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SEHS4653
Control System Analysis

Unit 3

Transient and Steady-state Responses Analysis

(Reference: [1] chapter 5-1 to 5-3, 5-7 to 5-8)

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Introduction

- First step in analyzing a control system was to derive a mathematical model of the system [Unit 2]
- Establish a basis of comparison of performance of various control systems
- Many design criteria are based on the response to such test signals or on the response of systems to changes in initial conditions
- Commonly used test input signals are step functions, ramp functions, acceleration functions, impulse functions, sinusoidal functions, and white noise
- Once a control system is designed on the basis of test signals, the performance of the system in response to actual inputs is generally satisfactory

Introduction

Transient Response and Steady-State Response

- Transient Response, $c_{tr}(t)$: from the initial state to the final state
- Steady-state Response, $c_{ss}(t)$: system output behaves as $t \longrightarrow \infty$
- The system (total) response, $c(t)$,

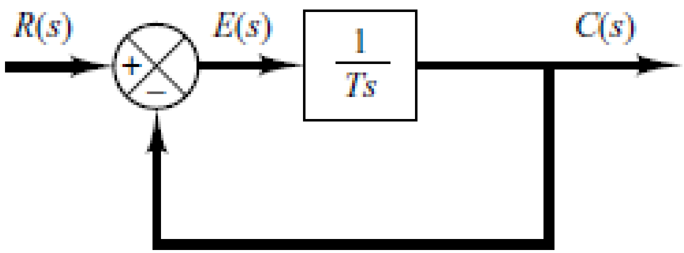
$$c(t) = c_{tr}(t) + c_{ss}(t)$$

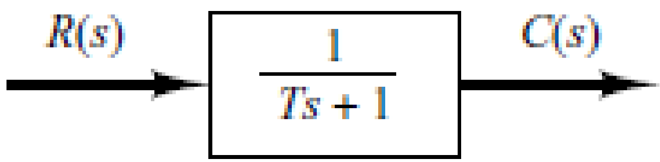
Absolute Stability, Relative Stability, and Steady-State Error

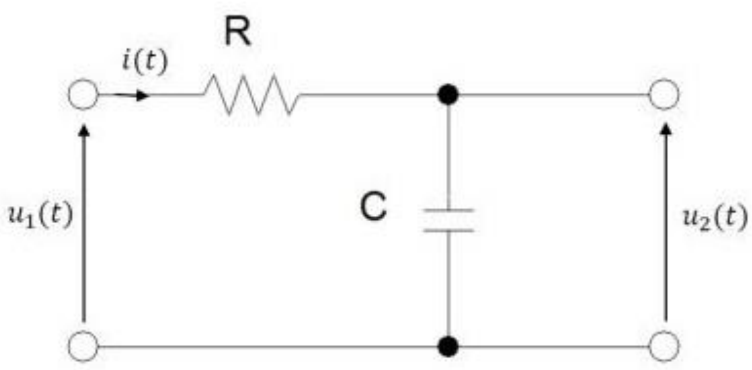
- The most important characteristic of the dynamic behavior of a control system is absolute stability – that is, whether the system is stable or unstable.
- **Stable**: if the output eventually comes back to its equilibrium state when the system is subjected to an initial condition.
- **Critically stable**: if oscillations of the output continues forever
- **Unstable**: if the output diverges without bound from its equilibrium state when it is subjected to an initial condition.
- If the output of a system at steady state does not exactly agree with the input, the system is said to have **steady-state error**

First-Order Systems

- Typical first-order systems include *RC* circuit, thermal system or the like







$$\frac{C(s)}{R(s)} = \frac{1}{Ts + 1}$$

First-Order Systems

Unit-Step Response

- The Laplace transform the **unit-step** function is $1/s$

$$C(s) = \left(\frac{1}{Ts+1}\right)\left(\frac{1}{s}\right) = \frac{1}{s} - \frac{1}{s + \left(\frac{1}{T}\right)}$$

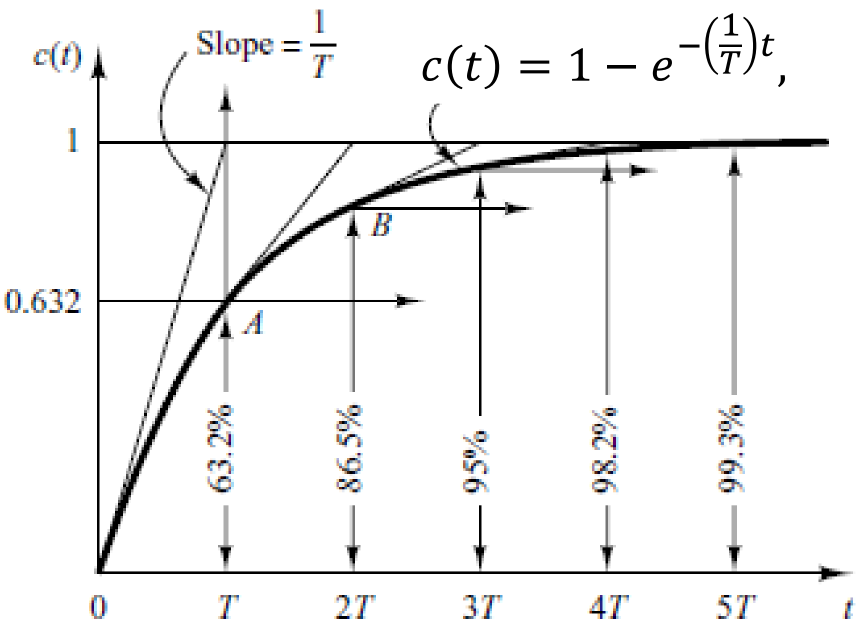
- Taking inverse Laplace transform, we have,

$$c(t) = 1 - e^{-\left(\frac{1}{T}\right)t}, \qquad \text{for } t \geq 0$$

- At $t = 0$, $c(t) = 0$
- At $t \longrightarrow \infty$, $c(t) = 1$
- At $t = T$, $c(t) = 1 - e^{-1} = 0.632 = 63.2\%$
- T is called time constant. The smaller T, the faster the system response

First-Order Systems

Unit-Step Response



$$c(t) = 1 - e^{-\left(\frac{1}{T}\right)t}, \quad \text{for } t \geq 0$$

- The slope of the tangent line at $t = 0$,

$$\left.\frac{dc}{dt}\right|_{t=0} = \frac{1}{T} e^{-\frac{t}{T}}\Big|_{t=0} = \frac{1}{T}$$

- It decreases monotonically from $1 / T$ at $t = 0$ to zero at $t = \infty$

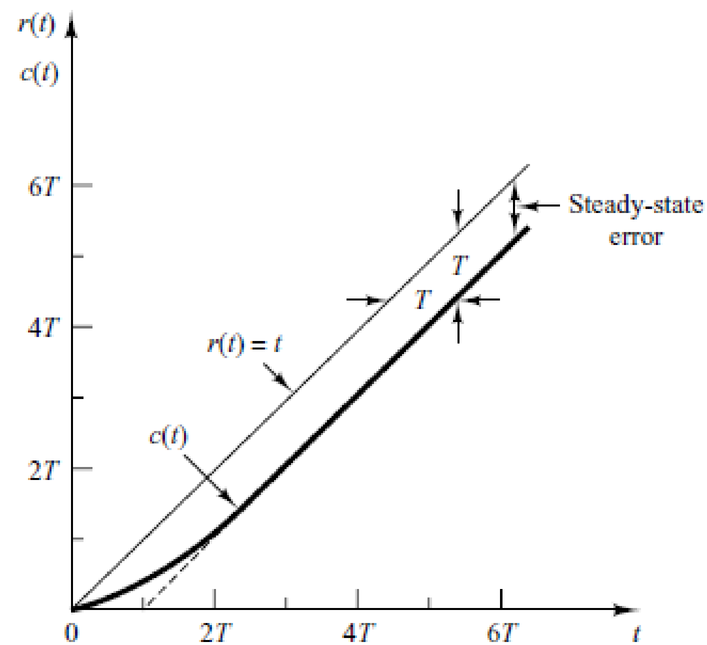
- Although the steady state is reached mathematically only after an infinite time. In practice, however, a reasonable estimate of the response time is the length of time the response curve needs to reach and stay within the 2% line of the final value, or 4 time constants

First-Order Systems

Unit-Ramp Response

- The Laplace transform the **unit-ramp** function is $1/s^2$

$$C(s) = \left(\frac{1}{Ts+1}\right)\left(\frac{1}{s^2}\right) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$



- Taking inverse Laplace transform, we have,

$$c(t) = t - T + Te^{-\left(\frac{1}{T}\right)t}, \qquad \text{for } t \geq 0$$

- The error signal, $e(t) = r(t) - c(t) = T\left(1 - e^{-t/T}\right)$
- At $t \to \infty,\ e(\infty) = T$

The error in following the unit-ramp input is equal to T for sufficiently large t

First-Order Systems

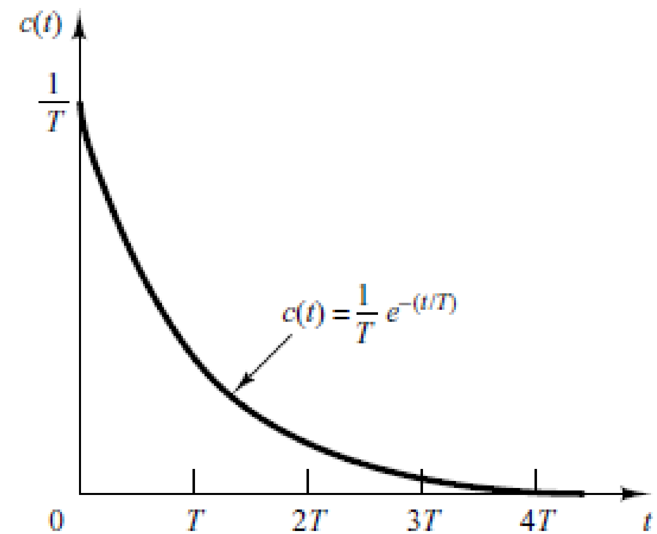
Unit-Impulse Response

- The Laplace transform the **unit-impulse** function is 1,

$$C(s) = \left(\frac{1}{Ts+1}\right)(1)$$

- Taking inverse Laplace transform, we have,

$$c(t) = \frac{1}{T}e^{-\left(\frac{1}{T}\right)t}, \qquad \text{for } t \geq 0$$

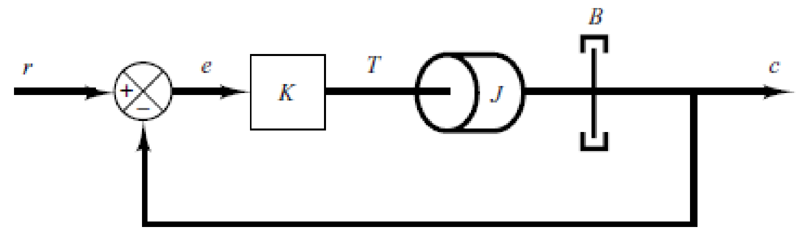


An Important Property of Linear Time-Invariant Systems

- The response to the derivative of an input signal can be obtained by differentiating the response of the system to the original signal
- The response to the integral of the original signal can be obtained by integrating the response of the system to the original signal and by determining the integration constant from the zero-output initial condition

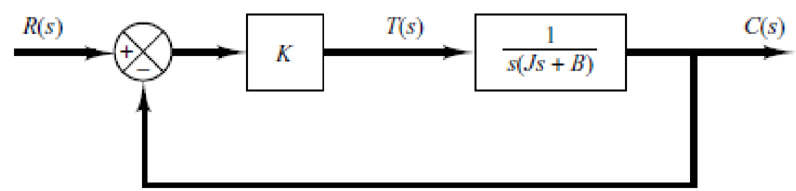
Second-Order Systems

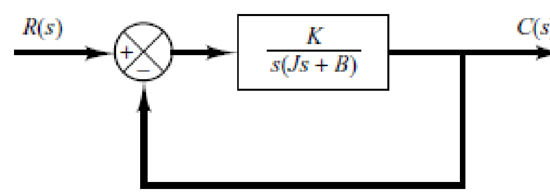
- Consider a servo system as an example of a second-order system
- The servo system shown consists of a proportional controller and load elements.
- Control the output position c in accordance with the input position r



$$J\ddot{c} + B\dot{c} = T$$

Inertia — Viscous-friction — Torque





- The transfer function is then,

$$Js^2C(s) + BsC(s) = T(s) \Rightarrow \frac{C(s)}{T(s)} = \frac{1}{Js^2 + Bs}$$

Second-Order Systems

- The closed-loop transfer function with the gain (K) of the proportional controller,

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} = \frac{\frac{K}{J}}{s^2 + \frac{B}{J}s + \frac{K}{J}}$$

- We can rewrite the closed-loop transfer function as,

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

- It is convenient to write, $\frac{K}{J} = \omega_n^2$, $\frac{B}{J} = 2\zeta\omega_n = 2\sigma$

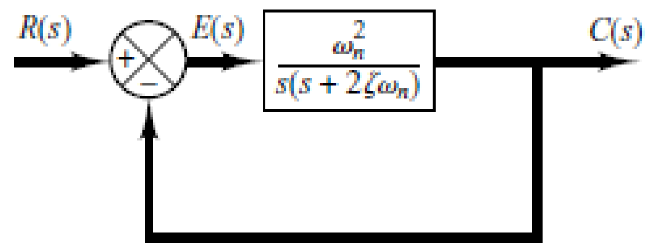
- where σ is called the *attenuation*; ω_n, the *undamped natural frequency*; and ζ, the *damping ratio* of the system. The damping ratio ζ is the ratio of the actual damping B to the critical damping $B_c = 2\sqrt{jK}$ or

$$\zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

Second-Order Systems

- In terms of ζ and ω_n, the system shown below can be modified and the closed-loop transfer function *C*(*s*) / *R*(*s*) can be written as,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



- This form is called the standard form of the second-order system
- The dynamic behavior of the second-order system can then be described in terms of two parameters ζ and ω_n
  - I. If ($0 < \zeta < 1$): the system is underdamped
  - II. If ($\zeta = 1$): the system is critically damped
  - III. If ($\zeta > 1$): the system is overdamped
  - IV. If ($\zeta = 0$): the transient response does not die out

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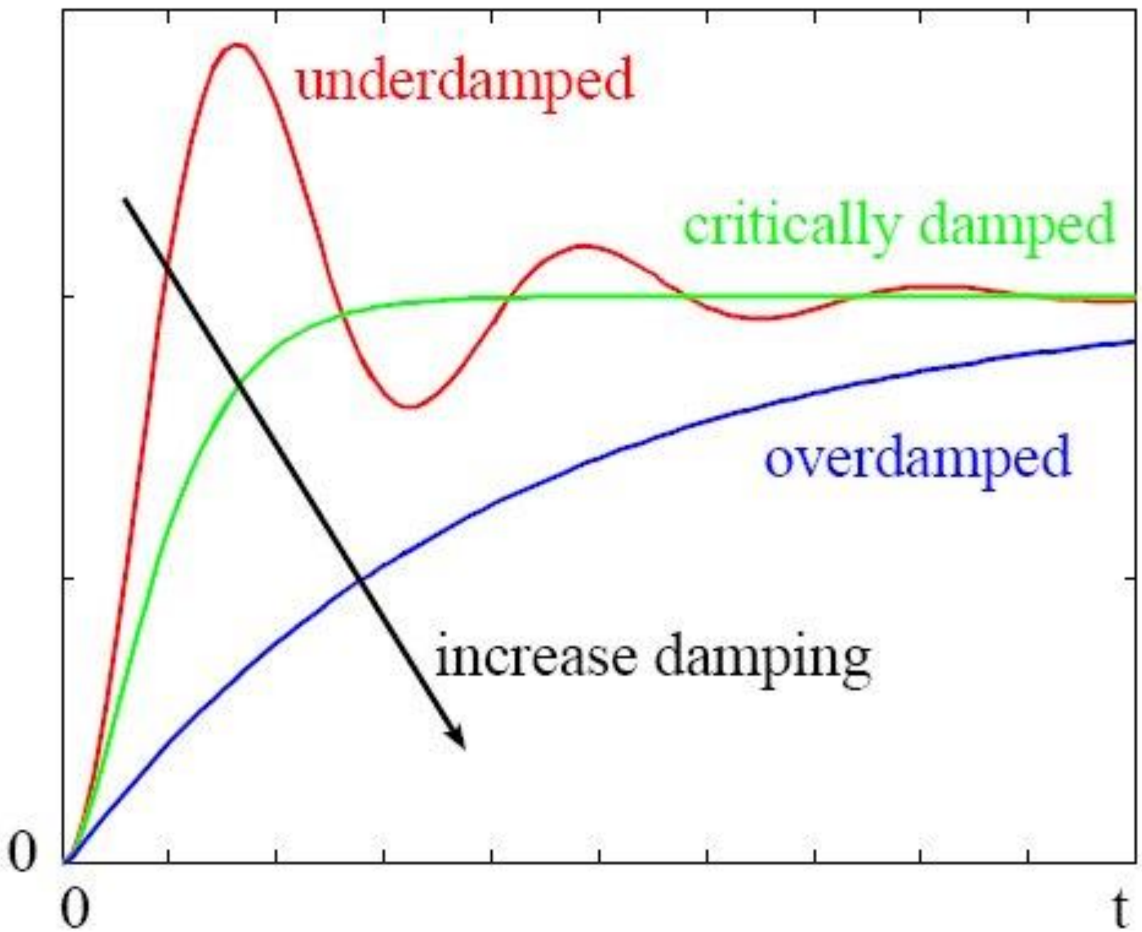
Second-Order Systems

Step response of a second-order system with different damping ratio



Second-Order Systems

(I) Underdamped Case ($0 < \zeta < 1$) :

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

- where $\omega_d = \omega_n\sqrt{1-\zeta^2}$: the damped natural frequency
- For a **unit-step input**, $C(s)$ can be written

$$C(s) = \frac{1}{s}\frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)} = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

- From the Laplace Transform Table, the output in time domain is,

$$c(t) = 1 - \underbrace{\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}}_{\text{damping}}\underbrace{\sin(\omega_d t + \phi)}_{\text{oscillation}} \quad \text{, where } \phi = \cos^{-1}\zeta$$

Second-Order Systems

(I) Underdamped Case ($0 < \zeta < 1$) (continued) :

- The error signal,

$$e(t) = r(t) - c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)$$

- At steady-state ($t \longrightarrow \infty$), no errors exists between the input and output
- If the damping ratio (ζ) is zero, the response becomes undamped,

$$c(t) = 1 - \frac{e^{-0\omega_n t}}{\sqrt{1-0^2}}\sin(\omega_n\sqrt{1-0^2}t + 90°) = 1 - \cos\omega_n t\,, \text{for } t \geq 0$$

- From the above equation, we see that ω_n represents the undamped natural frequency at which the system output would oscillate if the damping is zero
- Since $\omega_d = \omega_n\sqrt{1-\zeta^2}$, $\zeta \uparrow \Longrightarrow \omega_d \downarrow$. The response becomes overdamped and will not oscillate if $\zeta > 1$

Second-Order Systems

(II) Critically Damped Case ($\zeta = 1$) :

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \xrightarrow{\zeta = 1} \frac{\omega_n^2}{(s+\omega_n)^2}$$

- For a **unit-step input**, $c(t)$ will be

$$C(s) = \frac{1}{s}\frac{\omega_n^2}{(s+\omega_n)^2} \xrightarrow{\text{Inverse Laplace Transform}} c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), \qquad t \geq 0$$

Second-Order Systems

(III) Overdamped Case ($\zeta > 1$) :

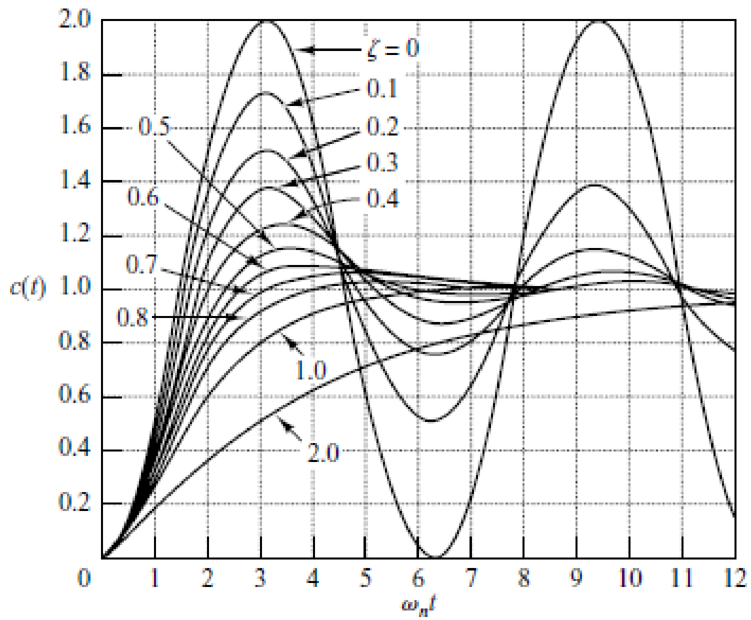
- $C(s)$ can be written with $R(s) = 1 / s$,

$$C(s) = \frac{1}{s}\frac{\omega_n^2}{\left(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)\left(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)}$$

- Taking inverse Laplace transform,

$$\begin{aligned} c(t) \\ &= 1 + \frac{1}{2\sqrt{\zeta^2 - 1}\left(\zeta + \sqrt{\zeta^2 - 1}\right)} e^{-\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} \\ &- \frac{1}{2\sqrt{\zeta^2 - 1}\left(\zeta - \sqrt{\zeta^2 - 1}\right)} e^{-\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} \end{aligned}$$

Second-Order Systems



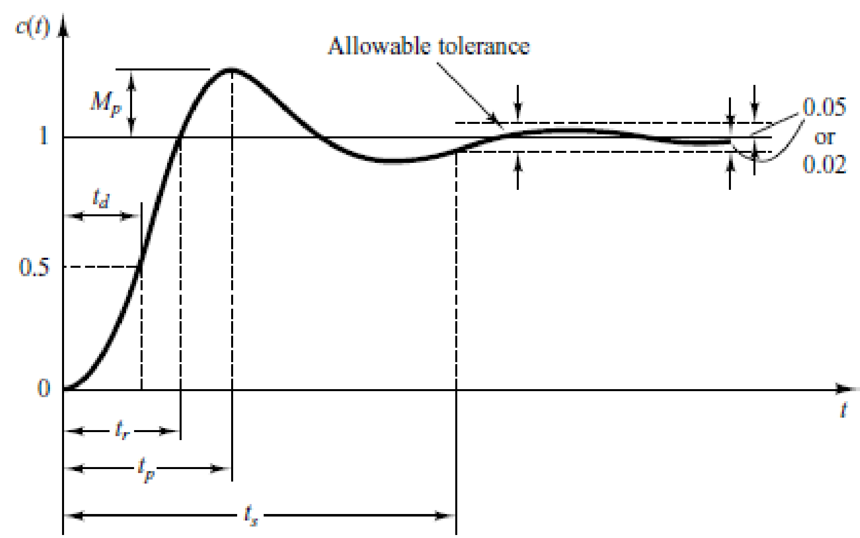
- An underdamped system with ζ between 0.5 and 0.8 gets close to the final value more rapidly than a critically damped or overdamped system
- <u>Among the systems responding without oscillation</u>, a critically damped system exhibits the fastest response
- An overdamped system is always sluggish (moving slowly) in responding to any inputs

Unit-step Response

Definition of Transient-response Specifications

- The performance characteristics of a control system are specified in terms of the transient response to a **unit-step input**, since it is easy to generate
- For comparing transient responses, zero initial condition will be used
- In specifying the transient-response characteristics of a control system to a unit-step input, it is common to specify the following:

1. **Delay time**, t_d: Time required for the response to reach half the final value the very first time
2. **Rise time**, t_r: Time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value
3. **Peak time**, t_p: Time required for the response to reach the first peak of the overshoot
4. **Settling time**, t_s: Time required for the response curve to reach and stay within a range about $\pm 2\%$ to $\pm 5\%$ of its final value
5. **Maximum (percent) overshoot**, M_p: Maximum peak value of the response curve measured from unity



$$M_p(\%) = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

Unit-step Response

Second-order Systems and Transient-response Specifications

- **Rise time, t_r** (0% to 100%)

$$c(t_r) = 1 \Rightarrow 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r + \phi) = 1$$

- Since $\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \neq 0$, we can obtain the following equation,

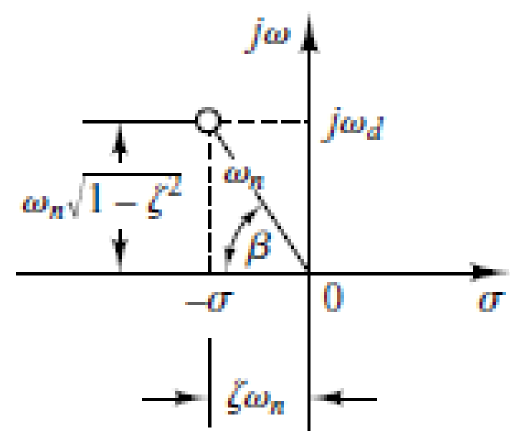
$$\sin(\omega_d t_r + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}) = 0 \Rightarrow \tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

- As $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\zeta\omega_n = \sigma$, we have

$$\tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$$

Then, the rise time is,

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d}$$



Unit-step Response

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

Second-order Systems and Transient-response Specifications

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

- **Peak time, t_p**
  - Obtained by differentiating $c(t)$ with respect to time and letting this derivative equal zero

$$\frac{dc(t)}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) + e^{-\zeta\omega_n t}\left(\omega_d \sin\omega_d t + \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right)$$

- The cosine terms cancel each other, $\frac{dc(t)}{dt}$, evaluated at $t = t_p$, can be simplified to,

$$\left.\frac{dc(t)}{dt}\right|_{t=t_p} = 0 = (\sin\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}$$

- Hence, $\sin(\omega_d t_p) = 0$ or $\omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots$
- Since the peak time corresponds to the first peak overshoot,

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

corresponds to one-half cycle of the frequency of damped oscillation

Unit-step Response

Second-order Systems and Transient-response Specifications

- **Maximum Overshoot, M_p**
  - It occurs at $t_p = \frac{\pi}{\omega_d}$. If the final output value is unity, then

$$M_p = c(t_p) - 1 = -e^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)}\left(\cos\omega_d\frac{\pi}{\omega_d} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d\frac{\pi}{\omega_d}\right) = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}$$

- The maximum percent overshoot is $e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \times 100\%$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Unit-step Response

Second-order Systems and Transient-response Specifications

- **Settling time, t_s**
  - Time corresponding to a $\pm 2\%$ or $\pm 5\%$ tolerance band
- The envelope curves of the transient response,

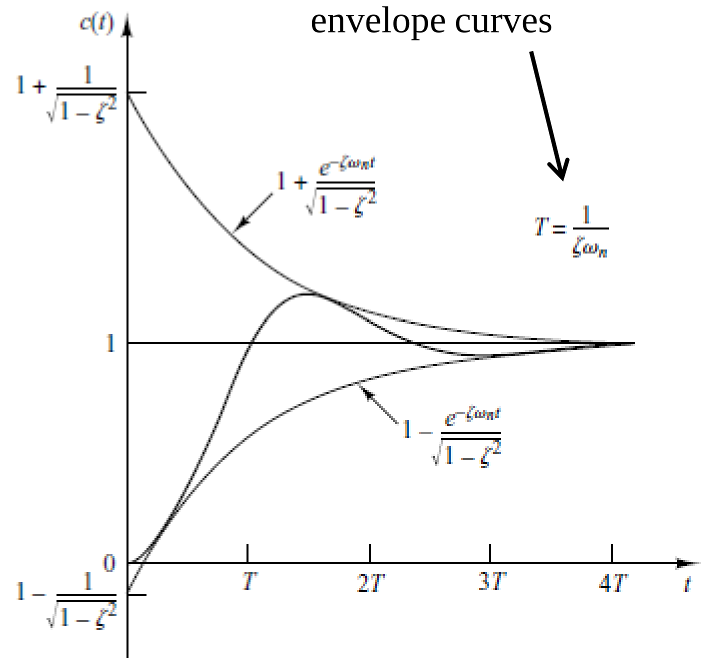
$$1 \pm \left(\frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \right)$$

- Hence, the settling time is commonly defined as,

$$t_s = 4T = \frac{4}{\zeta \omega_n} \quad \text{(2\% criterion)}$$

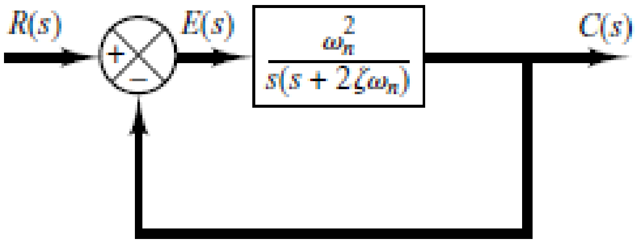
$$t_s = 3T = \frac{3}{\zeta \omega_n} \quad \text{(5\% criterion)}$$

T : Time constant of the envelope curves



Example 1

Consider the system shown below, where $\zeta = 0.6$ and $\omega_n = 5$ rad/s. Find the rise time t_r , peak time t_p , maximum overshoot M_p , and settling time t_s when the system is subjected to a unit-step input.



Answer:

$\omega_d = 5\sqrt{1-0.6^2} = 4\,, \sigma = (0.6)(5) = 3$

$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.9273 \text{ rad}$

Rise time, $t_r = \frac{\pi-\beta}{\omega_d} = \frac{\pi-0.9273}{4} = 0.554 \text{ s}$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4} = 0.785 \text{ s}$

Maximum overshoot, $M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} = e^{-\frac{0.6}{\sqrt{1-0.6^2}}\pi} = 0.0948$

The maximum percent overshoot is thus 9.48%

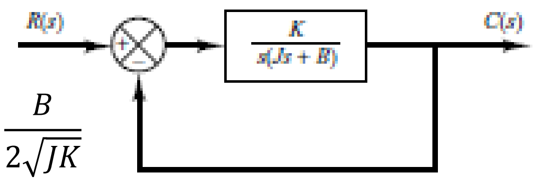
Settling time, $t_s = \frac{4}{\zeta\omega_n} = \frac{4}{(0.6)(5)} = 1.333 \text{ s}$ (for 2% criterion)

$t_s = \frac{3}{\zeta\omega_n} = \frac{3}{(0.6)(5)} = 1 \text{ s}$ (for 5% criterion)

System with Velocity Feedback

- Revisited the servo system in p.10
- The derivative of the output signal can be used to improve system performance
- In obtaining the derivative of the output position signal, it is desirable to use a tachometer instead of physically differentiating the output signal

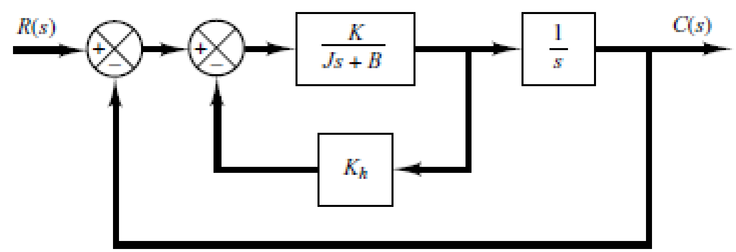
$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K}$$



$$\zeta = \frac{B}{2\sqrt{JK}}$$

- The velocity signal, together with the positional signal, is fed back to the input to produce the actuating error signal
- The transfer function of the servo system with velocity-feedback constant K_h can be written as,

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}$$



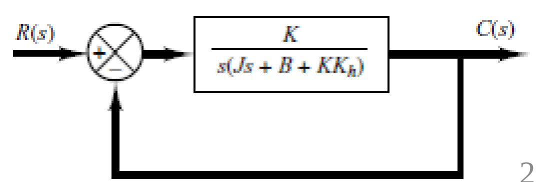
(a)

- The new damping ratio becomes,

$$\zeta = \frac{B + KK_h}{2\sqrt{KJ}}$$

- The undamped natural frequency is unchanged,

$$\omega_n = \sqrt{K/J}$$



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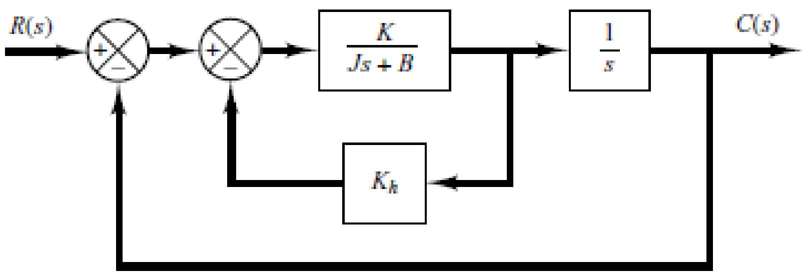
Example 2

For the system shown below, determine the values of gain K and velocity-feedback constant K_h so that the maximum overshoot in the unit-step response is 0.2 and the peak time is 1 sec. With these values of K and K_h, obtain the rise time and settling time (2%). Assume that $J = 1$ kgm^2 and $B = 1$ Nm/rad/sec.



Answer: $t_p = \frac{\pi}{\omega_d} = 1$ $\therefore \omega_d = \pi$

$$M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} = 0.2 \rightarrow \ln\left(e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}\right) = \ln 0.2$$

$$-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi = \ln 0.2 \rightarrow \left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi\right)^2 = (\ln 0.2)^2$$

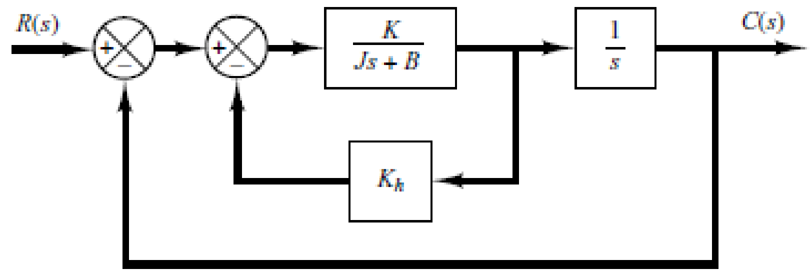
$$\frac{\zeta^2\pi^2}{1-\zeta^2} = (\ln 0.2)^2 \rightarrow \zeta^2\pi^2 = (1-\zeta^2)(\ln 0.2)^2 \rightarrow \zeta^2(\pi^2 + (\ln 0.2)^2) = (\ln 0.2)^2$$

$$\zeta = \pm\sqrt{\frac{(\ln 0.2)^2}{\pi^2 + (\ln 0.2)^2}} = 0.4559 \text{ or } -0.4559 \text{ (reject)}$$

$$\zeta = \sqrt{\frac{(\ln M_P)^2}{\pi^2 + (\ln M_P)^2}}$$

Example 2

Answer:



$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{\pi}{\sqrt{1-0.4559^2}} = 3.53 \text{ rad/s}$$

$$\omega_n = \sqrt{\frac{K}{J}} = \sqrt{\frac{K}{1}} = 3.53 \rightarrow \boldsymbol{K = 12.46\ \mathrm{Nm}}$$

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}$$

$$\zeta = \frac{B + KK_h}{2\sqrt{KJ}} \rightarrow 0.4559 = \frac{1 + 12.46K_h}{2\sqrt{(12.46)(1)}} \rightarrow \boldsymbol{K_h = 0.178\ \mathrm{s}}$$

Rise Time

$$\boldsymbol{t_r} = \frac{\pi - \beta}{\omega_d} = \frac{\pi - \cos^{-1}\zeta}{\pi} = \frac{\pi - \cos^{-1} 0.4559}{\pi} = \boldsymbol{0.651\ \mathrm{s}}$$

Settling Time (2%)

$$\boldsymbol{t_s\ (2\%)} = \frac{4}{\zeta\omega_n} = \frac{4}{(0.4556)(3.53)} = \boldsymbol{2.486\ \mathrm{s}}$$

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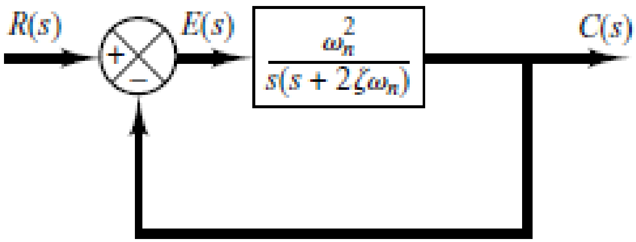
Unit-Impulse Response

- The **unit-impulse response** of the second-order system shown below is,

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



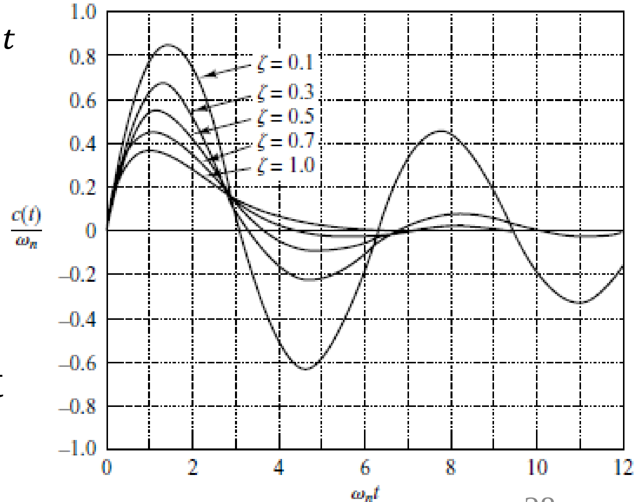
- Its inverse Laplace transform is,

1. $\mathbf{0 \le \zeta < 1}$: $c(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$

2. $\mathbf{\zeta = 1}$: $c(t) = \omega_n^2 t e^{-\omega_n t}$

3. $\mathbf{\zeta > 1}$:

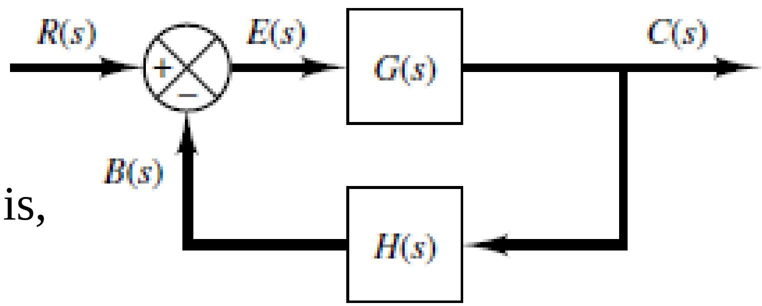
$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2-1}} e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} - \frac{\omega_n}{2\sqrt{\zeta^2-1}} e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}$$



- For the critically damped and overdamped cases, the responses is always positive or zero
- For the underdamped case, the response oscillates about zero and takes both positive and negative values

Steady-State Errors in Feedback Control Systems

- Any physical control system inherently suffers **steady-state error** in response to certain types of inputs, e.g. it may have no steady-state error to a step input, but may exhibit nonzero steady-state error to a ramp input ⇒ depends on the type of **open-loop transfer function** of the system



Steady-State Errors

- Consider the system beside, the transfer function is,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

- The transfer function between the error signal $e(t)$ and the input signal $r(t)$ is,

$$\frac{E(s)}{R(s)} = \frac{R(s)-H(s)C(s)}{R(s)} = 1 - \frac{H(s)C(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

- The steady-state error can be computed by using the final-value theorem,

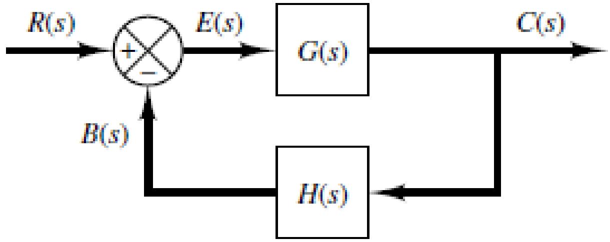
$$E(s) = \frac{1}{1+G(s)H(s)}R(s) \Rightarrow e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)H(s)}$$

Steady-State Errors in Unity-Feedback Control Systems

Static Position Error Constant K_p

- The steady-state error of the system for a **unit-step input** is,

$$e_{ss} = \lim_{s\to 0}\left(\frac{s}{1+G(s)H(s)}\right)\left(\frac{1}{s}\right) = \frac{1}{1+G(0)H(0)}$$



- The static position error constant K_p is defined by,

$$K_p = \lim_{s\to 0} G(s)H(s) = G(0)H(0)$$

- Thus, the steady-state error in terms of K_p is given by, $e_{ss} = \frac{1}{1+K_p}$

Static Velocity Error Constant K_v

- The steady-state error of the system with a **unit-ramp input** is,

$$e_{ss} = \lim_{s\to 0}\left(\frac{s}{1+G(s)H(s)}\right)\left(\frac{1}{s^2}\right) = \lim_{s\to 0}\frac{1}{sG(s)H(s)}$$

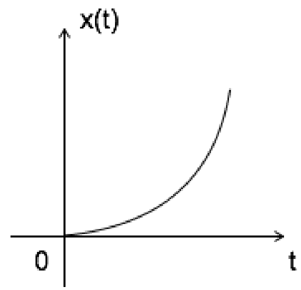
- The static velocity error constant K_v is defined by, $K_v = \lim_{s\to 0} sG(s)\,H(s)$
- Thus, the steady-state error in terms of K_v is given by, $e_{ss} = \frac{1}{K_v}$

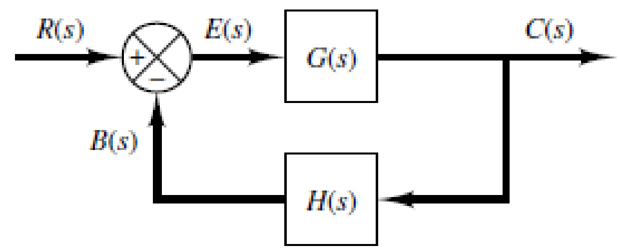
Steady-State Errors in Unity-Feedback Control Systems

Static Acceleration Error Constant K_a

- The steady-state error of the system for a **unit-parabolic input** (or acceleration input) is,

$$r(t) = \begin{cases} \frac{t^2}{2}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$





$$e_{ss} = \lim_{s \to 0} \left(\frac{s}{1 + G(s)H(s)} \right) \left(\frac{1}{s^3} \right) = \lim_{s \to 0} \frac{1}{s^2 G(s)H(s)}$$

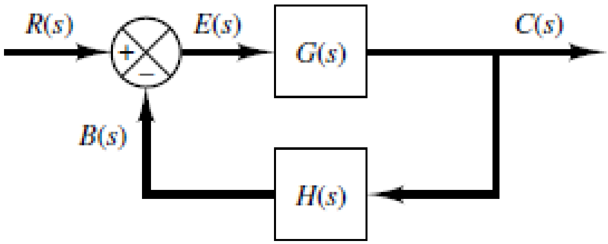
- The static acceleration error constant K_a is defined by,

$$K_a = \lim_{s \to 0} s^2 G(s)H(s)$$

- Thus, the steady-state error in terms of K_a is given by, $e_{ss} = \frac{1}{K_a}$

Steady-State Errors in Unity-Feedback Control Systems

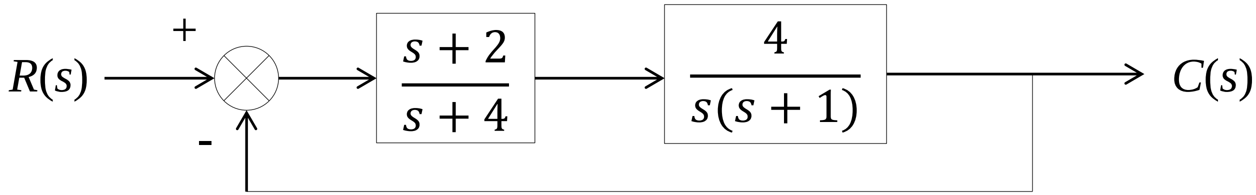
Summary



	Step Input $r(t) = 1$	**Ramp Input** $r(t) = t$	**Acceleration Input** $r(t) = t^2/2$
Static Position Error Constant K_p	$\lim_{s\to 0} G(s)H(s)$	-	-
Static Velocity Error Constant K_v	-	$\lim_{s\to 0} sG(s)\,H(s)$	-
Static Acceleration Error Constant K_s	-	-	$\lim_{s\to 0} s^2G(s)H(s)$
Steady State Error e_{ss}	$\frac{1}{1+K_p}$	$\frac{1}{K_v}$	$\frac{1}{K_a}$

Example 3

Find the steady state error for (a) a unit-step input; (b) a unit-ramp input; and (c) a unit parabolic input.



<u>Answer:</u>

(a)

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{4(s+2)}{s(s+4)(s+1)} = \infty \quad \Rightarrow \quad e_{ss} = \frac{1}{1+K_p} = 0$$

(b)

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} \frac{4s(s+2)}{s(s+4)(s+1)} = 2 \quad \Rightarrow \quad e_{ss} = \frac{1}{K_v} = 0.5$$

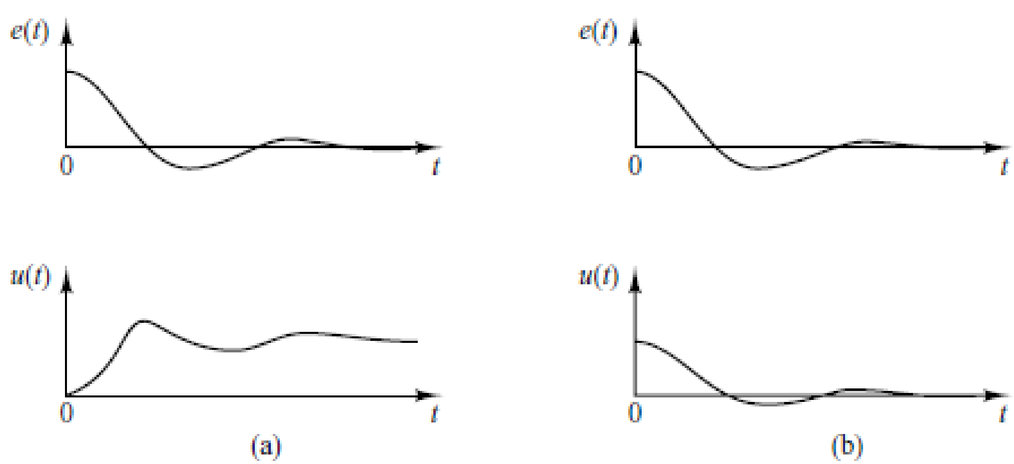
(c)

$$K_a = \lim_{s\to 0} s^2G(s)H(s) = \lim_{s\to 0} \frac{4s^2(s+2)}{s(s+4)(s+1)} = 0 \quad \Rightarrow \quad e_{ss} = \frac{1}{K_a} = \infty$$

Effects of Integral and Derivative Control Actions on System Performance

Integral Control Action

- In the integral control of a plant, the control signal — output signal from the controller — at any instant is the area under the actuating-error-signal curve up to that instant
- The control signal $u(t)$ can have a nonzero value when the actuating error signal $e(t)$ is zero as shown below
- This is impossible in the case of the proportional controller, since a nonzero control signal requires a nonzero actuating error signal as shown below

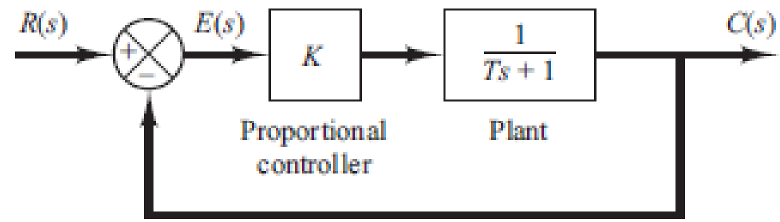


(a) Plots of $e(t)$ and $u(t)$ curves showing nonzero control signal (integral control)

(b) Plots of $e(t)$ and $u(t)$ curves showing zero control signal (proportional control)

Effects of Integral and Derivative Control Actions on System Performance

Proportional Control of Systems



- Consider the system shown beside, then

$$G(s) = \frac{K}{Ts+1} \text{ and } \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1+G(s)}$$

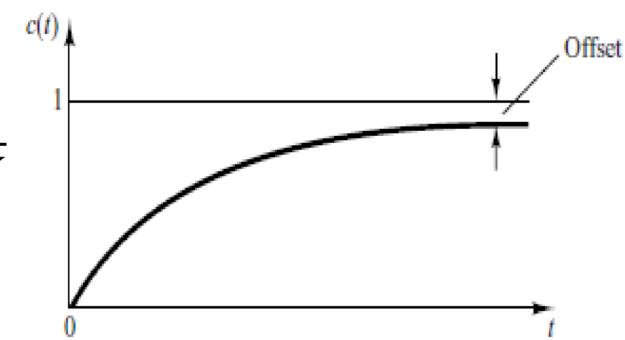
$$\therefore E(s) = \frac{1}{1+G(s)}R(s) = \frac{1}{1+\frac{K}{Ts+1}}R(s)$$

$R(s) = 1/s$ ⟹

$$E(s) = \left(\frac{Ts+1}{Ts+1+K}\right)\left(\frac{1}{s}\right)$$

- The steady-state error is,

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}(s)\left(\frac{Ts+1}{Ts+1+K}\right)\left(\frac{1}{s}\right) = \frac{1}{1+K}$$



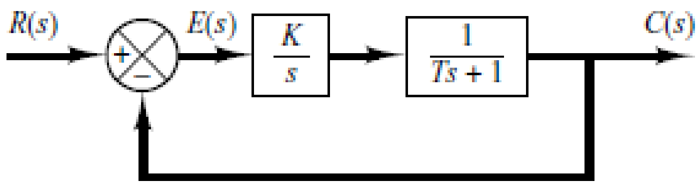
A system without an integrator in the feedforward path always has a steady-state error (called "offset") in the step response.

Effects of Integral and Derivative Control Actions on System Performance

Integral Control of Systems

- Consider the system shown beside, then

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(Ts+1)}}{1+\frac{K}{s(Ts+1)}} = \frac{K}{s(Ts+1)+K}$$



$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{s(Ts+1)}{s(Ts+1)+K} \Rightarrow E(s) = \frac{s(Ts+1)}{s(Ts+1)+K}R(s)$$

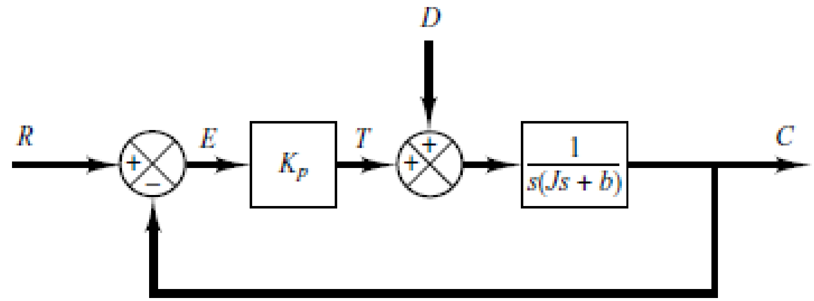
- The steady-state error for the unit-step response can be obtained by applying the final-value theorem,

$$\therefore e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\left(\frac{s^2(Ts+1)}{s(Ts+1)+K}\right)\left(\frac{1}{s}\right) = 0$$

- Integral control of the system thus eliminates the steady-state error in the response to the step input

Example 4

Consider the system shown below. The proportional controller delivers torque T to position the load element, which consists of moment of inertia (J) and viscous friction (b). Torque disturbance is denoted by D which is a step function of magnitude T_d . Determine the steady-state error if reference input is zero.



Answer:

Since $R(s) = 0$, there will be only one input $D(s)$. Hence the transfer function between $C(s)$ and $D(s)$ is,

$$\frac{C(s)}{D(s)} = \frac{\frac{1}{s(Js+b)}}{1+\frac{1}{s(Js+b)}K_p} = \frac{1}{Js^2+bs+K} \Rightarrow \therefore \frac{E(s)}{D(s)} = \frac{R(s)-C(s)}{D(s)} = -\frac{C(s)}{D(s)} = -\frac{1}{Js^2+bs+K}$$

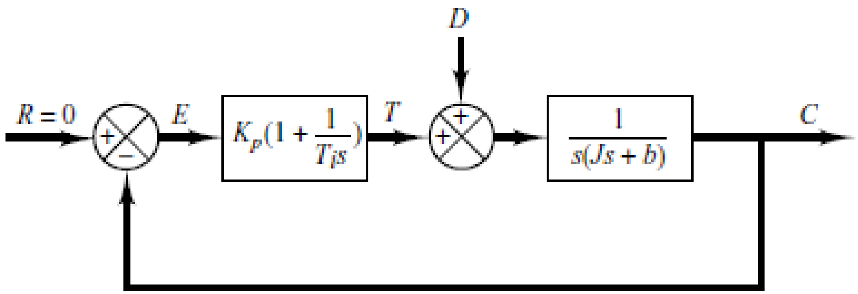
The steady-state error due to a step disturbance torque of magnitude T_d is given by

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0}(s)\left(-\frac{1}{Js^2+bs+K}\right)\left(\frac{T_d}{s}\right) = -\frac{T_d}{K}$$

Example 5

The proportional controller in Example 4 is now replaced by a proportional-plus-integral controller as shown below. Find the steady-state error of the system with the same condition of Example 4, i.e. $R(s) = 0$ and $D(s) = T_d / s$.

<u>Answer:</u>



$$E(s) = R(s) - C(s) \rightarrow \quad E(s) = -C(s)$$

$$\frac{E(s)}{D(s)} = -\frac{C(s)}{D(s)} = -\frac{\frac{1}{s(Js+b)}}{1 + \left(K_p\left(1 + \frac{1}{T_i s}\right)\right)\left(\frac{1}{s(Js+b)}\right)} = -\frac{\frac{1}{s(Js+b)}}{\frac{s^2 T_i(Js+b) + K_p(T_i s + 1)}{T_i(Js+b)s^2}}$$

$$\therefore \frac{E(s)}{D(s)} = -\frac{T_i s}{(Js+b)T_i s^2 + K_p(T_i s + 1)}$$

Example 5

Answer:

$$\therefore \frac{E(s)}{D(s)} = -\frac{T_i s}{s(Js+b)T_i s + K_p(T_i s + 1)} = -\frac{T_i s}{JT_i s^3 + bT_i s^2 + K_p T_i s + K_p}$$

Steady-state error

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0}(s)\left(-\frac{T_i s}{JT_i s^3 + bT_i s^2 + K_p T_i s + K_p}\right)\left(\frac{T_d}{s}\right)$$

$$e_{ss} = \frac{0}{K_p} = 0$$

Effects of Integral and Derivative Control Actions on System Performance

Derivative Control of Systems

- Derivative control action, when added to a proportional controller, obtaining a controller with high sensitivity
- It responds to the rate of change of the actuating error and can produce a significant correction before the magnitude of the actuating error becomes too large
- Derivative control thus anticipates the actuating error, initiates an early corrective action, and tends to increase the stability of the system
- Not affect the steady-state error directly, it adds damping to the system and thus permits the use of a larger value of the gain K, which will result in an improvement in the steady-state accuracy