First-order Systems: Unit-Step Response (p.6)

Method 1

Partial Fraction

$$C(s) = \left(\frac{1}{Ts+1}\right)\left(\frac{1}{s}\right) = \frac{A}{s} + \frac{B}{Ts+1}$$

$$\therefore \quad A(Ts+1) + Bs = 1$$

Put $s=0$, $\quad A = 1$

Put $s = -\frac{1}{T}$, $\quad -\frac{B}{T} = 1 \Rightarrow B = -T$

$$\therefore \quad C(s) = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

Taking Inverse Laplace Transform,

$$\mathcal{L}^{-1}[C(s)] = c(t) = 1 - e^{-\frac{1}{T}t}$$

Method 2

Laplace Transform Table

Rule 10

$$C(s) = \frac{1}{s(Ts+1)} = \frac{1/T}{s(s+1/T)}$$

$$\xrightarrow{\mathcal{L}^{-1}[\,]} \frac{1/T}{\frac{1}{T}}\left(1 - e^{-\frac{1}{T}t}\right)$$

$$c(t) = 1 - e^{-\frac{1}{T}t}$$

First-Order Systems: Unit-Ramp Response (p.8)

Method 1
Partial Fraction

$$C(s) = \left(\frac{1}{Ts+1}\right)\left(\frac{1}{s^2}\right) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{Ts+1}$$

$$\therefore A(Ts+1) + Bs(Ts+1) + Cs^2 = 1$$

$s=0$, $A=1$

$s=-\frac{1}{T}$, $(C)\left(-\frac{1}{T}\right)^2 = 1 \Rightarrow C = T^2$

$s=\frac{1}{T}$, $2A + \frac{2B}{T} + \frac{C}{T^2} = 1 \Rightarrow B = -T$

$$\therefore C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+\frac{1}{T}}$$

$$\mathcal{L}^{-1}[C(s)] = c(t) = t - T + Te^{-\frac{1}{T}t}$$

Rule 14

Method 2
Laplace Transform Table

$$C(s) = \frac{1}{s^2(Ts+1)} = \frac{1/T}{s^2\left(s+\frac{1}{T}\right)}$$

$$\xrightarrow{\mathcal{L}^{-1}[\,]} \frac{1/T}{(1/T)^2}\left(\frac{1}{T}t - 1 + e^{-\frac{1}{T}t}\right)$$

$$c(t) = T\left(\frac{1}{T}t - 1 + e^{-\frac{1}{T}t}\right)$$

$$= t - T + Te^{-\frac{1}{T}t}$$

First-order Systems : Unit-Impulse Response (p.8)

$$C(s) = \left(\frac{1}{Ts+1}\right)(1) = \frac{\frac{1}{T}}{s+\frac{1}{T}}$$

Rule 5

$$\mathcal{L}^{-1}[C(s)] = c(t) = \frac{1}{T} e^{-\frac{1}{T}t}$$

Second-Order Systems (p.11)

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} = \frac{\frac{K}{J}}{s^2 + \frac{B}{J}s + \frac{K}{J}}$$

← Quadratic Equation

roots

$$s = \frac{-\frac{B}{J} \pm \sqrt{\left(\frac{B}{J}\right)^2 - (4)(1)\left(\frac{K}{J}\right)}}{2(1)}$$

$$= \frac{-B}{2J} \pm \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}$$

$\therefore \frac{C(s)}{R(s)}$ can be written as,

$$\boxed{\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}}$$

Second-Order Systems (continued)

Since there are 3 natures of root which depends on the value of discriminant, i.e.,

$$\left(\frac{B}{2J}\right)^2 - \frac{K}{J} \begin{cases} > 0 \\ = 0 \Leftrightarrow \text{critical case} \\ < 0 \end{cases}$$

$$\therefore \left(\frac{B}{2J}\right)^2 = \frac{K}{J} \Rightarrow B^2 = 4JK \Rightarrow B = 2\sqrt{JK}$$

$= B_c$ = critical damping

If we use, $\frac{K}{J} = \omega_n^2$

$$\frac{B}{J} = 2\zeta\omega_n = 2\sigma \quad (\Rightarrow \sigma = \zeta\omega_n)$$

then we have,

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \quad (p.12)$$

Standard Form of 2nd order system

Now the roots become,

roots on p.3

$$s = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2} \Leftarrow s = -\frac{B}{2J} \pm \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}$$

$$= -\zeta\omega_n \pm \sqrt{\omega_n^2(\zeta^2 - 1)}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

SEHS4653 Control System Analysis

"Unit 3 Transient and Steady-State Response Analysis" Supplementary Information

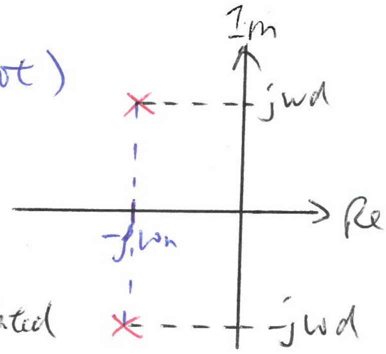
Second-Order Systems (Step Response) (p.12~13)

2 roots of a 2^{nd}-order systems, $s = -\zeta\omega_n \pm \sqrt{\boxed{\omega_n^2(\zeta^2-1)}}$

$\boxed{\text{Case I (underdamped)}: \ 0<\zeta<1}$

$\Rightarrow \sqrt{\ } < 0 \Rightarrow$ Complex roots (no real root)

(in complex conjugate)

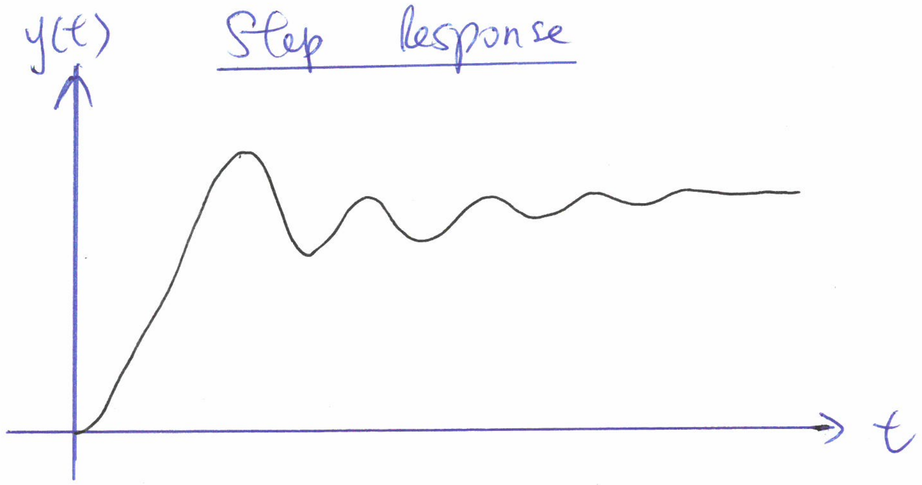


$\therefore s = -\zeta\omega_n \pm j\,\omega_n\sqrt{(1-\zeta^2)}$ make it > 0 $\therefore \sqrt{\ } > 0$ can be calculated

Let $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)}} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

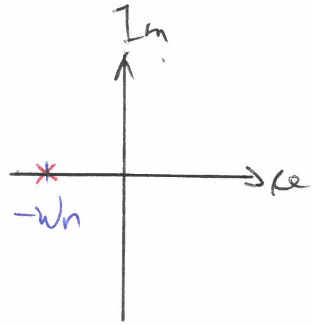
Step Response



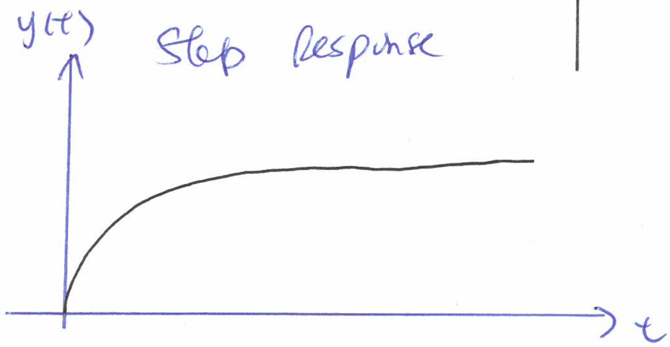
$$S = -\zeta W_n \pm W_n\sqrt{\zeta^2 - 1}$$

$\boxed{\text{Case II (Critically damped): } \zeta = 1}$

$\Rightarrow \sqrt{\ } = 0 \quad \Rightarrow$ double roots, $S = -W_n$



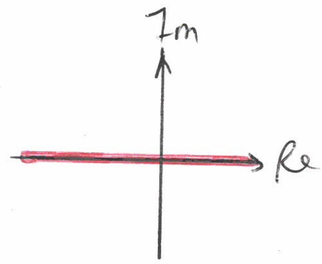
$$\boxed{\frac{C(s)}{R(s)} = \frac{W_n^2}{(S + W_n)^2}}$$



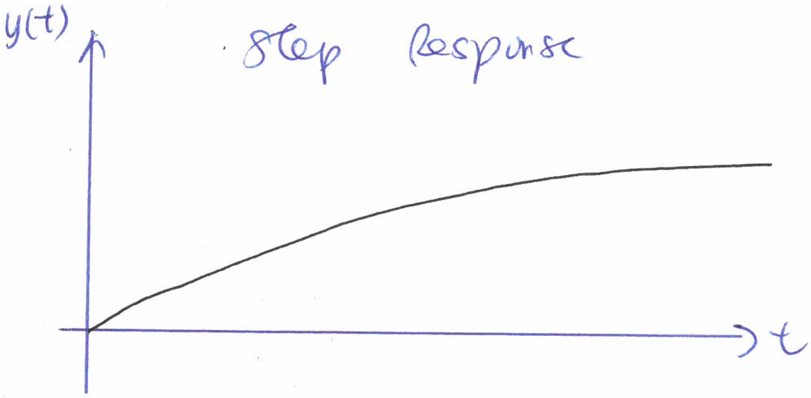
$\boxed{\text{Case III (Overdamped): } \zeta > 1}$

$\Rightarrow \sqrt{\ } > 0 \quad \Rightarrow$ distinct real roots

(any place on the real axis)



$$\boxed{\frac{C(s)}{R(s)} = \frac{W_n^2}{(S + \zeta W_n + W_n\sqrt{\zeta^2 - 1})(S + \zeta W_n - W_n\sqrt{\zeta^2 - 1})}}$$



Underdamped Case (p.14)   * Unit-Step Response *

$$C(s) = \left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{(\underbrace{s+\zeta\omega_n}_{a} + j\underbrace{\omega_d}_{b})(\underbrace{s+\zeta\omega_n}_{a} - j\underbrace{\omega_d}_{b})}\right) = \frac{\omega_n^2}{s\left[(s+\zeta\omega_n)^2 + \omega_d^2\right]}$$

$$(a+jb)\quad(a-jb) = a^2 + b^2 \qquad (\because j^2 = -1)$$

$$= \frac{A}{s} + \frac{Bs + C}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$0<\zeta<1$:

$\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore\ A\left[(s+\zeta\omega_n)^2 + \omega_d^2\right] + Bs^2 + Cs = \omega_n^2$$

Put $s=0$, $A(\zeta^2\omega_n^2 + \omega_d^2) = \omega_n^2$

$A(\zeta^2\omega_n^2 + \omega_n^2(1-\zeta^2))] = \omega_n^2 \quad \Rightarrow \quad \boxed{A=1}$

Put $s = -\zeta\omega_n$, $\omega_n^2(1-\zeta^2) + B\zeta^2\omega_n^2 - C\zeta\omega_n = \omega_n^2$

$\omega_n^2 - \omega_n^2\zeta^2 + B\zeta^2\omega_n^2 - C\zeta\omega_n = \omega_n^2$

$\Rightarrow \quad \underline{C = B\zeta\omega_n - \zeta\omega_n}$ (1)

Put $s = +\zeta\omega_n$; $4\zeta^2\omega_n^2 + \omega_n^2(1-\zeta^2) + B\zeta\omega_n^2 + C\zeta\omega_n = \omega_n^2$

$\Rightarrow \quad \underline{C = -B\zeta\omega_n - 3\zeta\omega_n}$ (2)

(1) = (2), $\boxed{B=-1}$, $\boxed{C = -2\zeta\omega_n}$

$$\therefore\ C(s) = \frac{1}{s} + \frac{-s - 2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

Underdamped case (Continued) $\quad 0<\zeta<1, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \quad C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \underbrace{\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}}_{\text{rule 16}} - \left(\frac{\zeta\omega_n}{\omega_d}\right)\underbrace{\left(\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right)}_{\text{rule 15}}$$

$$\mathcal{L}^{-1}[C(s)] = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$\therefore \quad \boxed{c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t}$$

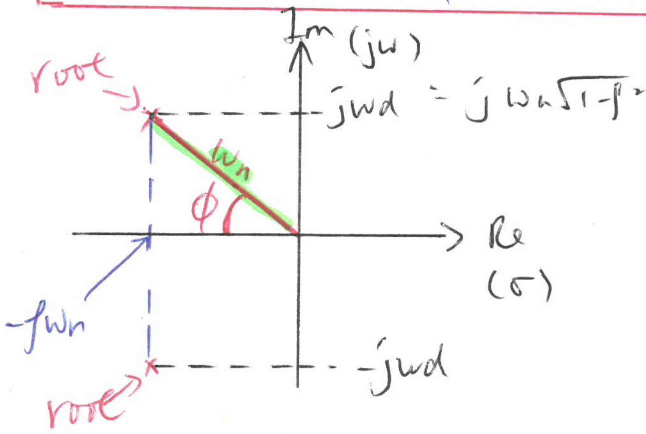
Another representation (shown on p. 21)

OR

$$C(s) = \frac{1}{s}\,\frac{\omega_n^2}{(s+\zeta\omega_n+j\omega_d)(s^2+\zeta\omega_n-j\omega_d)} = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n+\omega_n^2)}$$

Rule 19

$$\mathcal{L}^{-1}[C(s)] = \boxed{c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \phi\right) \qquad \phi = \cos^{-1}\zeta}$$



$$r^2 = (\zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2$$

$$r^2 = (\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)$$

$$r^2 = (\zeta\omega_n)^2 + \omega_n^2 - (\omega_n\zeta)^2$$

$$r^2 = \omega_n^2$$

Critically damped case (p.16) $\boxed{\text{*Unit-Step Response*}}$

$$C(s) = \frac{1}{s}\frac{W_n^2}{(s+W_n)^2} = \frac{A}{s} + \frac{B}{(s+W_n)^2} + \frac{C}{s+W_n}$$

$$\Rightarrow A(s+W_n)^2 + Bs + Cs(s+W_n) = W_n^2$$

Put $s=0$, $\quad AW_n^2 = W_n^2 \Rightarrow \boxed{A=1}$

Put $s=-W_n$, $\quad -BW_n = W_n^2 \Rightarrow \boxed{B=-W_n}$

Put $s=+W_n$, $\quad 4W_n^2 - W_n^2 + C(2W_n^2) = W_n^2 \Rightarrow \boxed{C=-1}$

$$\therefore C(s) = \frac{1}{s} + \underbrace{\frac{-W_n}{(s+W_n)^2}}_{\text{rule 6}} + \underbrace{\frac{-1}{s+W_n}}_{\text{rule 5}}$$

$$\mathcal{L}^{-1}[C(s)] = c(t) = 1 - W_nt\,e^{-W_nt} - e^{-W_nt}$$

$$\boxed{c(t) = 1 - e^{-W_nt}(1+W_nt)}$$

$$C(s) = \frac{W_n^2}{s(s+W_n)^2} \qquad \text{Rule 13}$$

$$\mathcal{L}^{-1}[C(s)] = c(t) = \frac{1}{W_n^2}(W_n^2)\left(1 - e^{-W_nt} - W_nt\,e^{-W_nt}\right)$$

$$\boxed{c(t) = 1 - e^{-W_nt}(1+W_nt)}$$

Over damped Case (p.17) $\boxed{\text{*Unit-Step Response*}}$

$$C(s) = \frac{1}{s}\frac{\omega_n^2}{(s+\underbrace{\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}}_{a})(s+\underbrace{\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}}_{b})} = \frac{\omega_n^2}{s(s+a)(s+b)}$$

Simplify the Computation

$$\therefore \quad C(s) = \frac{A}{s} + \frac{B}{s+a} + \frac{C}{s+b}$$

$$A(s+a)(s+b) + Bs(s+b) + Cs(s+a) = \omega_n^2$$

Put $s = 0$, $\quad A(ab) = \omega_n^2$

$$A(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})(\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}) = \omega_n^2$$

$$A(\zeta^2\omega_n^2 - \omega_n^2(\zeta^2-1)) = \omega_n^2 \quad \Rightarrow \quad \boxed{A = 1}$$

Put $s = -a$, $\quad B(-a)(-a+b) = \omega_n^2$

$$B = \frac{\omega_n^2}{(-a)(-a+b)}$$

$$B = \frac{\omega_n^2}{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1})}$$

$$= \frac{\omega_n^2}{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})(-2\omega_n\sqrt{\zeta^2-1})}$$

$$= \frac{\omega_n^2}{2\zeta\omega_n^2\sqrt{\zeta^2-1} + 2\omega_n^2(\sqrt{\zeta^2-1})^2}$$

B

$$\boxed{= \frac{1}{2\sqrt{\zeta^2-1}(\zeta+\sqrt{\zeta^2-1})}}$$

Overdamped case (continued)

Put $s=-b$, $\quad C(-b)(-b+a)=\omega_n^2 \quad ; \quad C=\dfrac{\omega_n^2}{(-b)(-b+a)}$

$$C=\frac{\omega_n^2}{(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})}$$

$$=\frac{\omega_n^2}{(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(2\omega_n\sqrt{\zeta^2-1})}$$

$$=\frac{\omega_n^2}{-2\zeta\omega_n^2\sqrt{\zeta^2-1}+2\omega_n^2(\sqrt{\zeta^2-1})^2}\;\boxed{=\frac{1}{2\sqrt{\zeta^2-1}(\sqrt{\zeta^2-1}-\zeta)}}$$

C

$$\therefore\ C(s)=\frac{1}{s}+\frac{B}{s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}}+\frac{C}{s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}}$$

$$=\frac{1}{s}+\frac{B}{s+\omega_n(\zeta+\sqrt{\zeta^2-1})}+\frac{C}{s+\omega_n(\zeta-\sqrt{\zeta^2-1})}$$

rule 5

$$\mathcal{L}^{-1}[C(s)]=c(t)=1+Be^{-\omega_n(\zeta+\sqrt{\zeta^2-1})t}+Ce^{-\omega_n(\zeta-\sqrt{\zeta^2-1})t}$$

$$\boxed{=1+\frac{1}{2\sqrt{\zeta^2-1}(\zeta+\sqrt{\zeta^2-1})}e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t}+\frac{1}{2\sqrt{\zeta^2-1}(\sqrt{\zeta^2-1}-\zeta)}e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t}}$$

SEHS4653 Control System Analysis

"Unit 3 Transient and Steady-State Response Analysis" Supplementary Information

Underdamped case (p.28)

Unit – Impulse Response

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$R(s) = 1 \quad [\because r(t) = \delta(t)]$

$$\therefore \quad C(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Rule 17

$$\mathcal{L}^{-1}[C(s)] = \boxed{c(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n\sqrt{1-\zeta^2}\, t}$$

Critically damped case (p.28)

Unit – Impulse Response

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2} \qquad \because \zeta = 1$$

$R(s) = 1$

Rule 6.

$$\mathcal{L}^{-1}[C(s)] = (\omega_n^2)\, t\, e^{-\omega_n t}$$

$$\therefore \quad \boxed{c(t) = \omega_n^2\, t\, e^{-\omega_n t}}$$

SEHS4653 Control System Analysis

"Unit 3 Transient and Steady-State Response Analysis" Supplementary Information

Overdamped case (p.28)

* Unit-Impulse Response *

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \underbrace{\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}}_{a})(s + \underbrace{\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}}_{b})}$$

$R(s) = 1$

2 distinct real roots $(\zeta > 1)$

$$\therefore \quad C(s) = \frac{\omega_n^2}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}$$

$$A(s+b) + B(s+a) = \omega_n^2$$

Put $s = -a$, $\quad A(-a+b) = \omega_n^2 \quad \Rightarrow \quad A = \frac{\omega_n^2}{-a+b}$

$$A = \frac{\omega_n^2}{-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}} = \boxed{\frac{\omega_n}{-2\sqrt{\zeta^2-1}}}$$

Put $s = -b$, $\quad B(-b+a) = \omega_n^2 \quad \Rightarrow \quad B = \frac{\omega_n^2}{-b+a}$

$$B = \frac{\omega_n^2}{-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}} = \boxed{\frac{\omega_n}{2\sqrt{\zeta^2-1}}}$$

$$\therefore \quad C(s) = \frac{A}{s+a} + \frac{B}{s+b}$$

Rule 5

$$\mathcal{L}^{-1}[C(s)] = c(t) = Ae^{-at} + Be^{-bt}$$

$$\boxed{c(t) = \frac{-\omega_n}{2\sqrt{\zeta^2-1}} e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}} e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t}}$$