s^3 & 2 & 4 & \\ s^2 & 1 & 5 & \\ s^1 & -6 & & \\ s^0 & 5 & & \end{array}$$

The number of changes in sign of the coefficients in the first column is 2

This means that there are two roots with positive real parts.

Hence, the system is unstable

Example 3

Use the Routh array to determine the stability of a system given the following characteristic equation

$$D(s) = s^3 + s^2 + 2s + 24$$

Answer:

The first 2 rows of the Routh array

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 24 \end{array}$$

The 3rd row of the array can be calculated as,

$$b_1 = \frac{(1)(2) - (1)(24)}{1} = -22$$

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 24 \\ s^1 & -22 & \end{array}$$

The 4th row of the array can be calculated as,

$$c_1 = \frac{(-22)(24) - (1)(0)}{-22} = 24$$

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 24 \\ s^1 & -22 & \\ s^0 & 24 & \end{array}$$

2 sign changes in the first column indicate 2 roots in the right half of s-plane. Hence, this system is **unstable**.

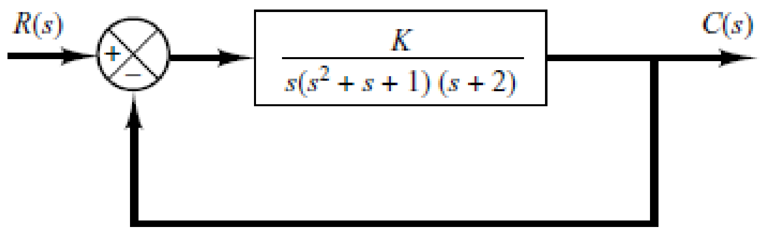
Example 4

Determine the range of K for a stable system with the characteristic equation as $D(s) = s^3 + 3s^2 + 3s + 1 + K$.

Answer:

Example 5

Determine the range of *K* which will result a stable system as shown below.



<u>Answer:</u>

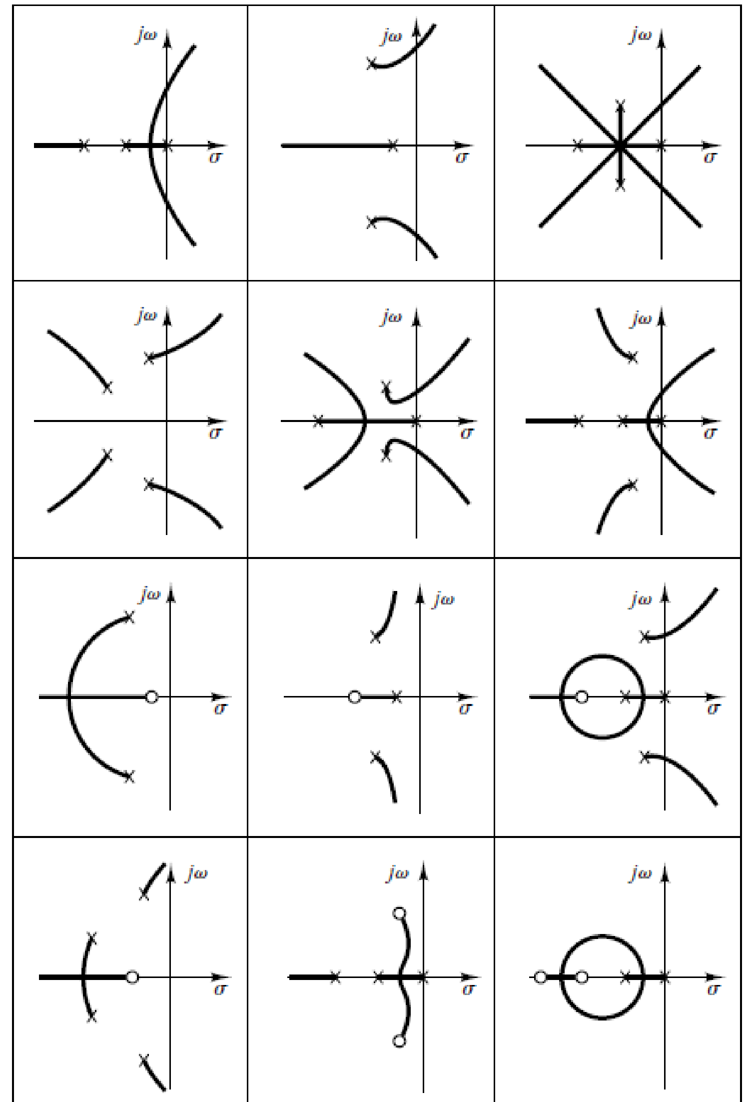
$$\Delta(s) = s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

Root Locus Method

- The basic characteristic of the transient response of a closed-loop system is closely related to the location of the closed-loop poles
- If the system has a variable loop gain (K), then the location of the closed-loop poles depends on the value of the loop gain chosen
- From the design viewpoint, in some systems, the selection of an appropriate gain values may move the closed-loop poles to desired locations. Otherwise, addition of a compensator to the system will become necessary
- Just finding the roots of the characteristic equation may be of limited value, because as the gain of the open-loop transfer function varies, the characteristic equation changes and the computations must be repeated
- A simple method for plotting the roots of the characteristic equation for all values of a system parameter was used extensively in control engineering, called the *root-locus method*
- By using the root-locus method, the designer can predict the effects on the location of the closed-loop poles of varying the gain value or adding open-loop poles and/or open-loop zeros

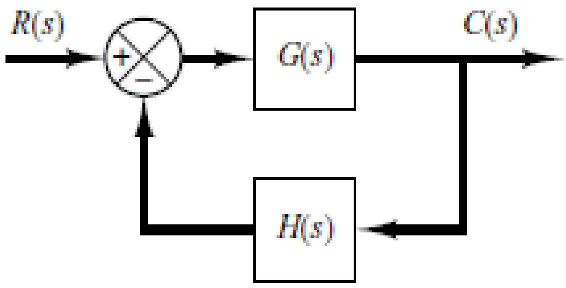
Root Locus Method

Open-loop pole-zero configurations and the corresponding root loci



Root Locus Method

Angle and Magnitude Conditions



- Consider the negative feedback system below, the closed

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

- The characteristic equation is obtained by setting the denominator of the above function equal to zero:

$$\Delta(s) = 1 + G(s)H(s) = 0 \quad \text{or } G(s)H(s) = -1$$

- The values of *s* that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles,

Angle Condition: $\angle G(s)H(s) = \pm 180°(2k + 1), \quad k = 0, 1, 2, \ldots$

Magnitude Condition: $|G(s)H(s)| = 1$

Root Locus Method

Angle and Magnitude Conditions

- A locus of the points in the complex plane satisfying the angle condition alone is the root locus
- The roots of the characteristic equation (the closed-loop poles) corresponding to a given value of the gain can be determined from the magnitude condition
- In many cases, $G(s)H(s)$ involves a **gain parameter** K, and the characteristic equation may be written as,

$$1 + \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0$$

- The root loci for the system are the loci of the closed-loop poles as the gain K is varied from zero to infinity
- The root loci are always symmetrical about the real axis
- Remember that the angles of the complex quantities originating from the open-loop poles and open-loop zeros to the test point s are measured in the counterclockwise direction

Root Locus Method

Angle and Magnitude Conditions

- For example, if $G(s)H(s)$ is given by,

$$G(s)H(s) = \frac{K(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$

(how about the root nature of the above equation?)

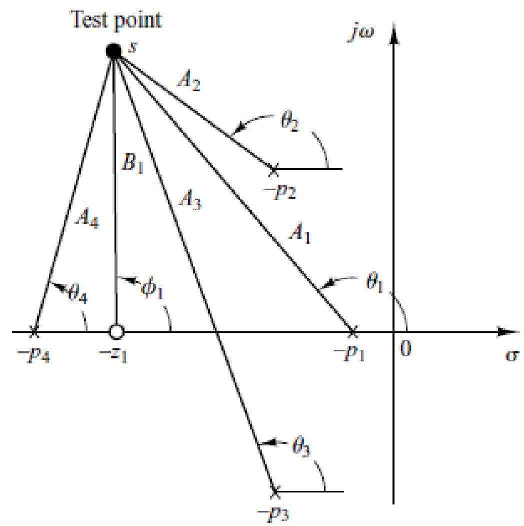
- The angle of $G(s)H(s)$ is then,

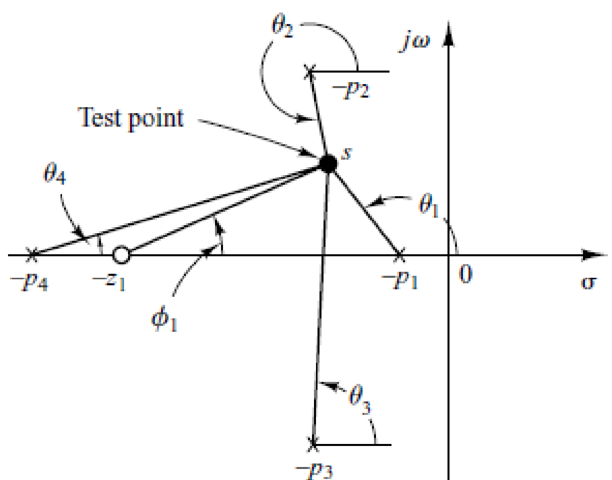
$$\angle G(s)H(s) = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$$

- The magnitude of $G(s)H(s)$ for this system is

$$|G(s)H(s)| = \frac{KB_1}{A_1 A_2 A_3 A_4}$$

where A_1 , A_2 , A_3 , A_4 , and B_1 are the magnitudes of the complex quantities $s+p_1$, $s+p_2$, $s+p_3$, $s+p_4$, and $s+z_1$, respectively

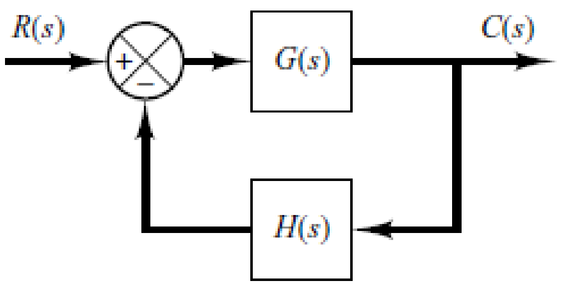




General Rules for Constructing Root Loci

- First, obtain the characteristic equation

$$1 + G(s)H(s) = 0$$



- Then rearrange this equation in the form of

$$1 + \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0$$

For $K > 0$

1. Locate the poles and zeros of $G(s)H(s)$ on the s plane

   The root-locus branches start from open-loop poles and terminate at zeros (finite zeros or zeros at infinity), as K increases from zero to infinity

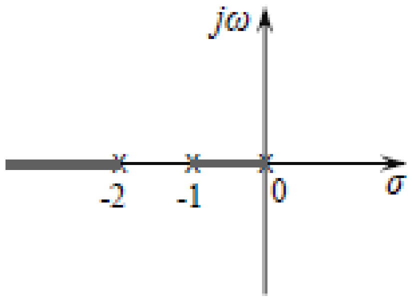
   Number of branches are equal to the number of roots of the characteristic equation

General Rules for Constructing Root Loci

2. Determine the root loci on the real axis

If the total number of real poles and real zeros to the right of this test point is odd, then this point lies on a root locus

Example:



- Select a test point, s, in each interval
- If select a test point on the negative real axis between 0 and –1,

$$\angle s = 180°, \qquad \angle(s+1) = \angle(s+2) = 0°$$

$$\therefore -\angle s - \angle(s+1) - \angle(s+2) = \pm 180°(2k+1), k = 0, 1, 2, 3 \ldots$$

- satisfies angle condition and this range forms a portion of the root locus

General Rules for Constructing Root Loci

3. Determine the asymptotes of root loci

The root loci for very large values of s must be asymptotic to straight lines whose angles (slopes) are given by

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{n-m} \quad (k = 0, 1, 2, \dots)$$

where n = no. of finite poles of $G(s)H(s)$ and m = no. of finite zeros of $G(s)H(s)$

All the asymptotes intersect at a point on the real axis. The point at which they do so is obtained by

$$s = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}$$

General Rules for Constructing Root Loci

4. Find the breakaway and break-in points

If a root locus lies between two adjacent open-loop poles on the real axis, then there exists at least one breakaway point between the two poles

If the root locus lies between two adjacent zeros on the real axis, then there always exists at least one break-in point between the two zeros

Suppose that the characteristic equation is given by $B(s) + KA(s) = 0$

The breakaway and break-in points can be determined from the roots of

$$\frac{dK}{ds} = -\frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)}$$

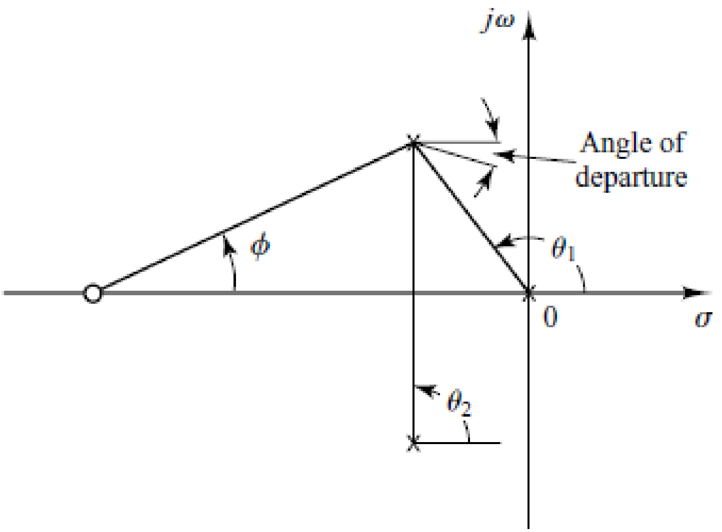
If a real root is NOT on the root locus portion of the real axis, this root is NOT the actual breakaway or break-in point

General Rules for Constructing Root Loci

5. Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero)

Angle of departure from a complex pole = 180°
– (sum of the angles of vectors to a complex pole in question from other poles)
+ (sum of the angles of vectors to a complex pole in question from zeros)

Angle of arrival at a complex zero = 180°
– (sum of the angles of vectors to a complex zero in question from other zeros)
+ (sum of the angles of vectors to a complex zero in question from poles)



Angle of departure = ?

General Rules for Constructing Root Loci

6. Find the points where the root loci may cross the imaginary axis
   (a) Use of Routh's stability criterion; OR
   (b) Put $s = j\omega$ in the characteristic equation, equating both the real part and the imaginary part to zero, and solving for ω and K

7. Determine the closed-loop poles or Draw the root locus

A particular point on each root-locus branch will be a closed-loop pole if the value of K at that point satisfies the magnitude condition, $|G(s)H(s)| = 1$

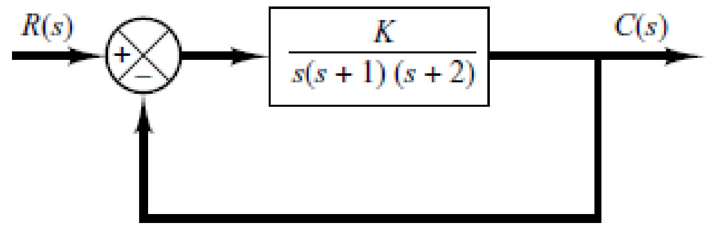
The value of K corresponding to any point s on a root locus can be obtained using the magnitude condition, or

$$K = \frac{\text{product of lengths between point } s \text{ to poles}}{\text{product of lengths between point } s \text{ to zeros}}$$

Example 6

Consider the negative feedback system shown below. For this system,

$$G(s) = \frac{K}{s(s+1)(s+2)} \quad ,H(s) = 1$$

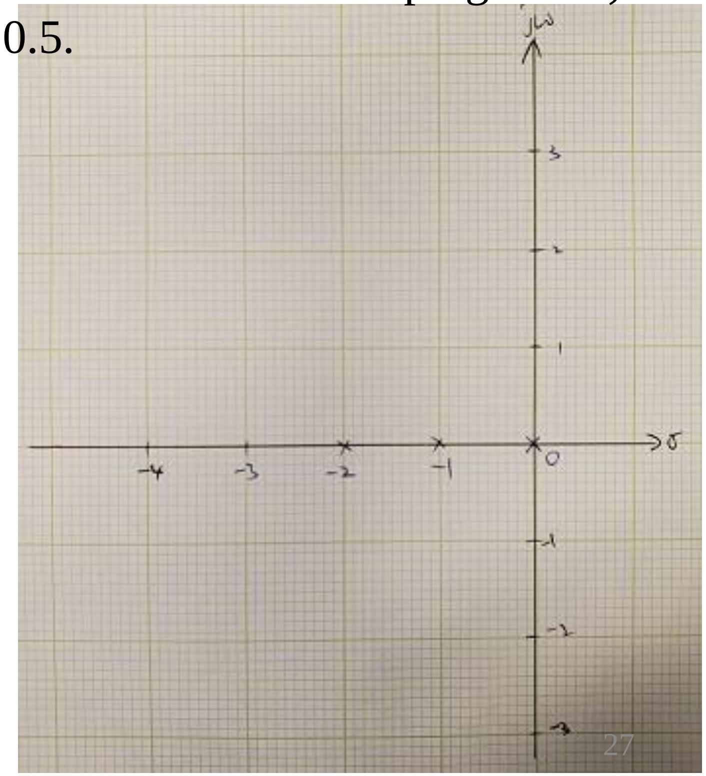


Sketch the root-locus plot and then determine the value of K such that the damping ratio ζ of a pair of dominant complex-conjugate closed-loop poles is 0.5.

<u>Answer:</u>

1. Locate the poles and zeros of $G(s)H(s)$ on the s plane

Pole: $s = 0, s = -1, s = -2$

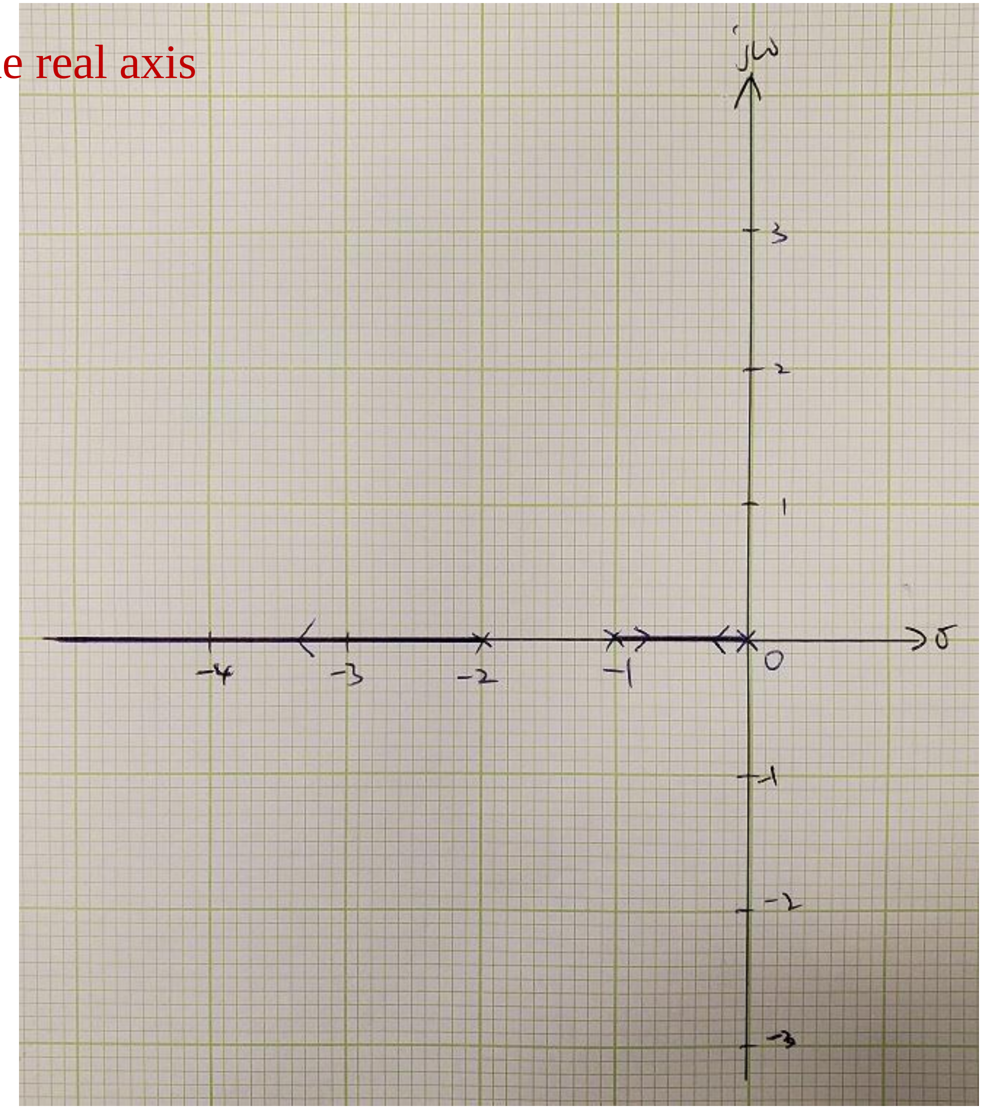


Example 6 (continued)

2. Determine the root loci on the real axis

$(-1, 0)$

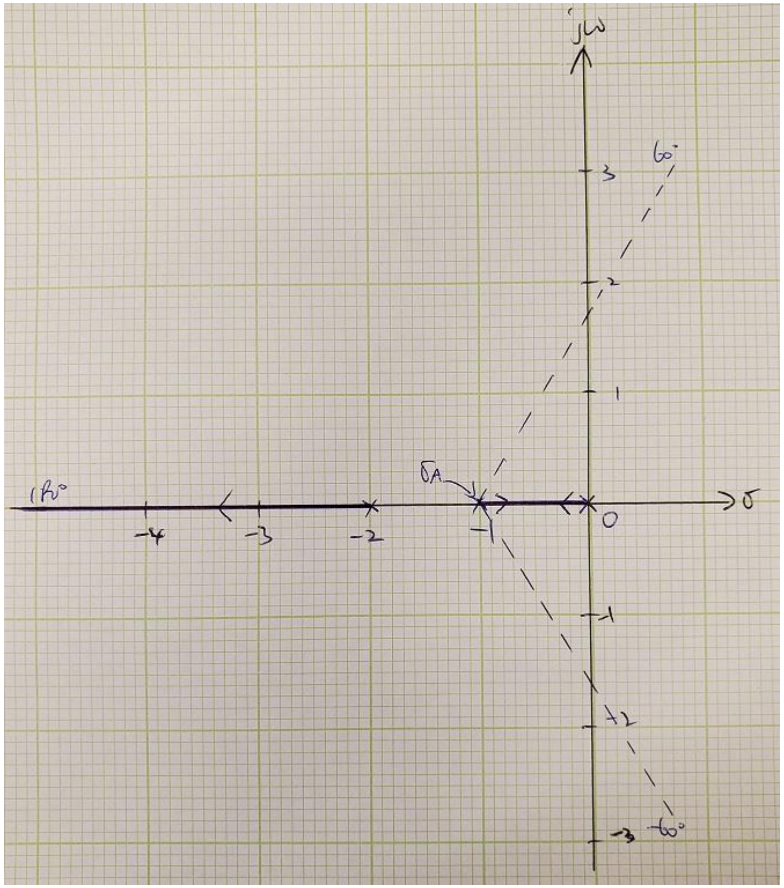
$(-\infty, -2)$



Example 6 (continued)

3. Determine the asymptotes of root loci

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{n-m} \quad (k = 0, 1, 2, \dots)$$



$$= \frac{\pm 180°(2k+1)}{3-0} = \pm 60°(2\text{k}+1)$$

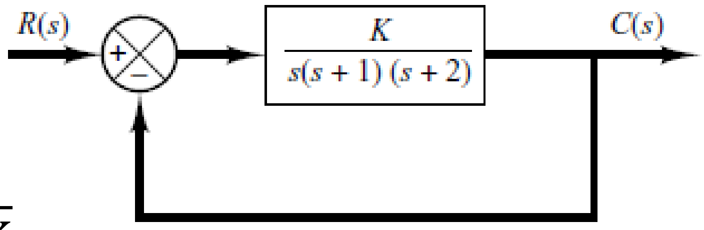
$$= \pm 60°, \pm 180°$$

$$s = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}$$

$$s = \frac{(0)+(-1)+(-2)}{3-0} = -1$$

Example 6 (continued)

4. Find the breakaway and break-in points



$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} = \frac{K}{s(s+1)(s+2)+K}$$

$$\because \frac{C(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

$$\Delta(s) = s(s+1)(s+2) + K = 0$$

$$\therefore \Delta(s) = 1 + G(s)H(s)$$

$$= s^3 + 3s^2 + 2s + K = 0$$

$$\therefore K = -s^3 - 3s^2 - 2s$$

Rejected and why?

$$\frac{dK}{ds} = -3s^2 - 6s - 2 = 0$$

$s = -0.423, -1.58$

Only breakaway point

$$K = -(-0.423)^3 - 3(-0.423)^2 - 2(-0.423) = 0.385$$

Example 6 (continued)

5. Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero)

Since there are neither complex pole(s) nor complex zero(s), this step can be omitted

Example 6 (continued)

6. Find the points where the root loci may cross the imaginary axis

Routh Array Method

The characteristic equation, $\Delta(s) = s^3 + 3s^2 + 2s + K = 0$

Using Routh array,

s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	
s^0	K	

For stability: $0 < K < 6$

The crossing points on the imaginary axis can then be found by solving the auxiliary equation obtained from the s^2 row; that is,

$$3s^2 + K = 3s^2 + 6 = 0 \implies s = \pm j\sqrt{2}$$

The frequencies at the crossing points on the imaginary axis are thus $\omega = \pm\sqrt{2}$.
The gain value corresponding to the crossing points is $K = 6$.

Example 6 (continued)

6. Find the points where the root loci may cross the imaginary axis

Equating term Method

Put $s = j\omega$ into $\Delta(s)$,

$$\Delta(j\omega) = (j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + K = 0$$

$$= -j\omega^3 - 3\omega^2 + 2j\omega + K = 0$$

Equating terms, we have

$$-j\omega^3 + 2j\omega = 0$$

$$-j\omega(\omega^2 - 2) = 0$$

$$\therefore \omega = 0, \pm\sqrt{2}$$

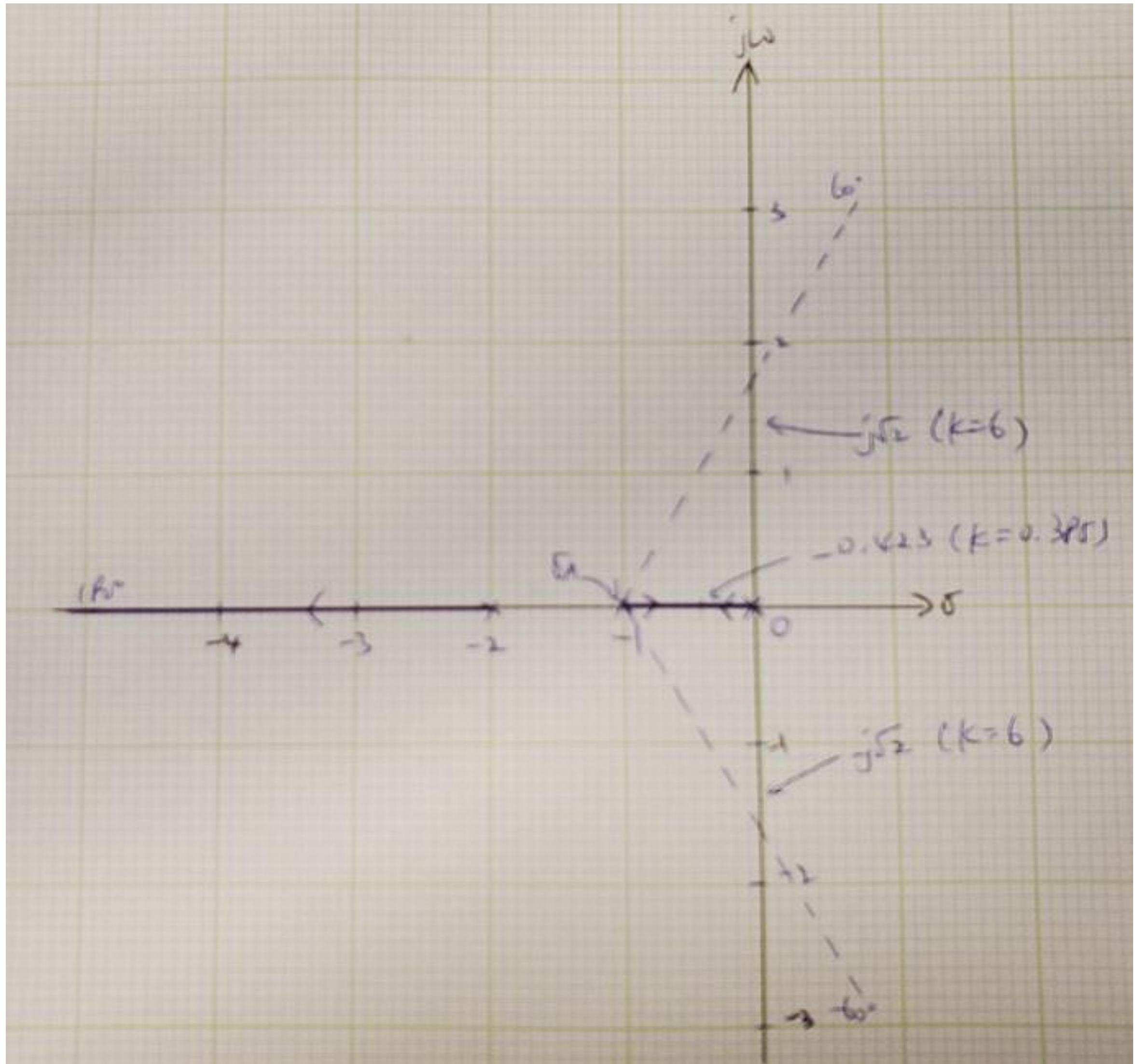
$$-3\omega^2 + K = 0$$

$$3\left(\sqrt{2}\right)^2 = K$$

$$\therefore K = 6$$

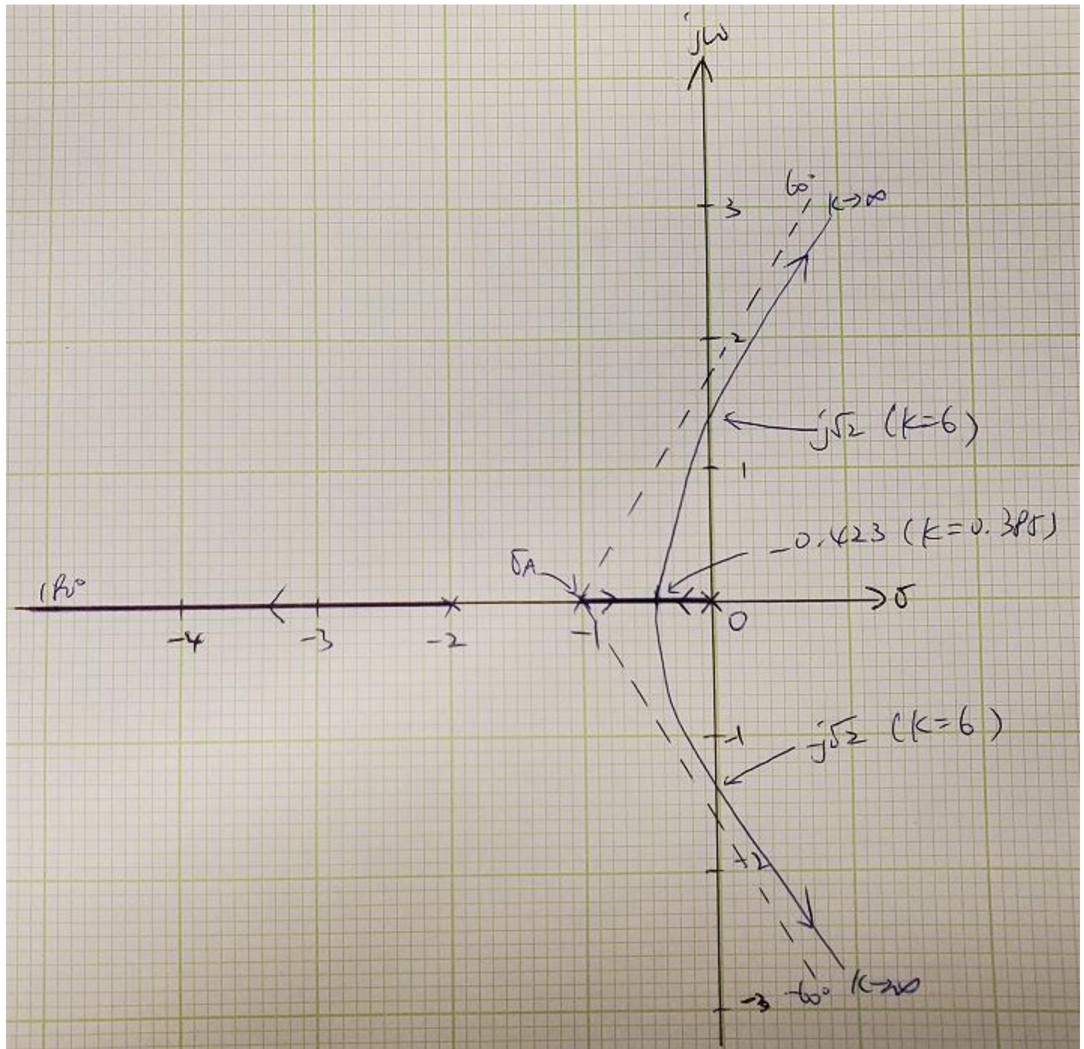
Example 6 (continued)

6. Find the points where the root loci may cross the imaginary axis



Example 6 (continued)

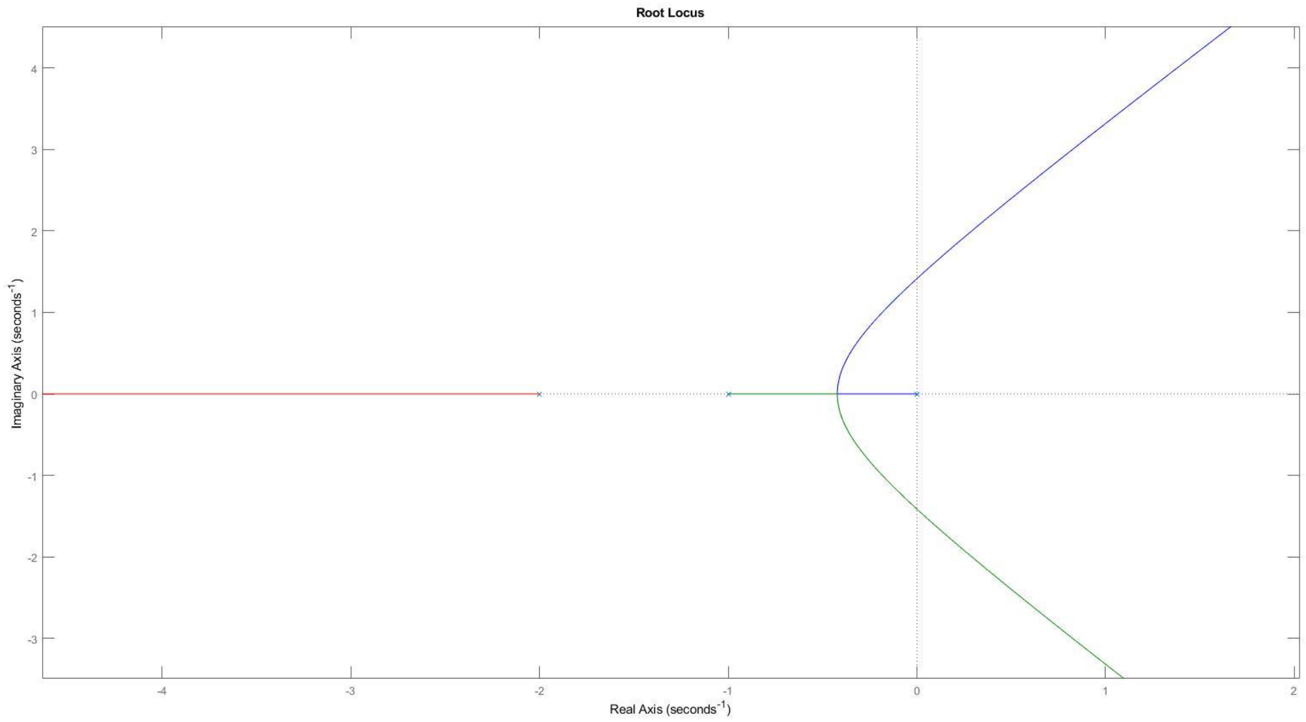
7. Determine the closed-loop poles or Draw the root locus



Example 6 (continued)

7. Determine the closed-loop poles or Draw the root locus

From Matlab

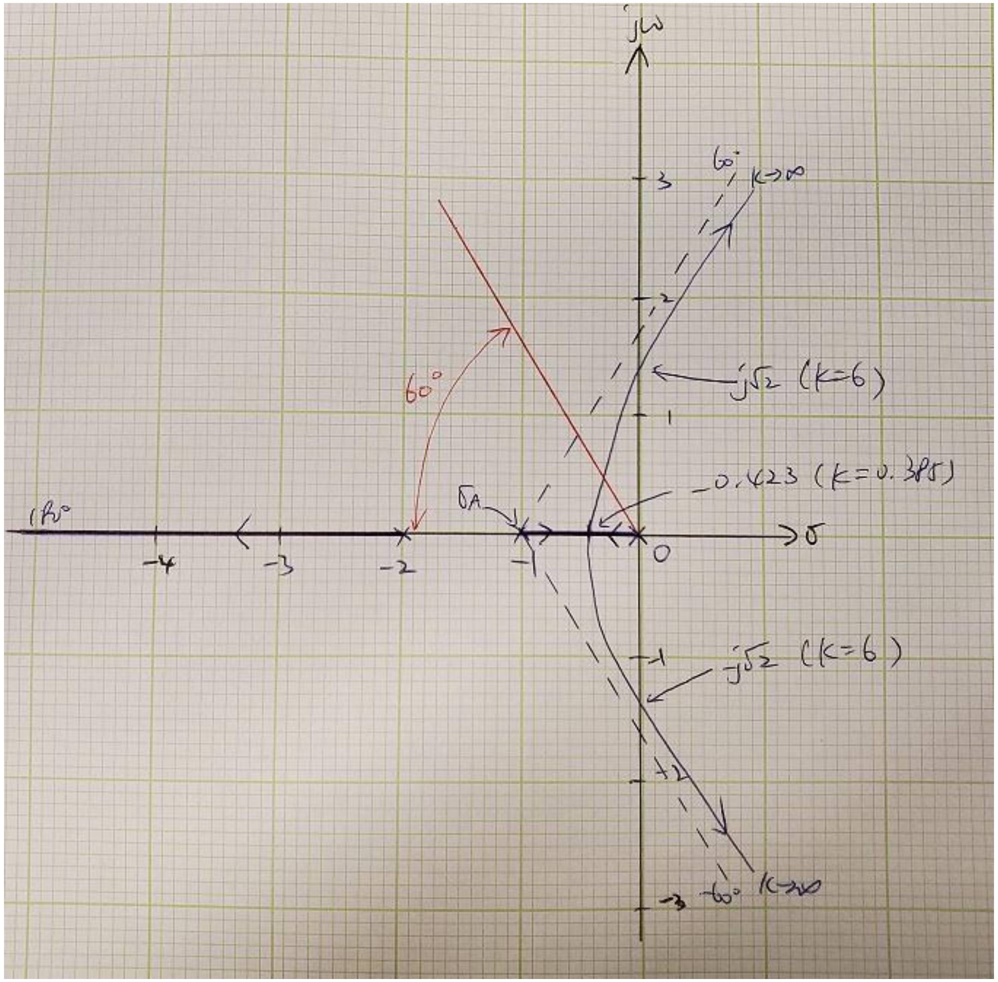


Example 6 (continued)

Determine the value of K if damping ratio is 0.5

$$\phi = \cos^{-1}\zeta = \cos^{-1}0.5 = 60°$$

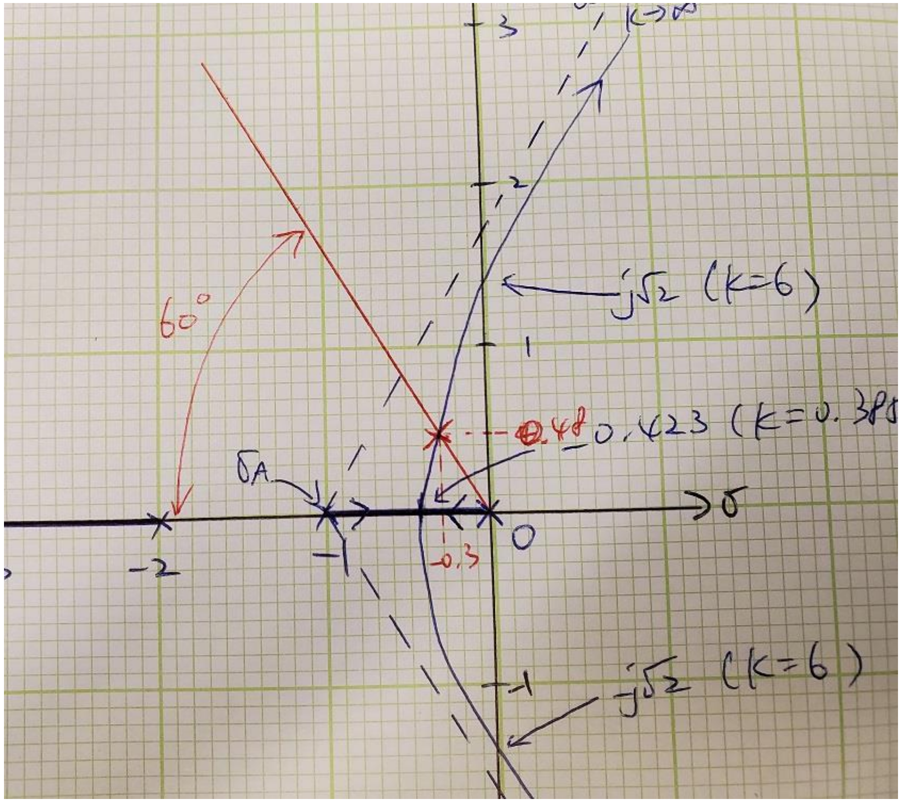
Draw a line from origin with 60° for cutting the root locus



Example 6 (continued)

Then, locate the closed-loop pole that intersects to the root locus $s = -0.3 + j0.48$

Substitute to $\Delta(s)$ for finding K, we have $s^3 + 3s^2 + 2s + K = 0$ $K = 0.85$



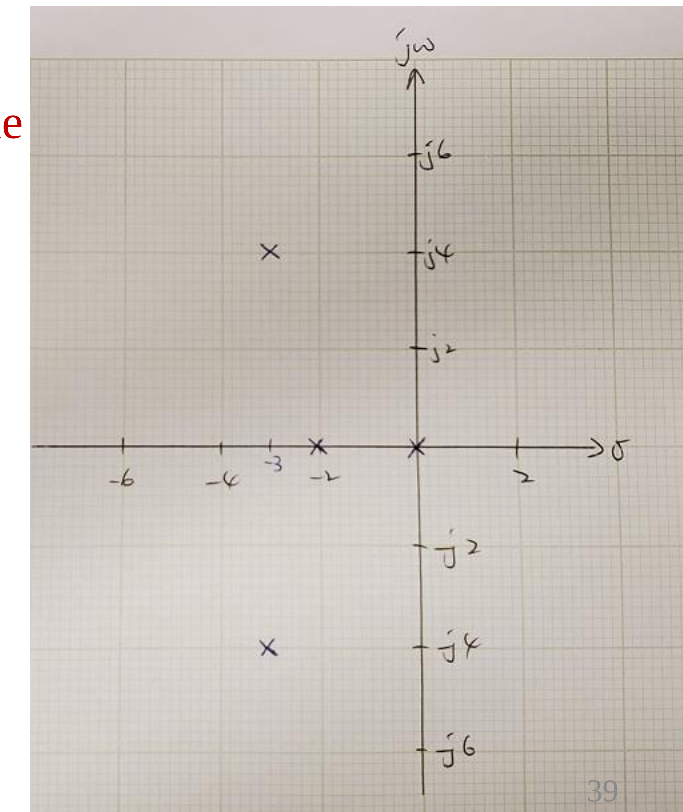
Example 7

Plot the root locus for the open-loop transfer function of a control system given by,

$$G(s) = \frac{K}{s(s+2)(s^2+6s+25)}$$

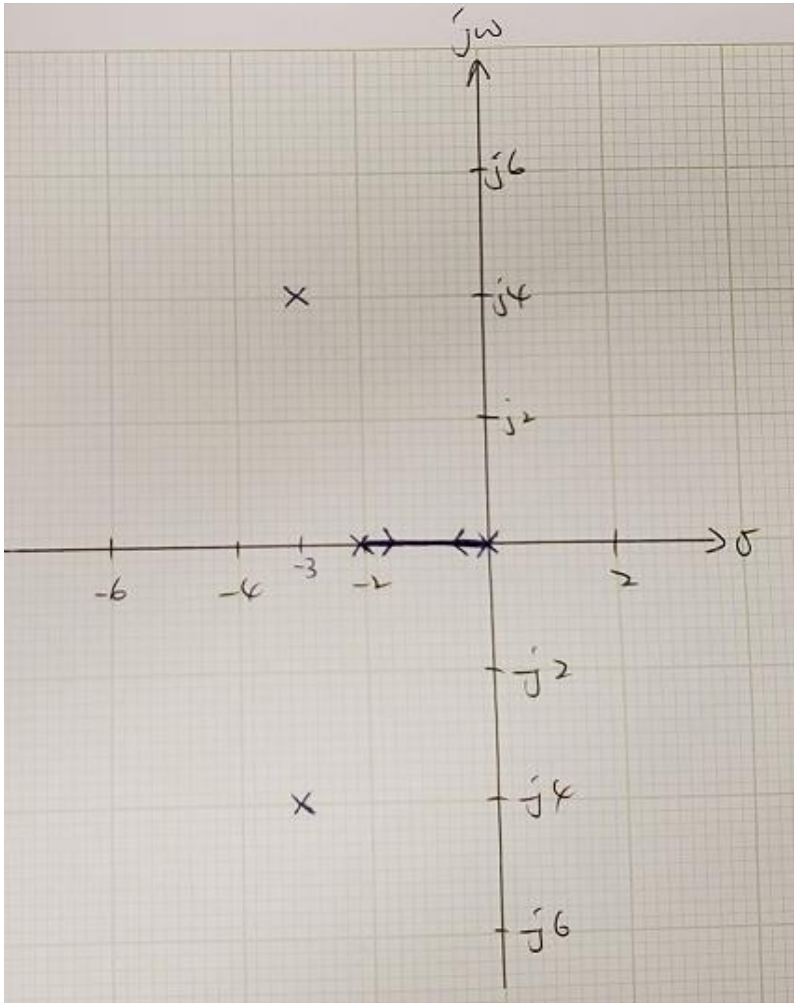
<u>Answer:</u>

1. Locate the poles and zeros of $G(s)H(s)$ on the s plane



Example 7 (continued)

2. Determine the root loci on the real axis

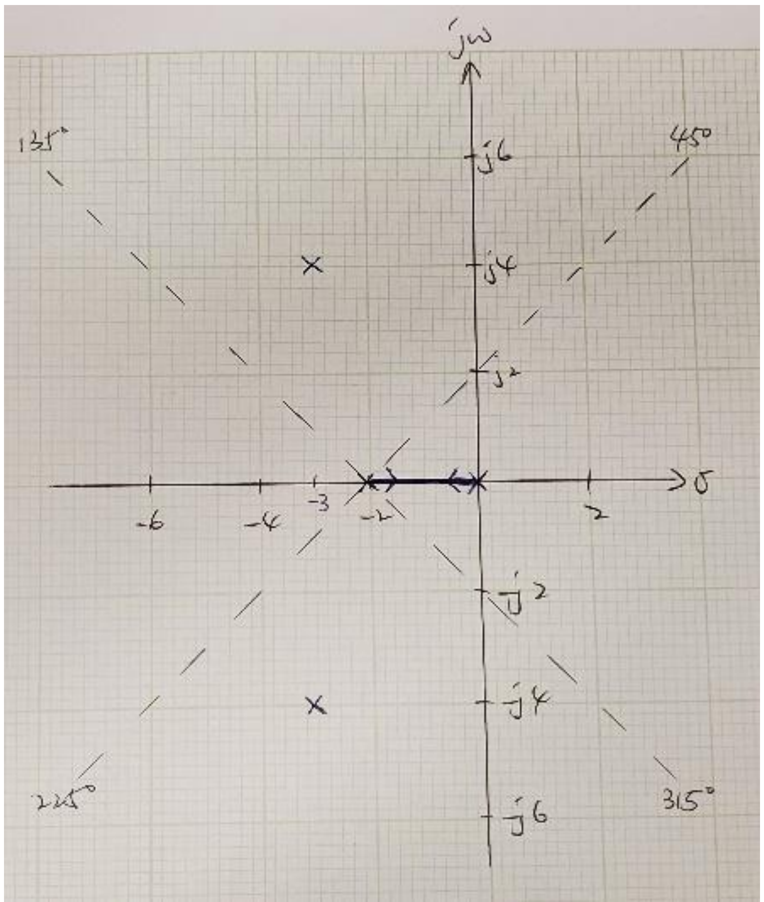


$(-2, 0)$

Example 7 (continued)

3. Determine the asymptotes of root loci

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{n-m} \quad (k = 0, 1, 2, \dots)$$



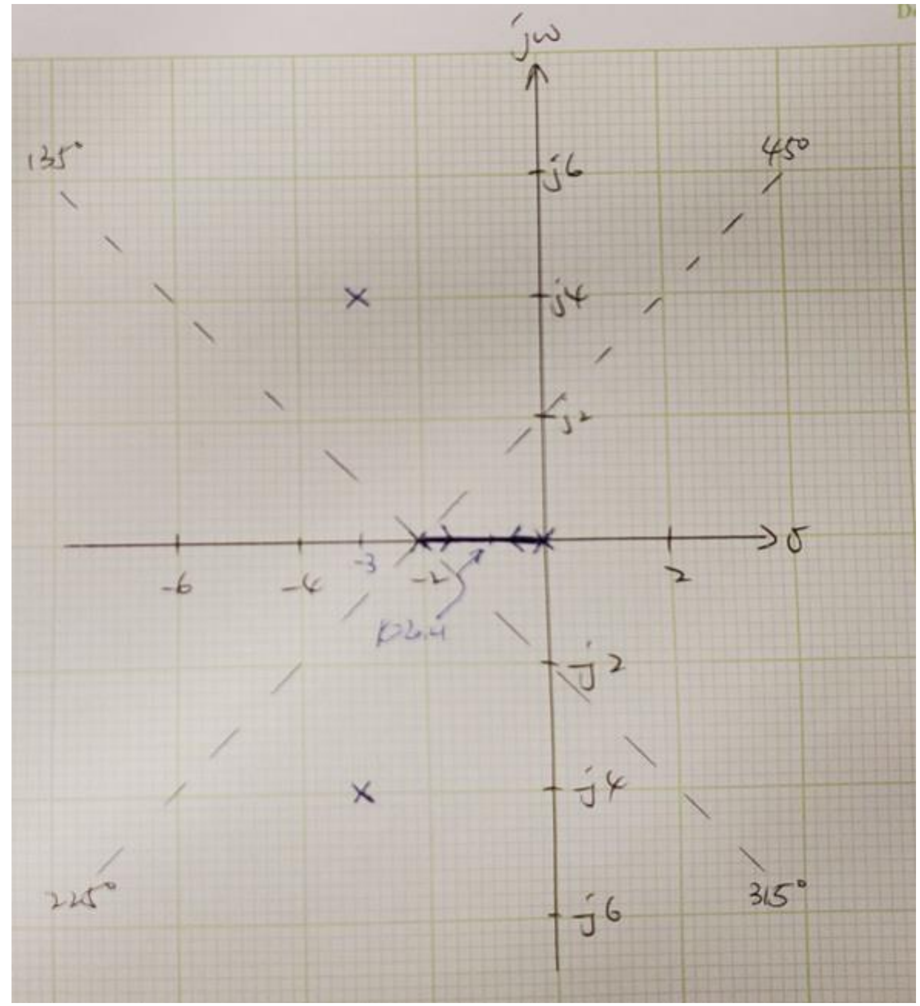
$$s = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}$$

Example 7 (continued)

4. Find the breakaway and break-in points

$$\Delta(s) = s(s+2)(s^2+6s+25)+K = 0$$

$$= s^4 + 8s^3 + 37s^2 + 50s + K = 0$$



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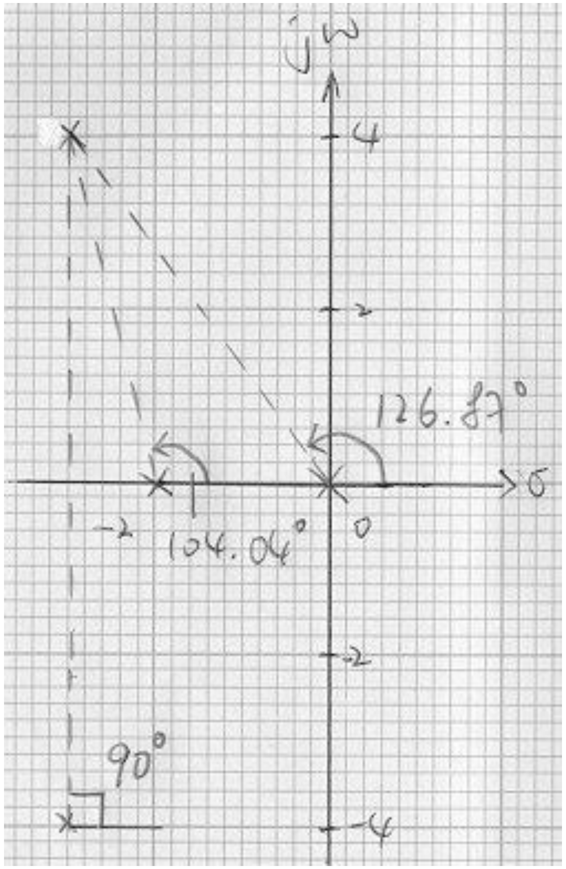
Example 7 (continued)

5. Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero)

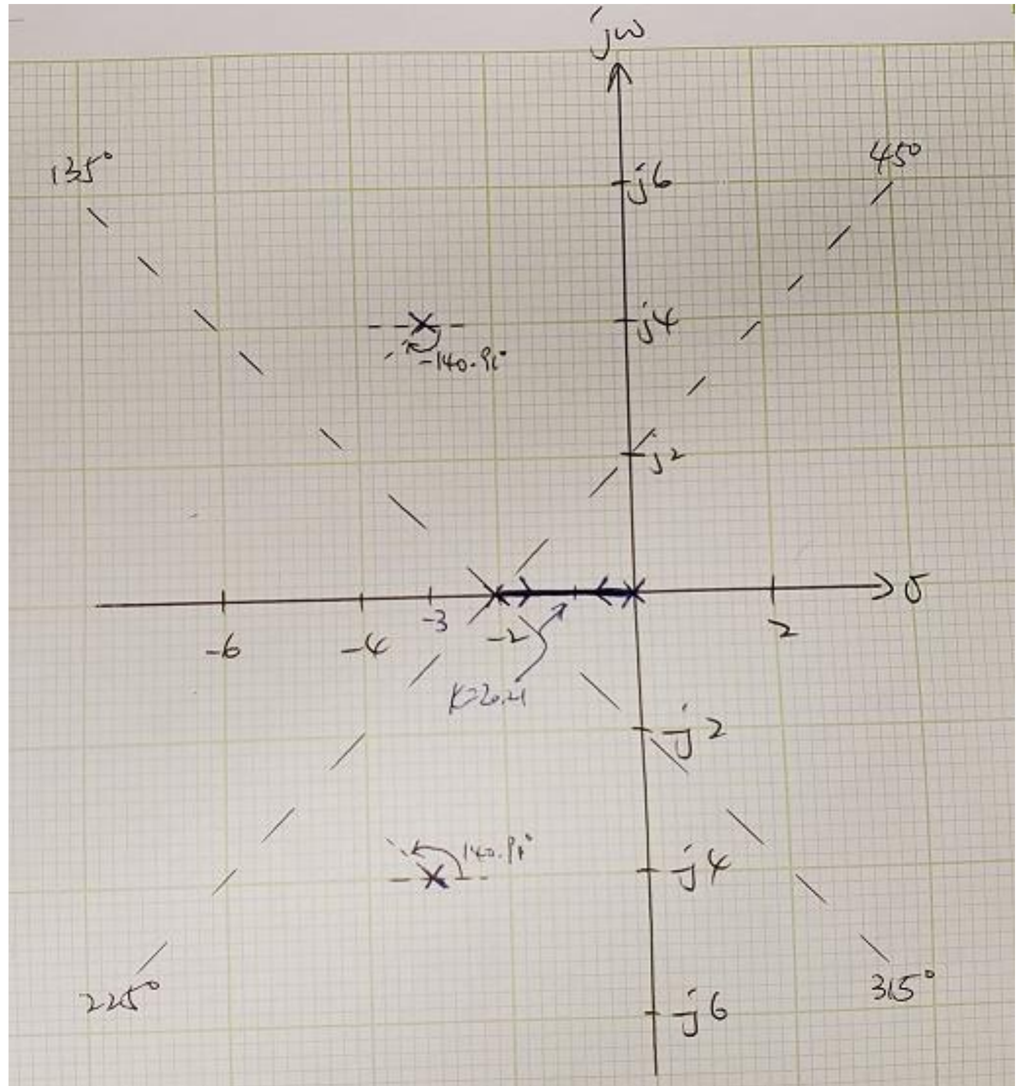
Angle of departure from a complex pole = 180°
– (sum of the angles of vectors to a complex pole in question from other poles)
+ (sum of the angles of vectors to a complex pole in question from zeros)

Angle of arrival at a complex zero = 180°
– (sum of the angles of vectors to a complex zero in question from other zeros)
+ (sum of the angles of vectors to a complex zero in question from poles)



Example 7 (continued)

5. Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero)



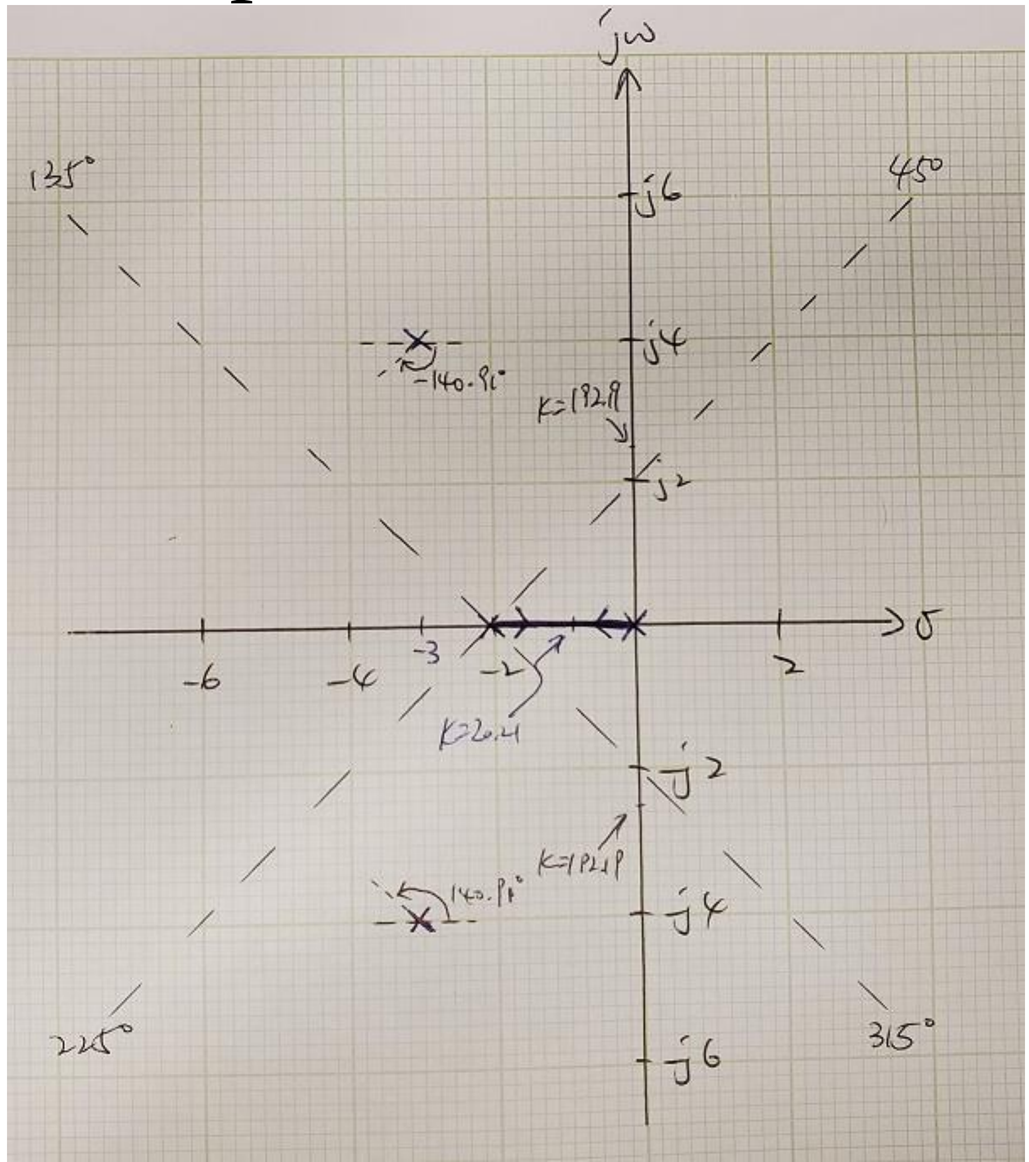
Example 7 (continued)

6. Find the points where the root loci may cross the imaginary axis

$$\Delta(s) = s^4 + 8s^3 + 37s^2 + 50s + K$$

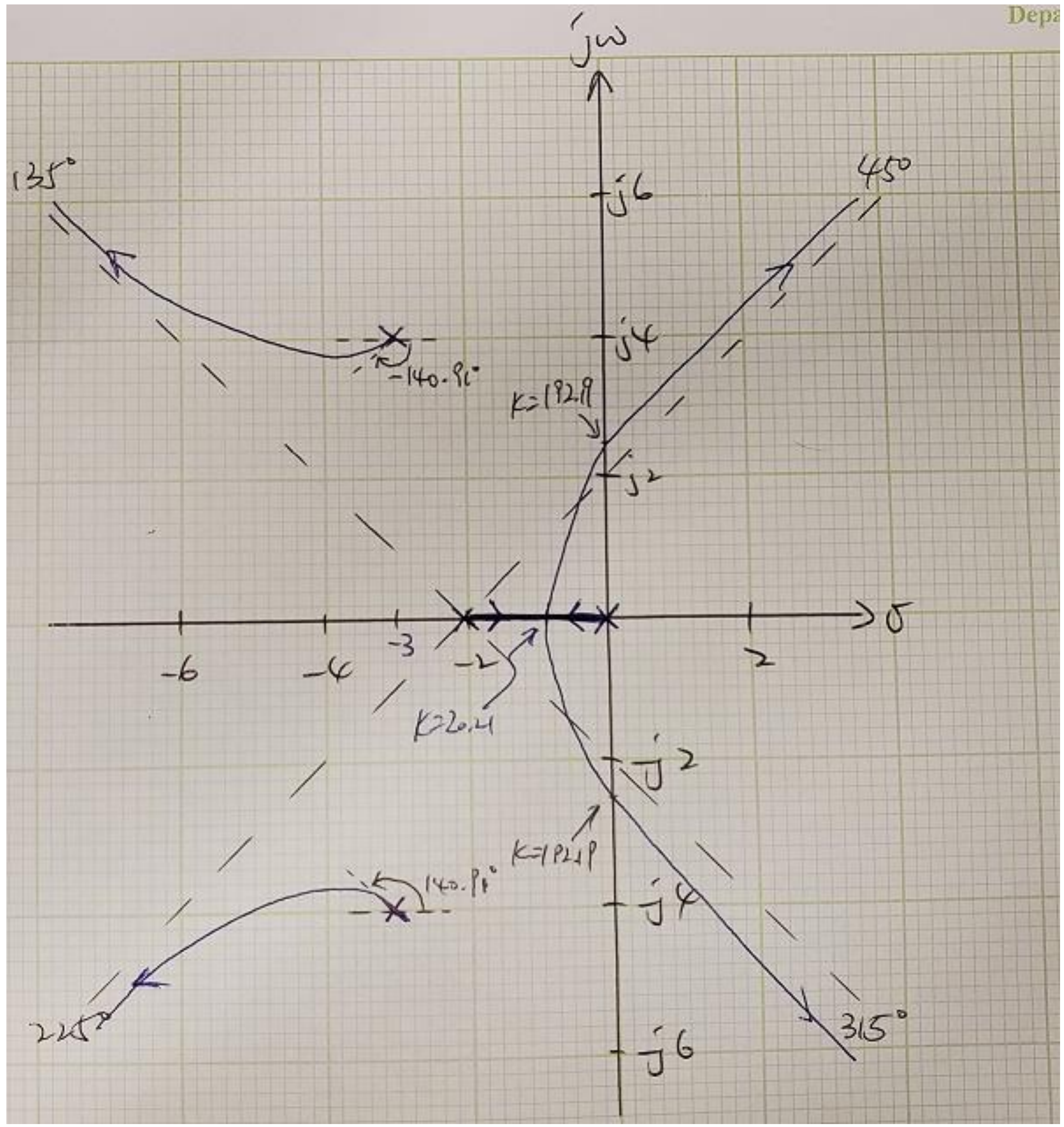
Example 7 (continued)

6. Find the points where the root loci may cross the imaginary axis



Example 7 (continued)

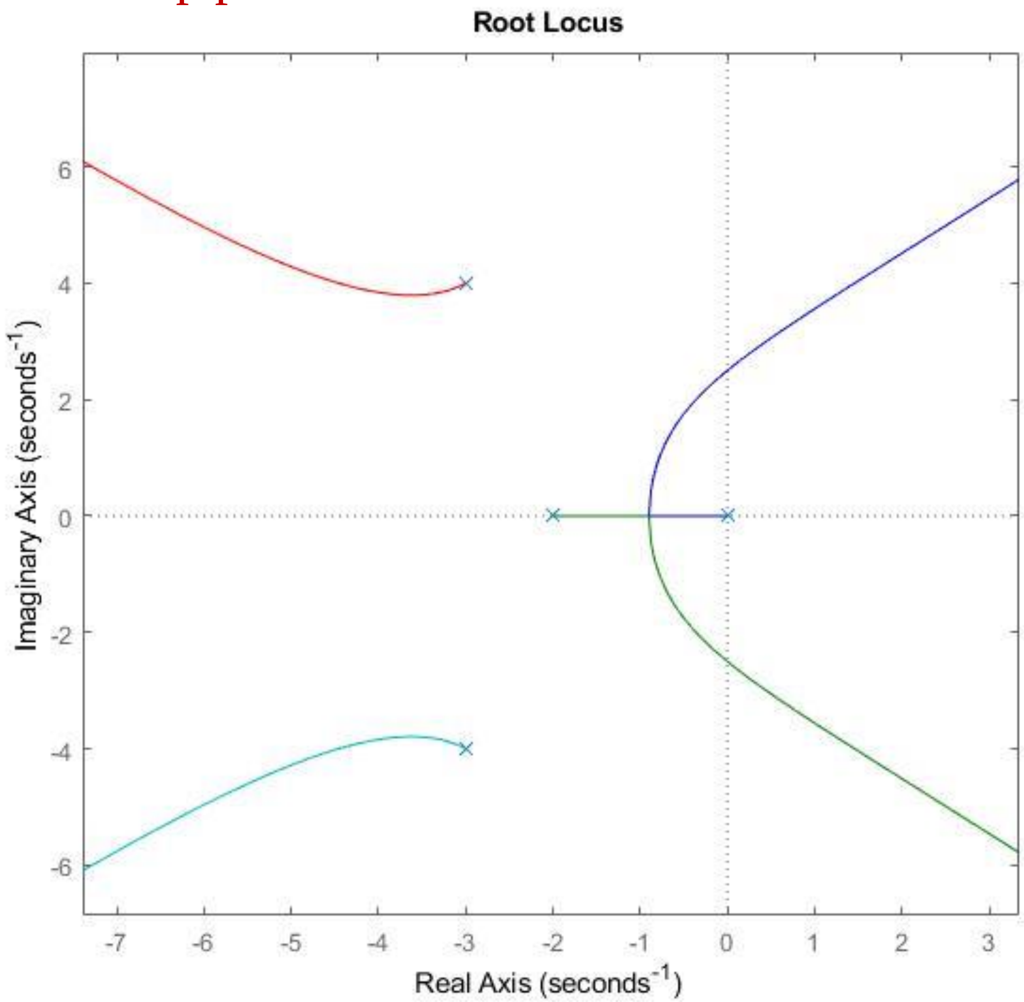
7. Determine the closed-loop poles or Draw the root locus



Example 7 (continued)

7. Determine the closed-loop poles

From Matlab



Root-Locus Approach to Control Systems Design

- In building a control system, a proper modification of the plant dynamics may be a simple way to meet the performance specifications. This, however, may not be possible because the plant may be fixed and not modifiable. Then we must adjust parameters other than those in the fixed plant

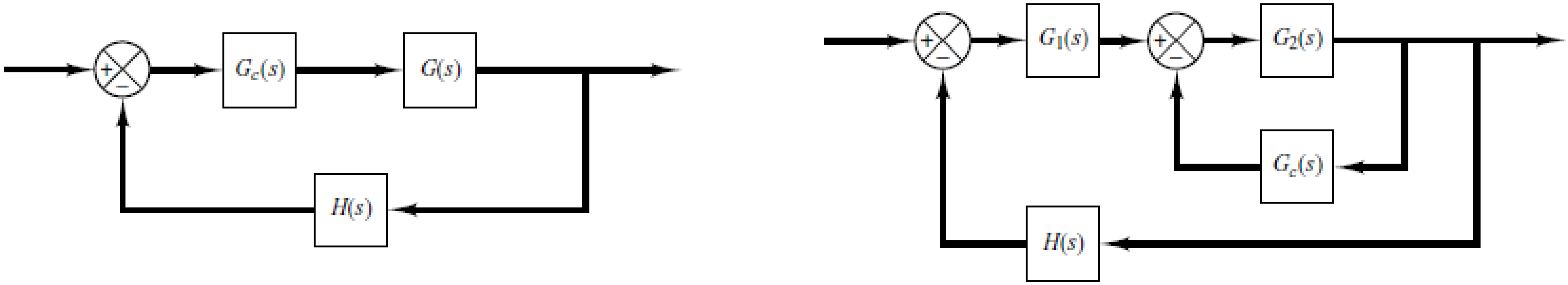
- In practice, the root-locus plot of a system may indicate that the desired performance cannot be achieved just by the adjustment of gain. Then it is necessary to reshape the root loci to meet the performance specifications

- The design by the root-locus method is based on reshaping the root locus of the system by adding poles and zeros to the system's open-loop transfer function and forcing the root loci to pass through desired closed-loop poles in the *s* plane

- If other than a gain adjustment is required, we must modify the original root loci by inserting a suitable compensator

Root-Locus Approach to Control Systems Design

- Series Compensation [Unit 7] and Parallel (or Feedback) Compensation



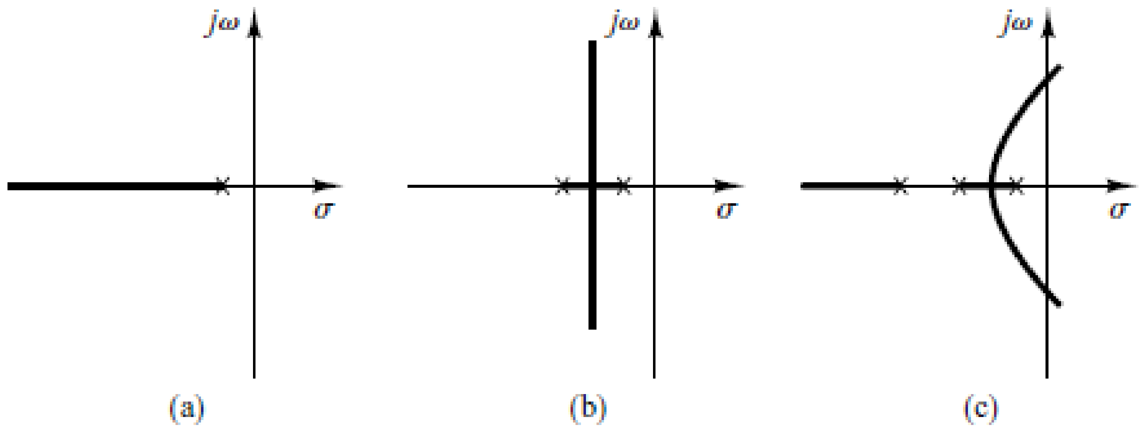
- If a sinusoidal input is applied to the input of a network, and the steady-state output (which is also sinusoidal) has a phase lead, then the network is called a lead network
- If the steady-state output has a phase lag, then the network is called a lag network
- In a lag–lead network, both phase lag and phase lead occur in the output but in different frequency regions
- A compensator having a characteristic of a lead network, lag network, or lag–lead network is called a lead compensator, lag compensator, or lag–lead compensator

Root-Locus Approach to Control Systems Design

Effects of the Addition of Poles

- The addition of a pole to the open-loop transfer function has the effect of pulling the root locus to the right, tending to lower the system's relative stability and to slow down the settling of the response

- Figures below show examples of root loci illustrating the effects of the addition of a pole to a single-pole system and the addition of two poles to a single-pole system



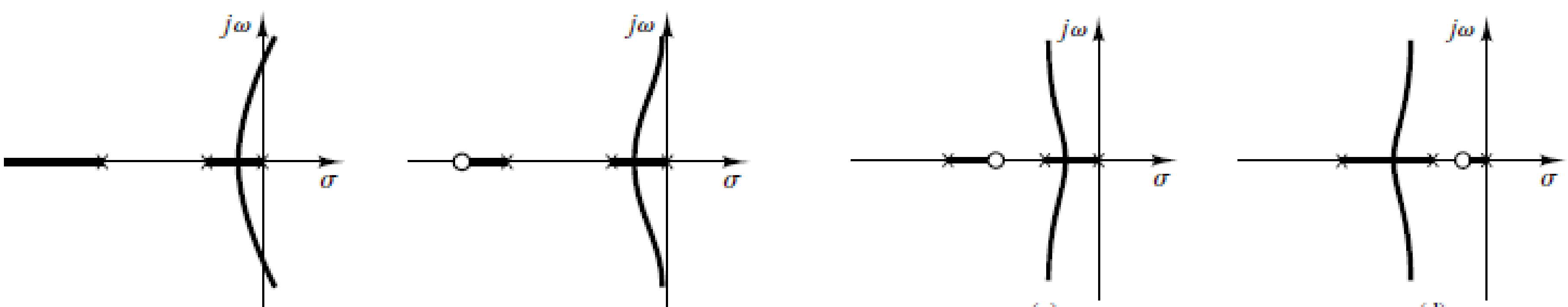
(a) Root-locus plot of a single-pole system; (b) root-locus plot of a two-pole system; (c) root-locus plot of a three-pole system.

Root-Locus Approach to Control Systems Design

Effects of the Addition of Zeros

- The addition of a zero to the open-loop transfer function has the effect of pulling the root locus to the left, tending to make the system more stable and to speed up the settling of the response

- Figure (a) below shows the root loci for a system that is stable for small gain but unstable for large gain. Figures (b), (c), and (d) show root-locus plots for the system when a zero is added to the open-loop transfer function



(a) (b) (c) (d)