SEHS4653
Control System Analysis

Unit 2
Mathematical Modelling of Dynamic Systems

(Reference: [1] chapter 2-1 to 2-3 and 3; [2] chapter 3-2)

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- Transfer Function and Impulse-Response Function
- Block Diagrams
- Modelling of Automatic Controllers
- Modelling of Mechanical Systems
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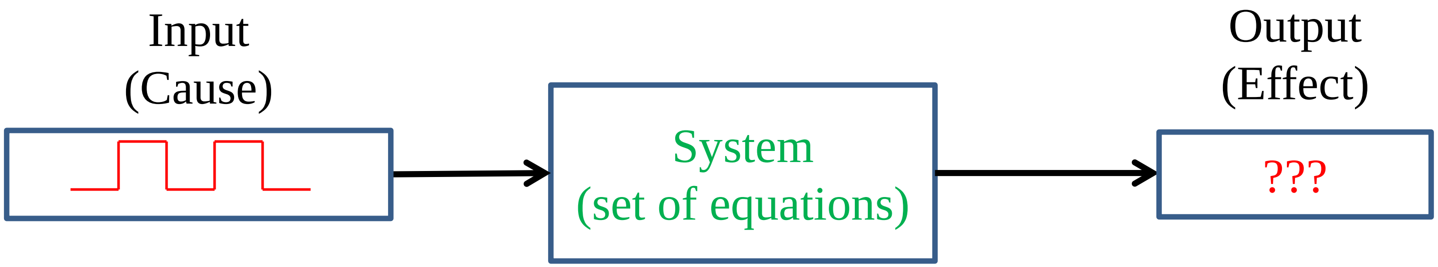
Introduction

- In studying control systems it must be able to model dynamic systems in *mathematical terms* and analyze their *dynamic characteristics*
- A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the system accurately, or at least fairly well
- The dynamics of many systems, whether they are mechanical, electrical, thermal, economic, biological, and so on, may be described in terms of differential equations
- Throughout the subject we assume that the *principle of causality* applies to the systems considered



Introduction

- Mathematical Models
  - May assume many different forms, depending on the particular system and the particular circumstances
  - In optimal control problems, it is advantageous to use state-space representations [Unit 7]
  - For the transient-response or frequency-response analysis of single-input, single-output, linear, time-invariant systems, the transfer-function representation [this Unit] may be more convenient

Introduction

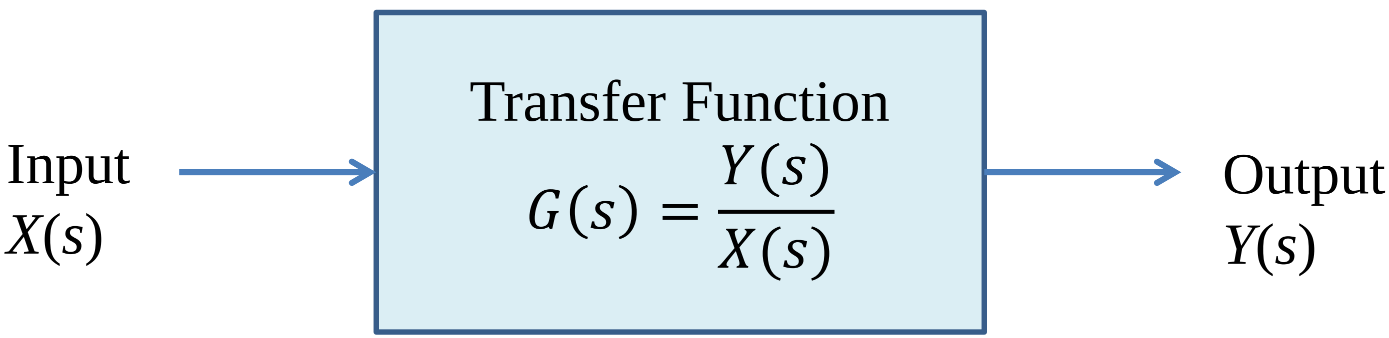
- Linear Systems
  - The *principle of superposition* applies that the response produced by the simultaneous application of two different forcing functions is the sum of the two individual responses
  - In an experimental investigation of a dynamic system, if cause and effect are proportional, thus implying that the principle of superposition holds

- Linear Time-Invariant Systems and Linear Time-Varying Systems
  - A differential equation is *linear* if the coefficients are constants or functions only of the independent variable
  - Linear time-invariant differential equations are constant-coefficient differential equations. Such systems are called linear time-invariant (LTI) systems
  - Systems that are represented by differential equations whose coefficients are functions of time are called linear time-varying (LTV) systems.

Transfer Function and Impulse-Response Function

Transfer Function

- commonly used to characterize the input-output relationships of components or systems that can be described by linear, time-invariant, differential equations
- is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero



Transfer Function and Impulse-Response Function

Consider the linear time-invariant system defined by the following differential equation,

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y^{(1)} + a_n y = b_0 x^{(m)} + b_1 x^{(m-1)} + \cdots + b_{m-1} x^{(1)} + b_m x, (n \geq m)$$

where *y* is the output and *x* is the input of the system

$$\text{Transfer fucntion} = G(s) = \left.\frac{\mathcal{L}[\text{output}]}{\mathcal{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$

$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

- Highest power of *s* in the denominator of the transfer function is equal to *n*, the system is called an *nth-order system*

Transfer Function and Impulse-Response Function

Comments on Transfer Function

1. It is a mathematical model expressed in differential equation that relates the output variable to the input variable

2. It is a property of a system itself, independent of the magnitude and nature of the input or driving function

3. It includes the units necessary to relate the input to the output; however, it does not provide any information concerning the physical structure of the system.

4. It is used to determine the output (or response) from different kind of inputs

5. If the transfer function of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system. Once established, a transfer function gives a full description of the dynamic characteristics of the system, as distinct from its physical description $\Rightarrow$ System Identification (outside the scope of this subject)

Transfer Function and Impulse-Response Function

Impulse-Response Function, $\delta(t)$

- Consider the output (response) of a linear time-invariant system (LTI) to a unit-impulse input when the initial conditions are zero. Since the Laplace transform of the unit-impulse function is unity, the Laplace transform of the output of the system is,

$$Y(s) = G(s)X(s) = G(s)$$

- The inverse Laplace transform of output gives the impulse response of the system,

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[G(s)] = g(t)$$

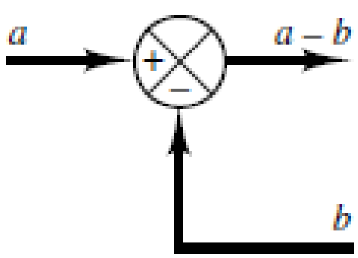
- The function $g(t)$ is also called the weighting function of the system
- The Laplace transform of the *impulse-response function* gives the transfer function
- It is possible to obtain complete information about the dynamic characteristics of the system by exciting it with an impulse input and measuring the response

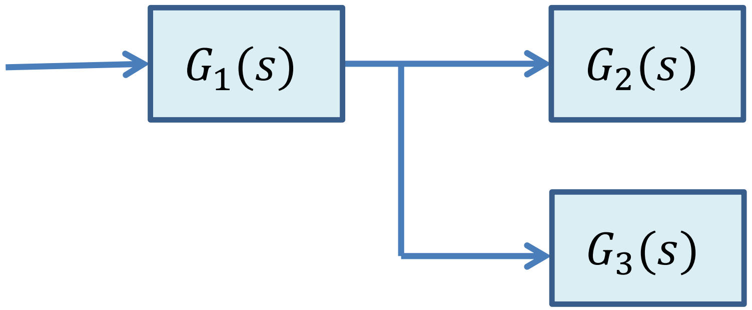
Block Diagrams

- A control system may consist of a number of components which is commonly represented by block diagram
- A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals
- In a block diagram all system variables are linked to each other through functional blocks
- The functional block (or simply block) is a symbol for the mathematical operation on the input signal to the block that produces the output
- The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals $\Rightarrow$ a *unilateral property*

Block Diagrams

- *Summing Point*: A circle with a cross is the symbol that indicates a summing operation. The plus or minus sign at each arrowhead indicates whether that signal is to be added or subtracted.
- *Branch Point*: A point from which the signal from a block goes concurrently to other blocks or summing points.

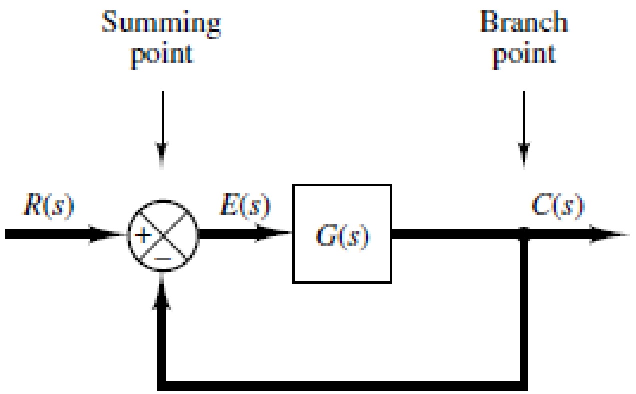


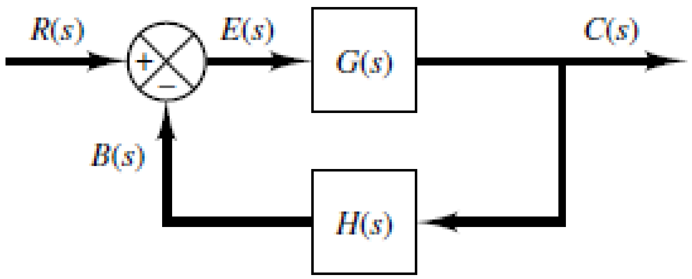


Block Diagrams

Block diagram of a Closed-loop System

- The output $C(s)$ is fed back to the summing point, where it is compared with the reference input $R(s)$.
- $C(s) =$
- It is necessary to convert the form of the output signal to that of the input signal (same dimension or unit)
- This conversion is accomplished by the feedback element whose transfer function is $H(s)$
- The feedback signal that is fed back to the summing point for comparison with the input is, $B(s) = H(s)C(s)$





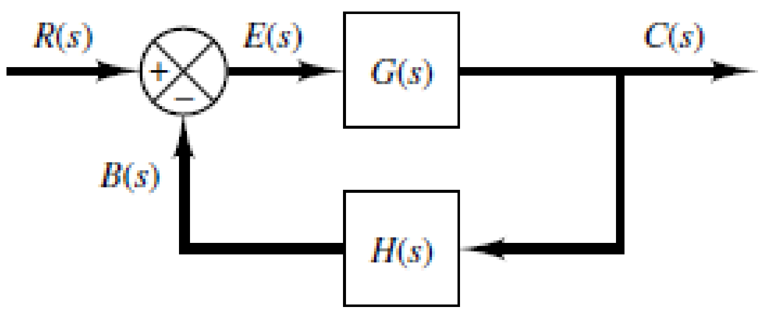
Block Diagrams

Open-Loop Transfer Function & Feedforward Transfer Function

B(*s*): Feedback Signal; *E*(*s*): Actuating Error Signal

$$\text{Open} - \text{loop transfer function} = \frac{B(s)}{E(s)} = G(s)H(s)$$

$$\text{Feedforward transfer function} = \frac{C(s)}{E(s)} = G(s)$$



If the feedback transfer function *H*(*s*) is unity, then the open-loop transfer function and the feedforward transfer function are the same

Block Diagrams

Closed-Loop Transfer Function

$$\text{Closed} - \text{loop transfer function} = \frac{\mathcal{L}[\text{Output}]}{\mathcal{L}[\text{Input}]} = \frac{C(s)}{R(s)}$$

$$C(s) = G(s)E(s)$$

$$E(s) = R(s) - B(s) = R(s) - H(s)C(s)$$

$$C(s) = G(s)[R(s) - H(s)C(s)]$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

From the above closed-loop transfer function,

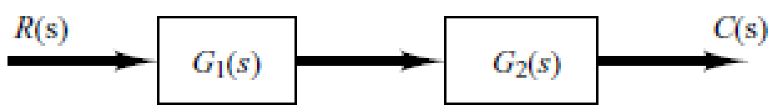
$$C(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)$$

Example 1

Determine the transfer function $C(s) / R(s)$ of the below systems

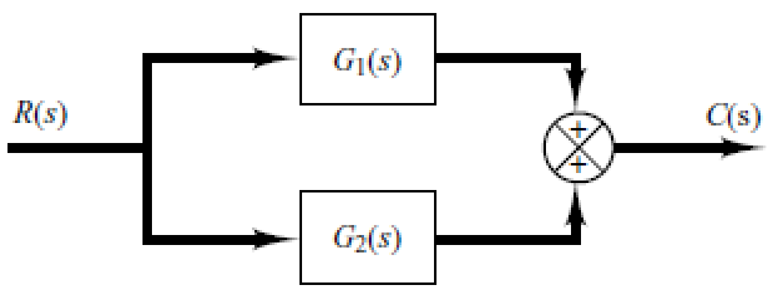
Cascaded

(a)



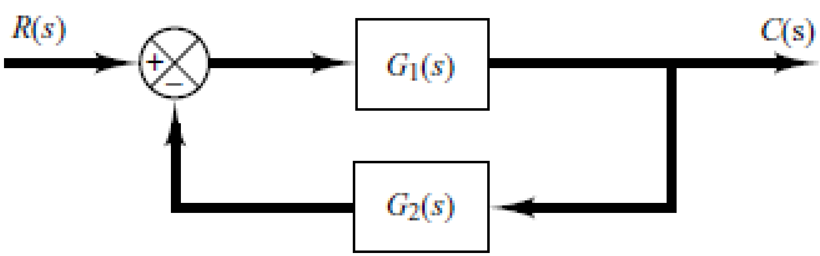
Parallel

(b)



Feedback

(c)



<u>Answer:</u>

$$\frac{C(s)}{R(s)} = G_1(s)G_2(s)$$

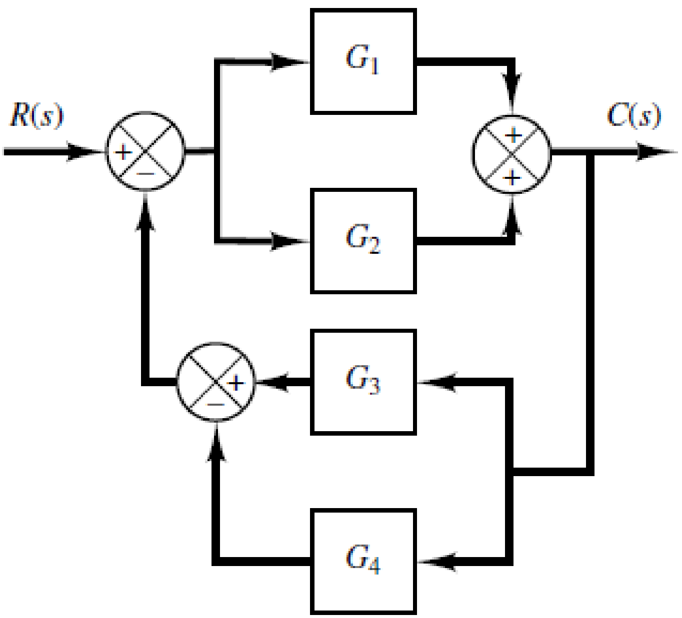
$$C(s) = R(s)G_1(s) + R(s)G_2(s)$$

$$\frac{C(s)}{R(s)} = G_1(s) + G_2(s)$$

$$\frac{C(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

Example 2

Obtain the closed-loop transfer function $C(s) / R(s)$.



Answer:

$$\frac{C(s)}{R(s)} = \frac{G_1 + G_2}{1 + (G_1 + G_2)(G_3 - G_4)}$$

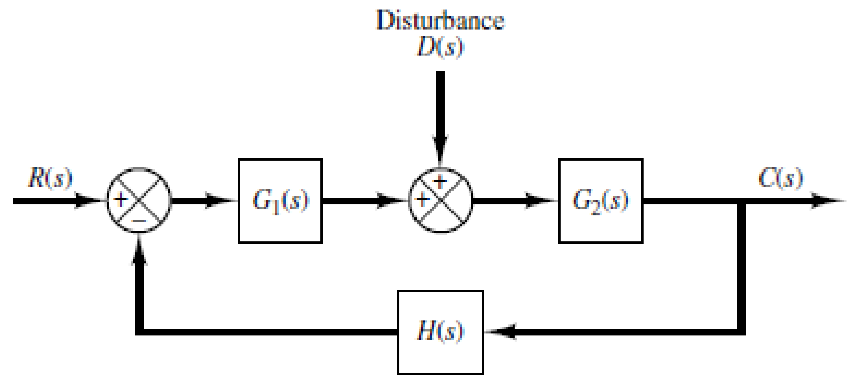
Block Diagrams

Closed-loop system subjected to a disturbance

- Apply superposition,

$$\frac{C_D(s)}{D(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$



- The response to the simultaneous application of the reference input and disturbance can be obtained by adding the two individual responses, $C(s) = C_R(s) + C_D(s)$

Block Diagrams

Procedures for drawing a block diagram

1. Write the equations that describe the dynamic behavior of each component

2. Then take the Laplace transforms of these equations, assuming zero initial conditions

3. Represent each Laplace-transformed equation individually in block form

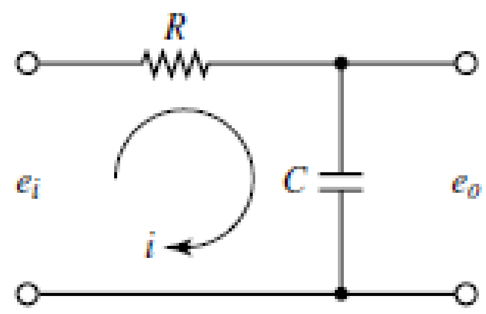
4. Assemble the elements into a complete block diagram.

Block Diagrams

Consider the *RC* circuit:

1. Write the equations

$$i = \frac{e_i - e_0}{R} \quad \text{and} \quad e_0 = \frac{1}{C}\int i dt$$

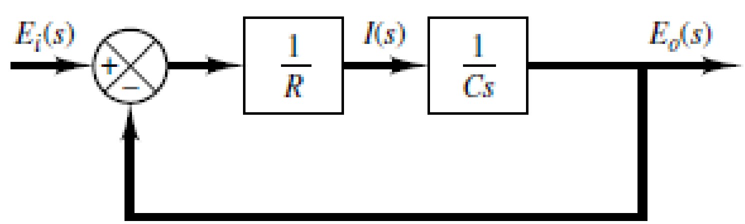


2. Taking Laplace Transform

$$I(s) = \frac{E_i(s) - E_0(s)}{R} \text{ and } E_0 = \frac{I(s)}{sC}$$

3. Represent in block form

4. Assemble all the blocks



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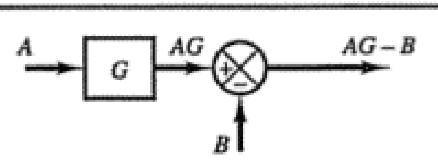
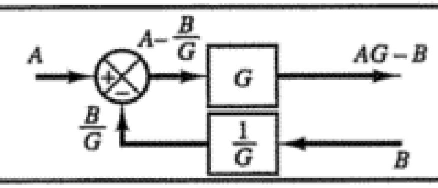
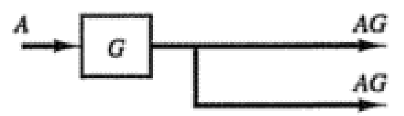
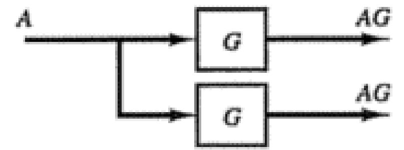
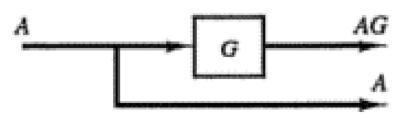
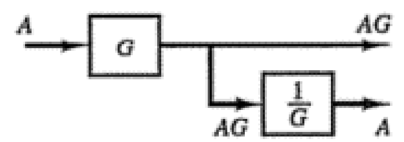
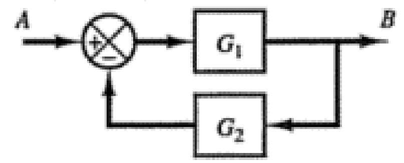
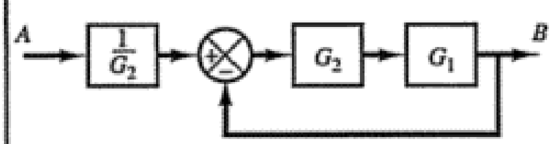
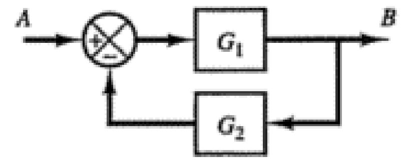
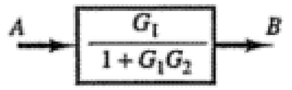
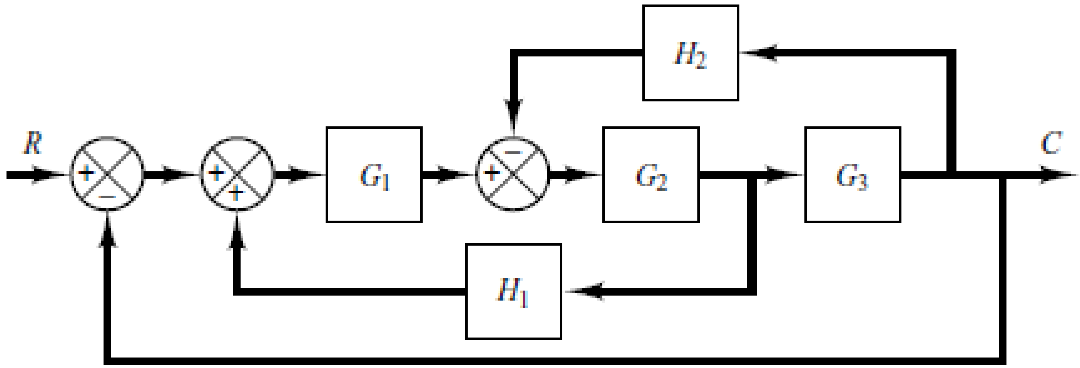
Block Diagrams

Block Diagram Reduction

	Original Block Diagrams	Equivalent Block Diagrams
1		
2		
3		
4		
5		

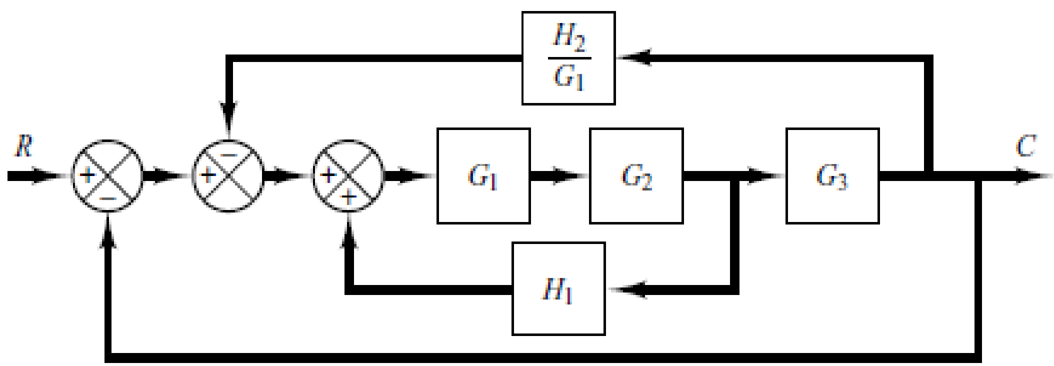
Example 3

Simplify the below block diagram.



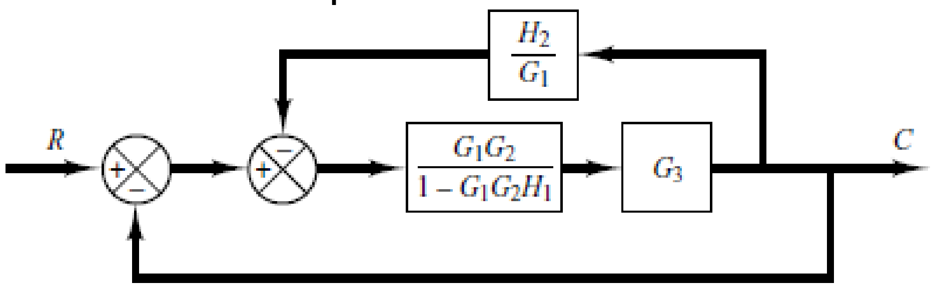
Answer:

Moving the summing point of the negative feedback loop containing H_2 outside the positive feedback loop containing H_1

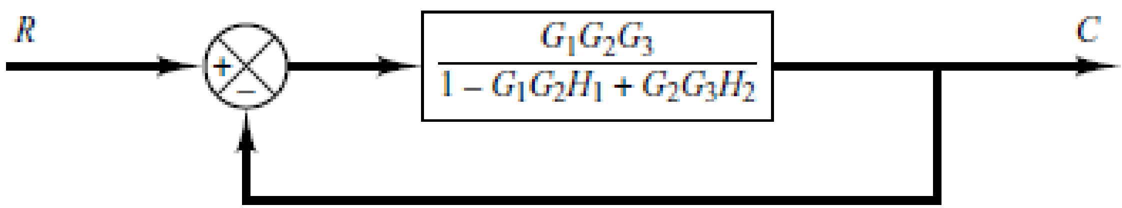


Example 3

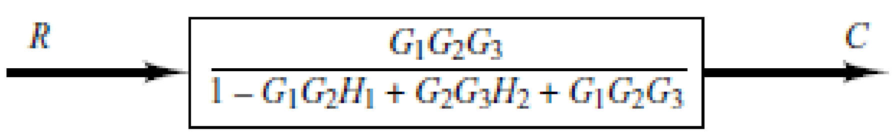
Eliminating the positive feedback loop



Eliminating the loop containing H_2 / G_1 gives,



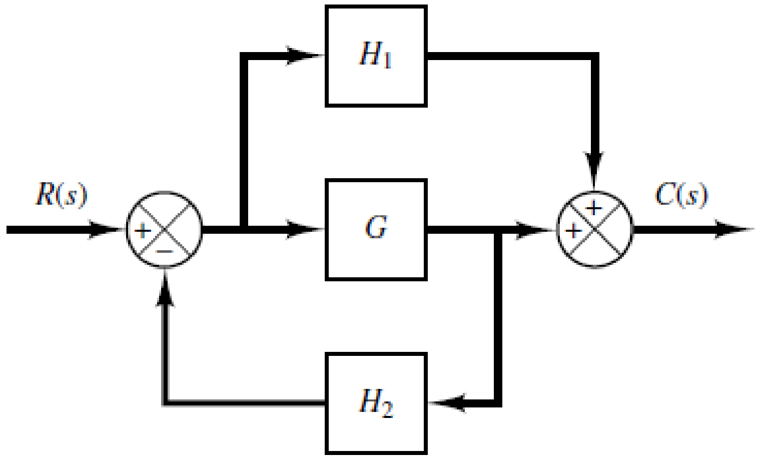
Finally, eliminating the feedback loop results,



Example 4

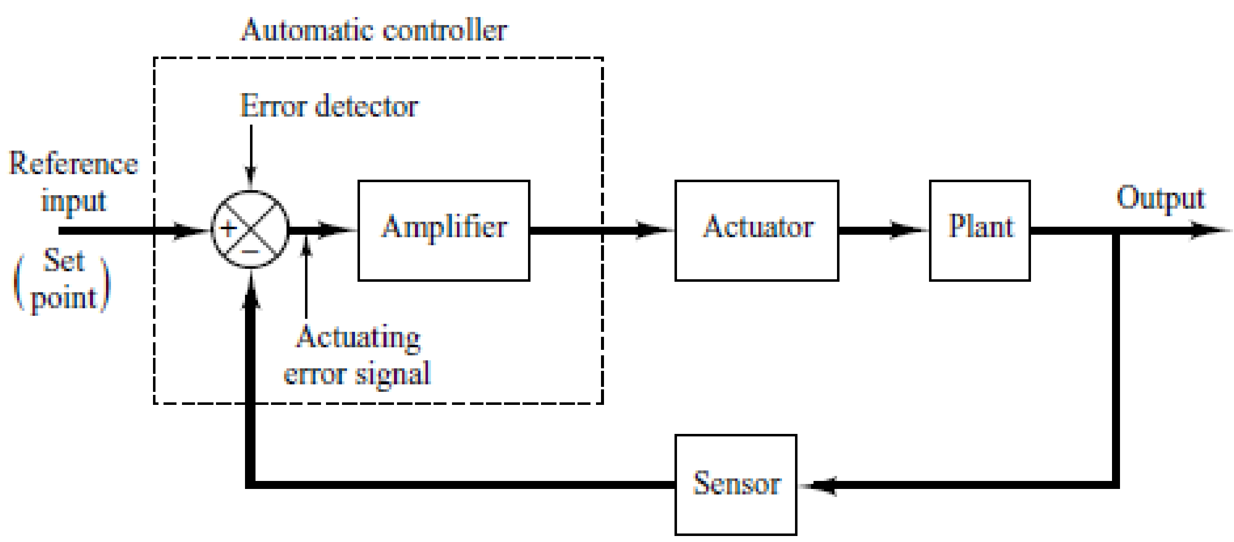
Simply the below block diagram.

<u>Answer:</u> $\frac{C(s)}{R(s)} = \frac{G + H_1}{1 + GH_2}$



Modelling of Automatic Controllers

- An automatic controller compares the actual value of the plant output with the reference input (desired value), determines the deviation, and produces a control signal that will reduce the deviation to zero or to a small value



Automatic Controllers

- Classifications of Industrial Controllers
    1. Two-position or on–off controllers
    2. Proportional (P) controllers
    3. Integral (I) controllers
    4. Proportional-plus-integral (PI) controllers
    5. Proportional-plus-derivative (PD) controllers
    6. Proportional-plus-integral-plus-derivative (PID) controllers

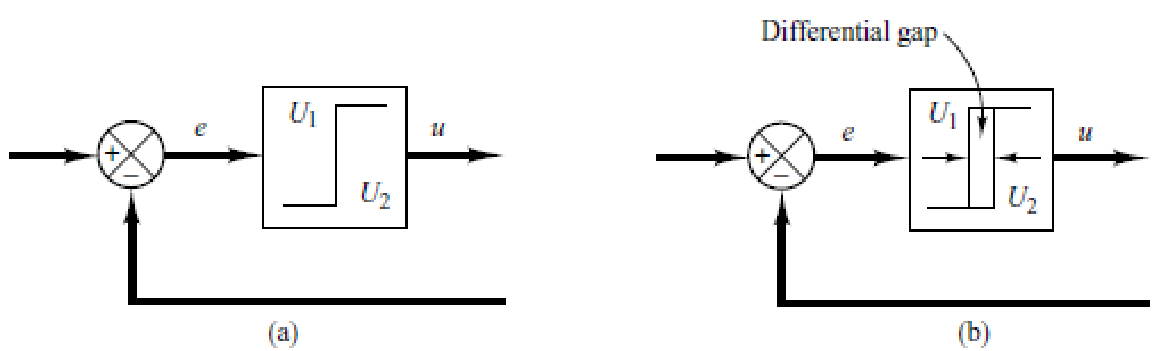
- Controllers may also be classified according to the kind of power employed in the operation, such as pneumatic controllers, hydraulic controllers, or electronic controllers

Automatic Controllers

- Two-Position or On-Off Control Action
  - The actuating element has only two fixed positions, i.e. ON and OFF
  - It is relatively simple and inexpensive and, for this reason, is very widely used in both industrial and domestic control systems
  - Let the output signal from the controller be $u(t)$ and the actuating error signal be $e(t)$, so that

$$u(t)\begin{cases} = U_1, & \text{for } e(t) > 0 \\ = U_2, & \text{for } e(t) < 0 \end{cases}$$

Where U_1 and U_2 are constants



Automatic Controllers

- Proportional Control Action
  - For a controller with proportional control action, the relationship between the output of the controller $u(t)$ and the actuating error signal $e(t)$ is,

$$u(t) = K_p e(t)$$

  - or, in Laplace-transformed quantities,

$$\frac{U(s)}{E(s)} = K_p$$

where K_p is termed the proportional gain

  - Whatever the actual mechanism may be and whatever the form of the operating power, the proportional controller is essentially an amplifier with an adjustable gain

Automatic Controllers

- Integral Control Action
  - In a controller with integral control action, the value of the controller output $u(t)$ is changed at a rate proportional to the actuating error signal $e(t)$. That is,

$$\frac{d}{dt}u(t) = K_i e(t) \quad \text{or} \quad u(t) = K_i \int_0^t e(t)dt$$

where K_i is an adjustable constant

  - The transfer function of the integral controller is,

$$\frac{U(s)}{E(s)} = \frac{K_i}{s}$$

Automatic Controllers

- Proportional-Plus-Integral Control Action
  - The control action of a proportional-plus-integral controller is defined by

$$u(t) = K_p e(t) + \frac{K_p}{T_i}\int_0^t e(t)dt$$

where T_i is called the integral time

  - The transfer function of the controller is,

$$\frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s}\right)$$

Automatic Controllers

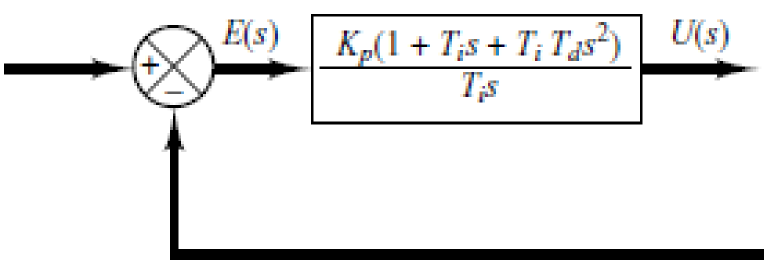
- Proportional-Plus-Integral-Plus-Derivative Control Action
  - The equation of a controller with this combined action is given by

$$u(t) = K_p e(t) + \frac{K_p}{T_i}\int_0^t e(t)dt + K_p T_d \frac{d}{dt} e(t)$$

where T_d is called the derivative time

  - The transfer function of the controller is,

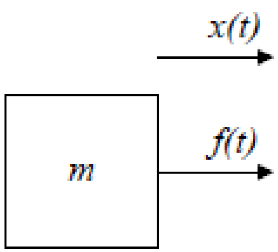
$$\frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$



Modelling of Mechanical Systems

- The fundamental law governing mechanical systems is Newton's second law: $ma = \sum F$, where m is the mass, a is the acceleration
- Linear Translation Motion: mass, linear spring, damper (provides viscous friction)

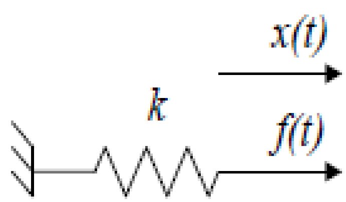
Mass



$$f(t) = m\frac{d^2}{dt^2}x(t)$$

$$F(s) = ms^2X(s)$$

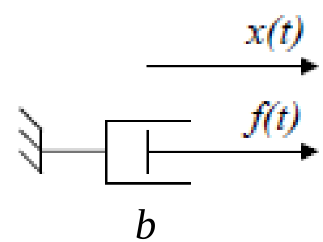
Linear Spring



$$f(t) = kx(t)$$

$$F(s) = kX(s)$$

Damper



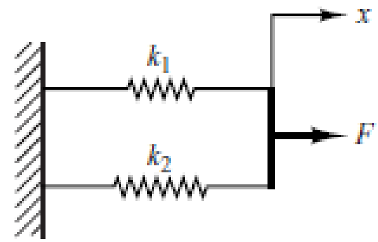
$$f(t) = b\frac{d}{dt}x(t)$$

$$F(s) = bsX(s)$$

Modelling of Mechanical Systems

Simple Spring Systems

- Springs in Parallel
  - The equivalent spring constant k_{eq} is obtained from $k_1x + k_2x = F = k_{eq}x$

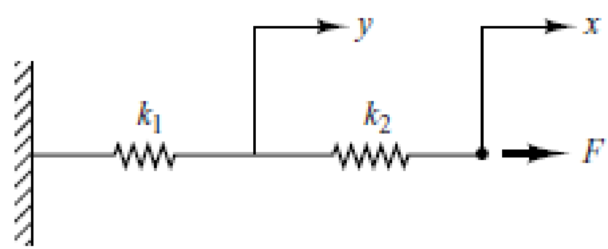


$$\Longrightarrow k_{eq} = k_1 + k_2$$

- Springs in Series
  - The force in each spring is the same, thus $k_1y = F, \quad k_2(x-y) = F$
  - Elimination of y from these two equations results in,

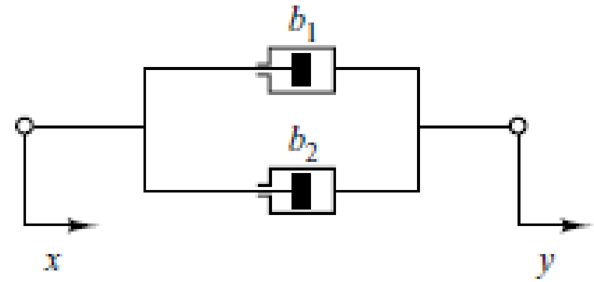
$$k_2\left(x - \frac{F}{k_1}\right) = F \quad or \quad k_2x = F + \frac{k_2}{k_1}F = \frac{k_1+k_2}{k_1}F$$

  - The equivalent spring constant k_{eq} for this case,



$$k_{eq} = \frac{F}{x} = \frac{k_1k_2}{k_1+k_2} = \frac{1}{\frac{1}{k_1}+\frac{1}{k_2}}$$

Modelling of Mechanical Systems



Simple Damper Systems

- Dampers in Parallel
  - The force f due to the dampers is,

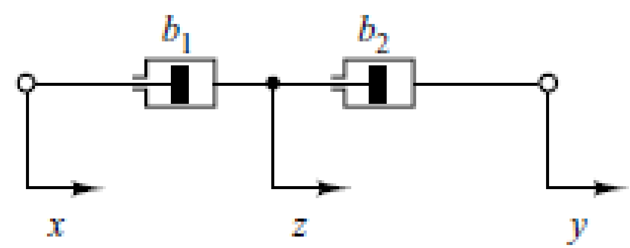
    $f = b_1(\dot{y}-\dot{x}) + b_2(\dot{y}-\dot{x}) = (b_1+b_2)(\dot{y}-\dot{x})$
  - The equivalent viscous-friction coefficient b_{eq} , $b_{eq} = b_1 + b_2$

- Dampers in Series
  - The force f due to the dampers, $f = b_1(\dot{z}-\dot{x}) = b_2(\dot{y}-\dot{z})$

    where z is the displacement of a point between damper b_1 and damper b_2
  - We have, $(b_1+b_2)\dot{z} = b_2\dot{y} + b_1\dot{x}$ $or\ \dot{z} = \frac{1}{b_1+b_2}(b_2\dot{y}+b_1\dot{x})$

$$f = b_2(\dot{y}-\dot{z}) = b_2\left[\dot{y} - \frac{1}{b_1+b_2}(b_2\dot{y}+b_1\dot{x})\right] = \frac{b_1 b_2}{b_1+b_2}(\dot{y}-\dot{x})$$

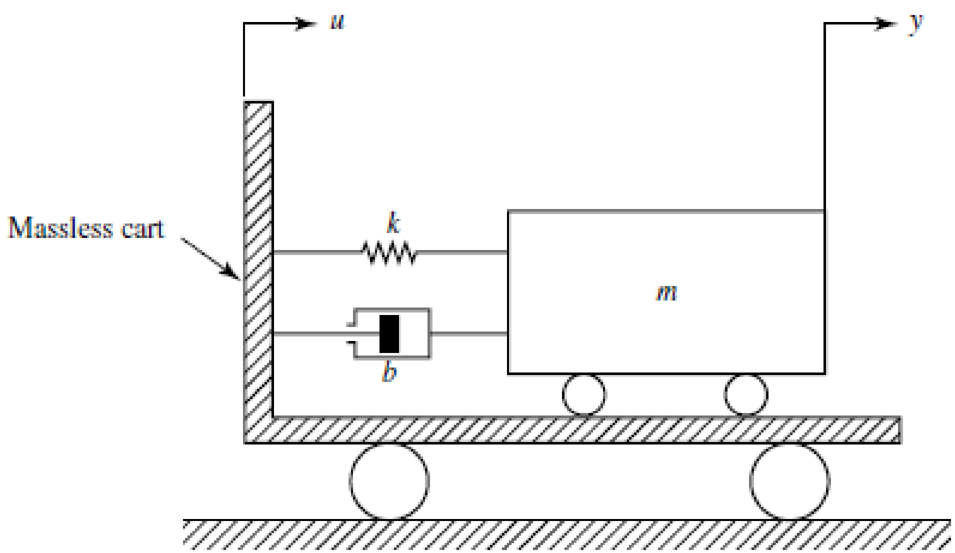
$$b_{eq} = \frac{f}{\dot{y}-\dot{x}} = \frac{b_1 b_2}{b_1+b_2} = \frac{1}{\frac{1}{b_1}+\frac{1}{b_2}}$$



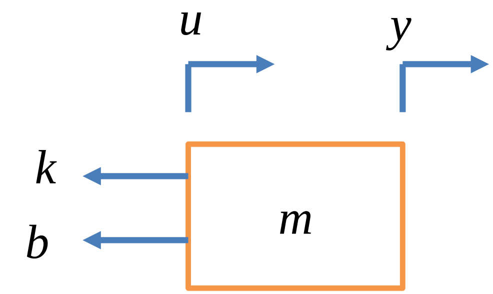
Example 5

Spring-Mass-Damper System

Consider the spring-mass-damper system mounted on a massless cart as shown below. In this system, $u(t)$ is the displacement of the cart and is the input to the system; and the displacement $y(t)$ of the mass is the output (the displacement is relative to the ground). In this system, m denotes the mass, b denotes the viscous-friction coefficient, and k denotes the spring.



Example 5

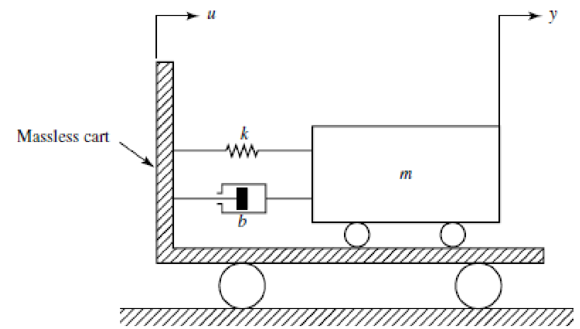


For translational systems, Newton's second law states that

$$ma = \sum F$$

Noting that the cart is massless, we obtain,

$$m\frac{d^2}{dt^2}y(t) = -b\left(\frac{d}{dt}y(t) - \frac{d}{dt}u(t)\right) - k[y(t) - u(t)]$$



Or $\quad m\ddot{y}(t) + b\dot{y}(t) + ky = b\dot{u}(t) + ku(t)$

Taking the Laplace transform and assuming zero initial condition, gives

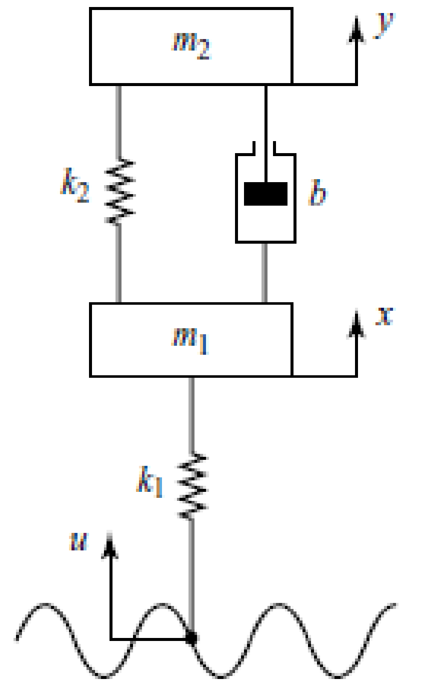
$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

Hence, the transfer function,

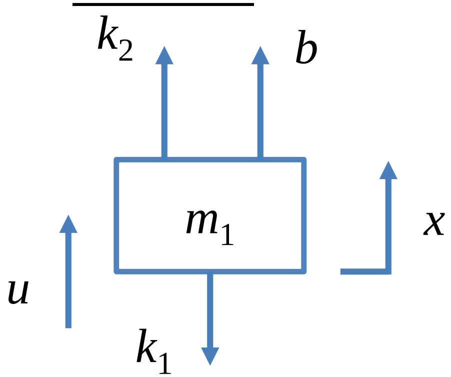
$$G(s) = \frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Example 6

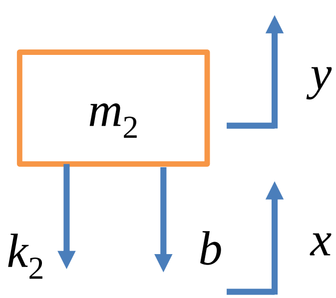
Obtain the transfer function $Y(s) / U(s)$ of the system below. The input u is a displacement input. (This is also a simplified version of an automobile or motorcycle suspension system.)



Answer:



$$m_1\ddot{x} = k_2(y - x) + b(\dot{y} - \dot{x}) - k_1(x - u)$$



$$m_2\ddot{x} = -k_2(y - x) - b(\dot{y} - \dot{x})$$

Example 6

Answer:

Rearranging the terms, we have

$m_1\ddot{x} + b\dot{x} + k_1 x + k_2 x = k_1 u + b\dot{y} + k_2 y$ and $m_2\ddot{y} + b\dot{y} + k_2 y = b\dot{x} + k_2 x$

Taking Laplace Transform (with zero initial condition),

$$m_1 s^2 X(s) + bsX(s) + k_1 X(s) + k_2 X(s) = k_1 U(s) + bsY(s) + k_2 Y(s)$$

$$m_2 s^2 Y(s) + bsY(s) + k_2 Y(s) = bsX(s) + k_2 X(s) \Rightarrow \frac{m_2 s^2 + bs + k_2}{bs + k_2} Y(s) = X(s)$$

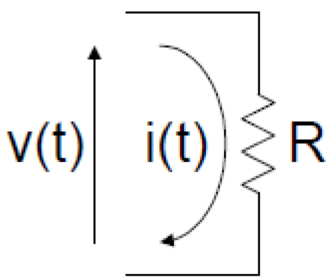
$$(m_1 s^2 + bs + k_1 + k_2)\left(\frac{m_2 s^2 + bs + k_2}{bs + k_2}\right) Y(s) = k_1 U(s) + [bs + k_2] Y(s)$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{k_1(bs + k_2)}{(m_1 s^2 + bs + k_1 + k_2)(m_2 s^2 + bs + k_2) - (bs + k_2)^2}$$

Modelling of Electrical Systems

- Basic laws governing electrical circuits are Kirchhoff's current law and voltage law
- Three basic elements: Resistor (*R*), Inductor (*L*) and Capacitor (*C*)

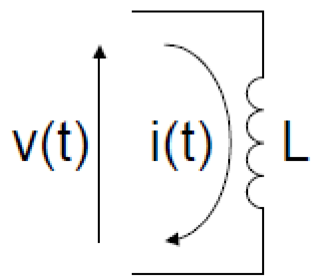
Resistor



$$v(t) = Ri(t)$$

$$V(s) = RI(s)$$

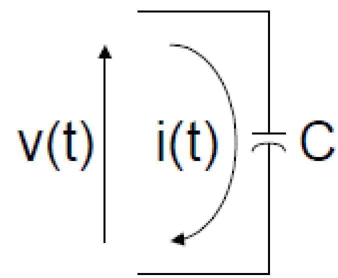
Inductor



$$v(t) = L\frac{d}{dt}i(t)$$

$$V(s) = sLI(s)$$

Capacitor



$$v(t) = \frac{1}{C}\int i(t)dt$$

$$V(s) = \frac{I(s)}{sC}$$

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Modelling of Electrical Systems

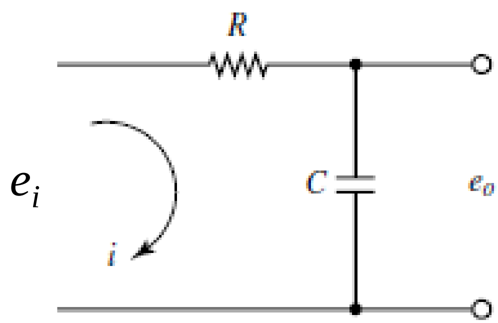
RC Circuit

Consider the electrical circuit.

Applying Kirchhoff's voltage law to the system, we obtain,

$$i(t)R + \frac{1}{C}\int i(t)dt = e_i(t)$$

$$\frac{1}{C}\int i(t)dt = e_0(t)$$



Taking Laplace transform with zero initial conditions,

$$RI(s) + \frac{I(s)}{sC} = E_i(s) \ \text{ and } \ \frac{I(s)}{sC} = E_0(s)$$

Then the transfer function of this system is,

$$\frac{E_0(s)}{E_i(s)} = \frac{\frac{1}{sC}I(s)}{\left[R + \frac{1}{sC}\right]I(s)} = \frac{1}{1 + sRC}$$

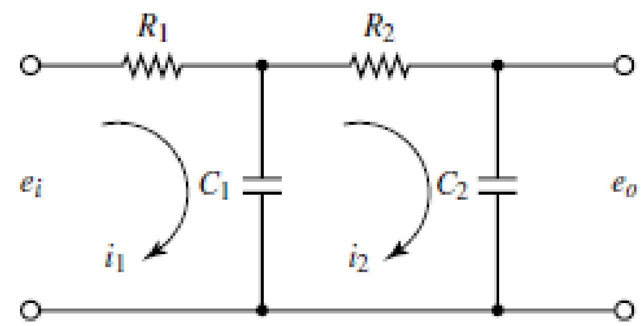
Modelling of Electrical Systems

Transfer Functions of Cascaded Elements (with loading effect)

Consider the system as shown below.

Assume that $e_i(t)$ is the input and $e_o(t)$ is the output.

The capacitances C_1 and C_2 are not charged initially.



The equations for this systems are,

$$\frac{1}{C_1}\int (i_1(t) - i_2(t))dt + R_1 i_1 = e_i(t)$$

$$\frac{1}{C_1}\int (i_1(t) - i_2(t))dt - R_2 i_2 - \frac{1}{C_2}\int i_2(t)\,dt = 0$$

$$\frac{1}{C_2}\int i_2(t)\,dt = e_0(t)$$

Laplace Transform

$$\frac{1}{sC_1}[I_1(s) - I_2(s)] + R_1 I_1(s) = E_i(s)$$

$$\frac{1}{sC_1}[I_1(s) - I_2(s)] - R_2 I_2(s) - \frac{1}{sC_2} I_2(s) = 0$$

$$\frac{1}{sC_2} I_2(s) = E_0(s)$$

Modelling of Electrical Systems

Transfer Functions of Cascaded Elements (with loading effect)

Eliminating $I_1(s)$ and $I_2(s)$ from the above equations, we find the transfer function between $E_o(s)$ and $E_i(s)$ to be

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{(R_1C_1s+1)(R_2C_2s+1)+R_1C_2s}$$

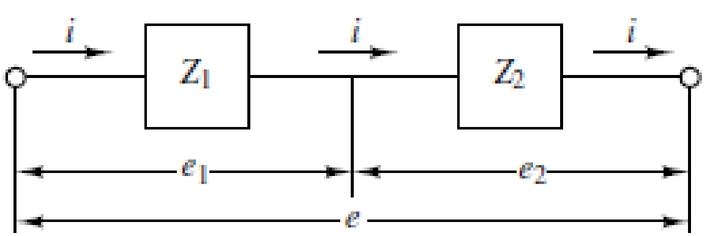
$$= \frac{1}{R_1C_1R_2C_2s^2+(R_1C_1+R_2C_2+R_1C_2)s+1}$$

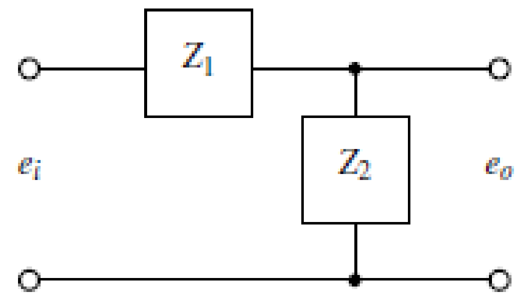
- The overall transfer function of the cascaded *RC* circuit is not just $\left(\frac{1}{R_1C_1s+1}\right)\left(\frac{1}{R_2C_2s+1}\right)$
- Derive the transfer function for an isolated circuit, the output is assumed to be unloaded, which means that no power is being withdrawn at the output

Modelling of Electrical Systems

Complex Impedances Method

- Write the Laplace-transformed equations directly, without writing the differential equations
- The complex impedance $Z(s)$ of a two-terminal circuit is the ratio of $E(s)$ to $I(s)$, so that $Z(s) = E(s) / I(s)$ assumes zero initial conditions
- For resistance R, capacitance C, or inductance L, then the complex impedance is given by R, $1 / Cs$, or Ls, respectively





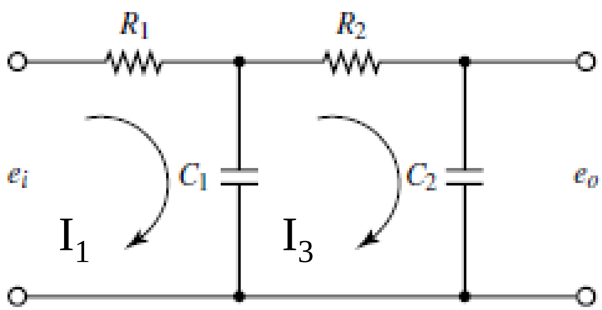
- Assume that the voltages $e_i(t)$ and $e_o(t)$ are the input and output of the circuit

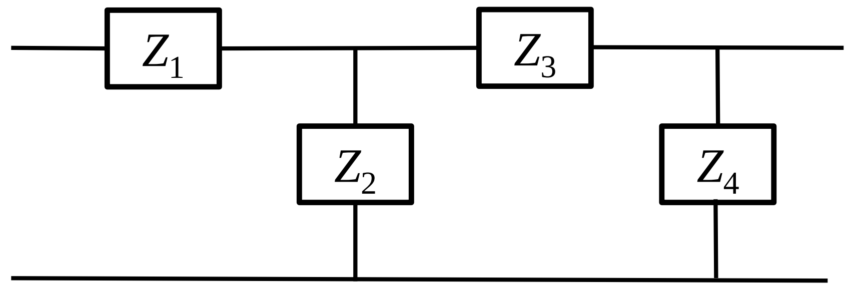
$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

Example 7

Revisit the Transfer Functions of Cascaded Elements (with loading effect)

p.40





$$Z_1 = R_1$$

$$Z_2 = \frac{1}{sC_1}$$

$$Z_3 = R_2$$

$$Z_4 = \frac{1}{sC_2}$$

$$I_1 = I_2 + I_3$$

$$E_i(s) = I_1 Z_1 + I_2 Z_2$$

$$I_2 Z_2 = I_3 Z_3 + I_3 Z_4$$

$$E_o(s) = I_3 Z_4$$

Example 7

Revisit the Transfer Functions of Cascaded Elements (with loading effect) p.40

$E_o(s) = Z_4 I_3$

$E_i(s) = Z_1 I_1 + Z_2 I_2$

$I_1 = I_2 + I_3$

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_4 I_3}{Z_1 I_1 + Z_2 I_2} = \frac{Z_4 I_3}{Z_1(I_2 + I_3) + Z_2 I_2}$$

$Z_2 I_2 = (Z_3 + Z_4)\, I_3$

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_4 I_3}{Z_1 I_3 + (Z_1 + Z_2) I_2} = \frac{Z_4 I_3}{Z_1 I_3 + (Z_1 + Z_2)\left(\frac{Z_3 + Z_4}{Z_2}\right) I_3}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2 Z_4}{Z_1 Z_2 + (Z_1 + Z_2)(Z_3 + Z_4)} = \frac{\left(\frac{1}{sC_1}\right)\left(\frac{1}{sC_2}\right)}{R_1\left(\frac{1}{sC_1}\right) + \left(R_1 + \frac{1}{sC_1}\right)\left(R_2 + \frac{1}{sC_2}\right)}$$

$$\therefore \frac{E_o(s)}{E_i(s)} = \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1}$$

Modelling of Electrical Systems

Transfer Functions of Non-loading Cascaded Elements

- The transfer function of a system consisting of two non-loading cascaded elements can be obtained by eliminating the intermediate input and output
- The transfer functions of the elements are,

$$G_1(s) = \frac{X_2(s)}{X_1(s)} \quad \text{and} \quad G_2(s) = \frac{X_3(s)}{X_2(s)}$$

- If the input impedance of the second element is infinite, the output of the first element is not affected by connecting it to the second element. Then the transfer function of the whole system becomes

$$G(s) = \frac{X_3(s)}{X_1(s)} = \frac{X_2(s)}{X_1(s)}\frac{X_3(s)}{X_2(s)} = G_1(s)G_2(s)$$

Modelling of Electrical Systems

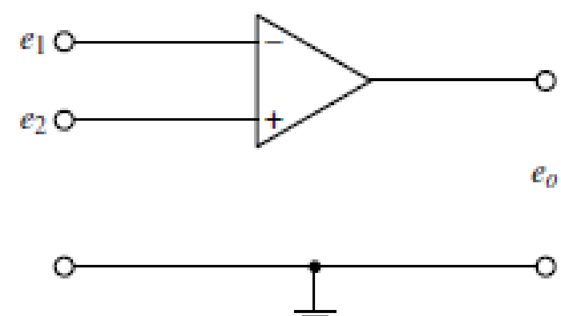
Electronic Controllers

- An electronic controller is a device that compares the input signal with that of a predetermined control point value and determines the appropriate amount of output signal required by the final control element to provide corrective action within a control loop.
- Operational amplifier is one of the key components of electronic controllers

Operational Amplifiers

- Operational amplifiers (op amps), are frequently used to amplify signals in sensor circuits and filters used for compensation purposes

$$e_o = K(e_2 - e_1) = -K(e_1 - e_2)$$

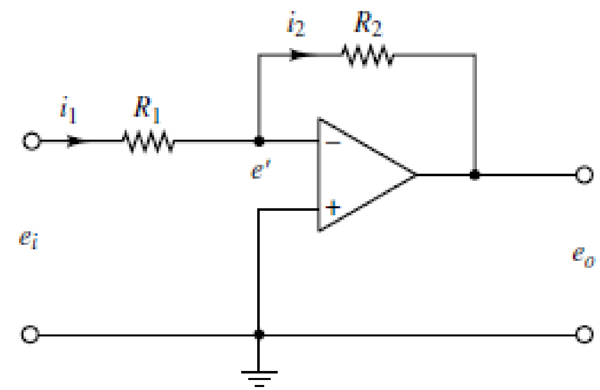


where the inputs e_1 and e_2 may be DC or AC signals and K is the differential gain (voltage gain)

Modelling of Electrical Systems

Inverting Amplifier

- Consider the operational-amplifier circuit



$$i_1 = \frac{e_i - e'}{R_1}, \qquad i_2 = \frac{e' - e_o}{R_2}$$

- Since only a negligible current flows into the amplifier, the current i_1 must be equal to current i_2 . Thus,

$$\frac{e_i - e'}{R_1} = \frac{e' - e_o}{R_2}$$

- Since $K(0 - e') = e_0$ and $K \gg 1$, e' must be almost zero, or $e' \approx 0$. Hence, we have

$$\frac{e_i}{R_1} = \frac{-e_o}{R_2} \quad or \quad e_o = -\frac{R_2}{R_1}e_i \qquad \boxed{\frac{E_o(s)}{E_i(s)} = -\frac{R_2}{R_1}}$$

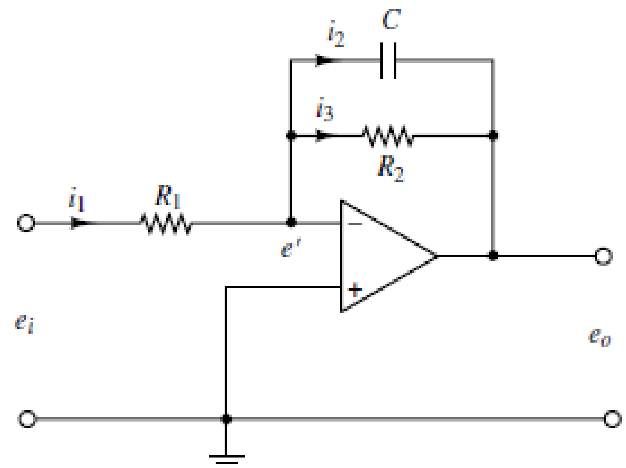
- Thus, the circuit shown is an inverting amplifier. If $R_1 = R_2$, then the op-amp circuit shown acts as a *sign inverter*

Example 8

Obtain the transfer function of an electrical circuit involving an operational amplifier.

<u>Answer:</u>

$$i_1 = \frac{e_i - e'}{R_1}, \qquad i_2 = C\frac{d}{dt}(e' - e_o), \qquad i_3 = \frac{e' - e_o}{R_2}$$



The current flowing into the amplifier is negligible, we have $i_1 = i_2 + i_3$. Hence,

$$\frac{e_i - e'}{R_1} = C\frac{d}{dt}(e' - e_o) + \frac{e' - e_o}{R_2}$$

Since $e' \approx 0$, we have $\frac{e_i}{R_1} = C\frac{d}{dt}(-e_o) + \frac{-e_o}{R_2}$

Taking the Laplace transform of this equation, assuming the zero initial condition, we have

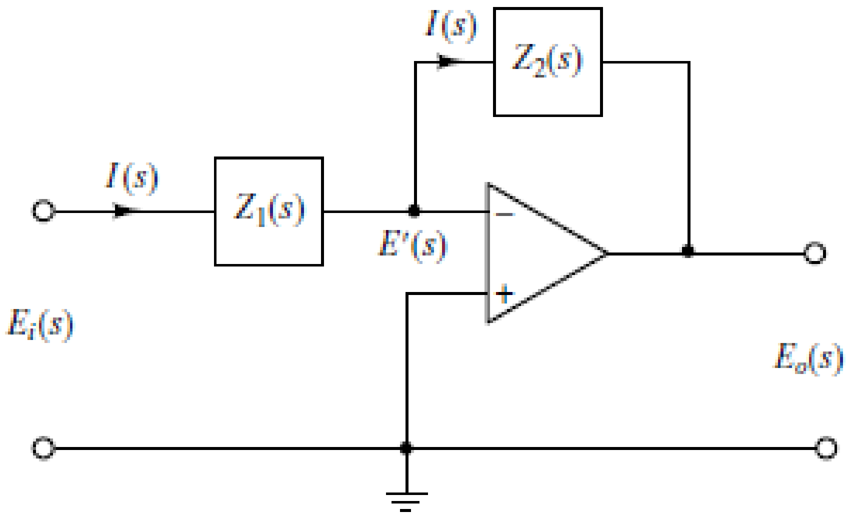
$$\frac{E_i(s)}{R_1} = -sCE_o(s) - \frac{E_o(s)}{R_2} \Longrightarrow \boxed{\frac{E_o(s)}{E_i(s)} = -\frac{R_2}{R_1(sCR_2 + 1)}}$$

Modelling of Electrical Systems

Impedance Method to Obtaining Transfer Functions

Consider the op-amp circuit

$$\frac{E_i(s) - E'(s)}{Z_1(s)} = \frac{E'(s) - E_o(s)}{Z_2(s)}$$

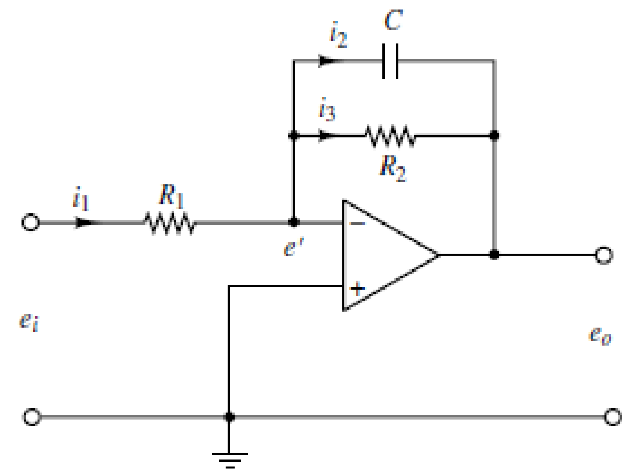


Since $E'(s) \approx 0$, we have

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

Same as p.47

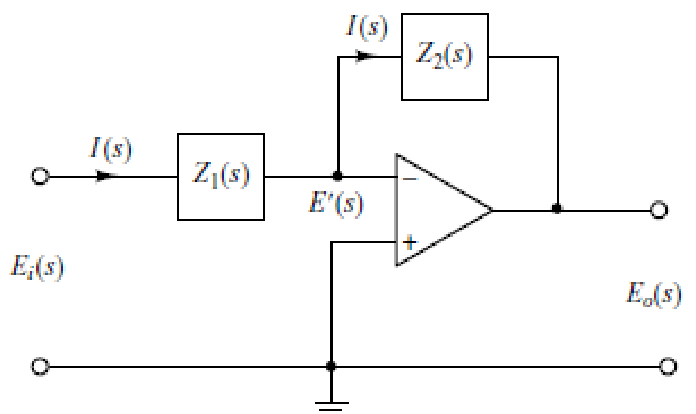
Revisit Example 8



Answer:

$$Z_1(s) = R_1, \qquad Z_2(s) = \frac{1}{Cs + \frac{1}{R_2}} = \frac{R_2}{R_2Cs + 1}$$

The transfer function $E_o(s) / E_i(s)$ is, therefore, obtained as



$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R_2}{R_1(sCR_2 + 1)}$$

Of course, the transfer function is the same as before

Modelling of Electrical Systems

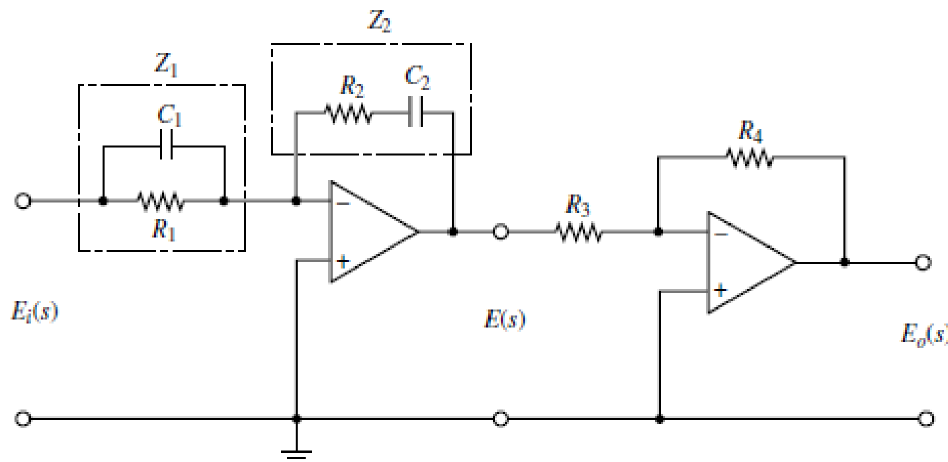
PID Controller Using Operational Amplifiers

- An electronic proportional-plus-integral-plus-derivative controller (a PID controller) using operational amplifiers
- The transfer function,

$$\frac{E(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

- where

$$Z_1 = \frac{R_1}{R_1C_1s+1} \qquad Z_2 = \frac{R_2C_2s+1}{C_2s}$$



- Thus,

$$\frac{E(s)}{E_i(s)} = -\left(\frac{R_2C_2s+1}{C_2s}\right)\left(\frac{R_1C_1s+1}{R_1}\right)$$

- The transfer function of the PID controller is,

$$\frac{E_o(s)}{E_i(s)} = \frac{E_o(s)}{E(s)}\frac{E(s)}{E_i(s)} = \left[-\frac{R_4}{R_3}\right]\left[-\left(\frac{R_2C_2s+1}{C_2s}\right)\left(\frac{R_1C_1s+1}{R_1}\right)\right]$$

Modelling of Electrical Systems

PID Controller Using Operational Amplifiers

- Hence,

$$\frac{E_o(s)}{E_i(s)} = \frac{R_4(R_1C_1 + R_2C_2)}{R_3R_1C_2}\left[1 + \frac{1}{(R_1C_1 + R_2C_2)s} + \frac{R_1C_1R_2C_2}{R_1C_1 + R_2C_2}s\right]$$

- When a PID controller is expressed as

$$\frac{E_o(s)}{E_i(s)} = K_p\left(1 + \frac{1}{T_is} + T_ds\right)$$

$$K_p = \frac{R_4(R_1C_1 + R_2C_2)}{R_3R_1C_2}$$

$$T_i = R_1C_1 + R_2C_2$$

$$T_d = \frac{R_1C_1R_2C_2}{R_1C_1 + R_2C_2}$$

K_p: Proportional gain
T_i: Integral time
T_d: Derivative time

- When a PID controller is expressed as

$$\frac{E_o(s)}{E_i(s)} = K_p + \frac{K_i}{s} + K_ds$$

$$K_P = \frac{R_4(R_1C_1 + R_2C_2)}{R_3R_1C_2}$$

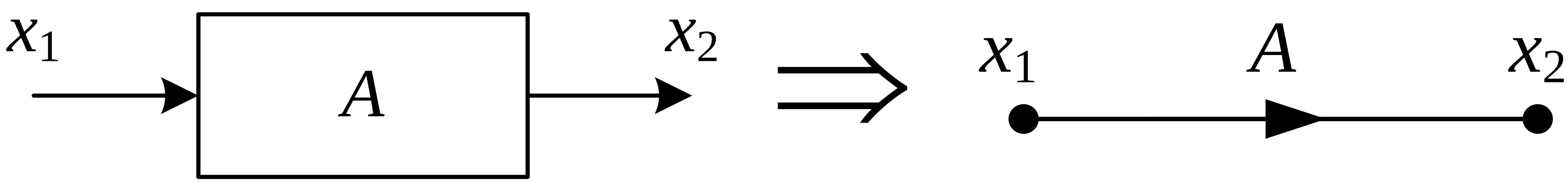
$$K_i = \frac{R_4}{R_3R_1C_2}$$

$$K_d = \frac{R_4R_2C_1}{R_3}$$

K_p: Proportional gain
K_i: Integral gain
K_d: Derivative gain

Signal Flow Graphs

- SFG is another pictorial representation of a system



- Every variable becomes a node and every transmission function ***A*** is designated by a branch
- Thus, ***A*** represents the system transfer function

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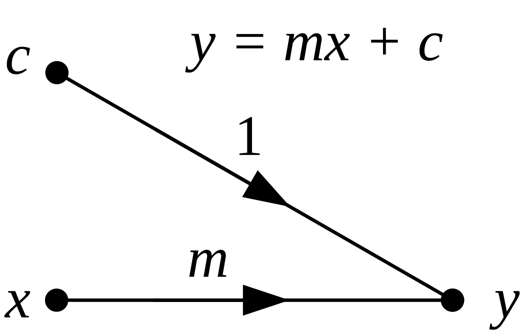
Signal Flow Graphs

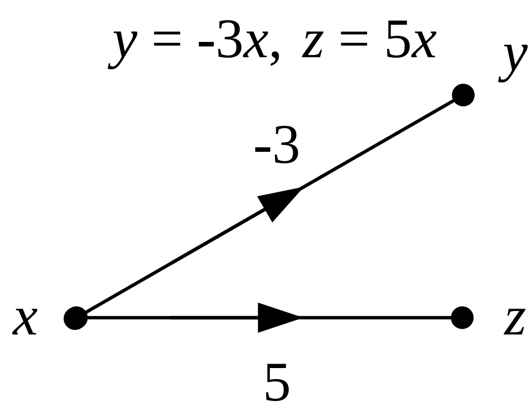
- Signal flow graph algebra

Addition



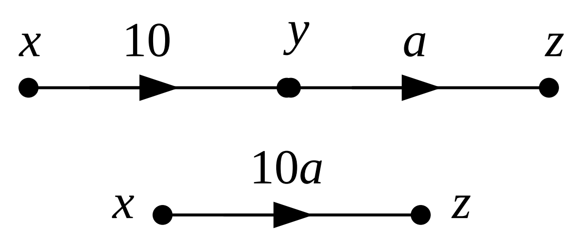
The variable at a node is equal to the sum of all signal entering the node

Transmission



The variable designated by a node is transmitted on every branch leaving the node

Multiplication



Cascades are reduced as in block diagrams

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SPEED

Signal Flow Graphs

Properties

1. SFG applies only to linear systems
2. The equations for which an SFG is drawn must be algebraic equations in the form of cause-and-effect
3. Nodes are used to represent variables. Normally, the nodes are arranged from *left to right*, from the input to the output, following a succession of cause-and-effect relations through the system
4. Signals travel along branches only in the direction described by the arrows of the branches.
5. The branch directing from node x_k to x_j represents the dependence of x_j upon x_k , but not the reverse
6. A signal x_k traveling along a branch between x_k and x_j is multiplied by the gain (A_{kj}) of the branch, so a signal $A_{kj}x_k$ is delivered at x_j

Example 9

Construct the signal flow graph of a system described by the following set of algebraic equations:

$$x_2 = A_{12}x_1 + A_{32}x_3$$
$$x_3 = A_{23}x_2 + A_{43}x_4$$
$$x_4 = A_{24}x_2 + A_{34}x_3 + A_{44}x_4$$
$$x_5 = A_{25}x_2 + A_{45}x_4$$

Answer:

Signal Flow Graphs

Definitions

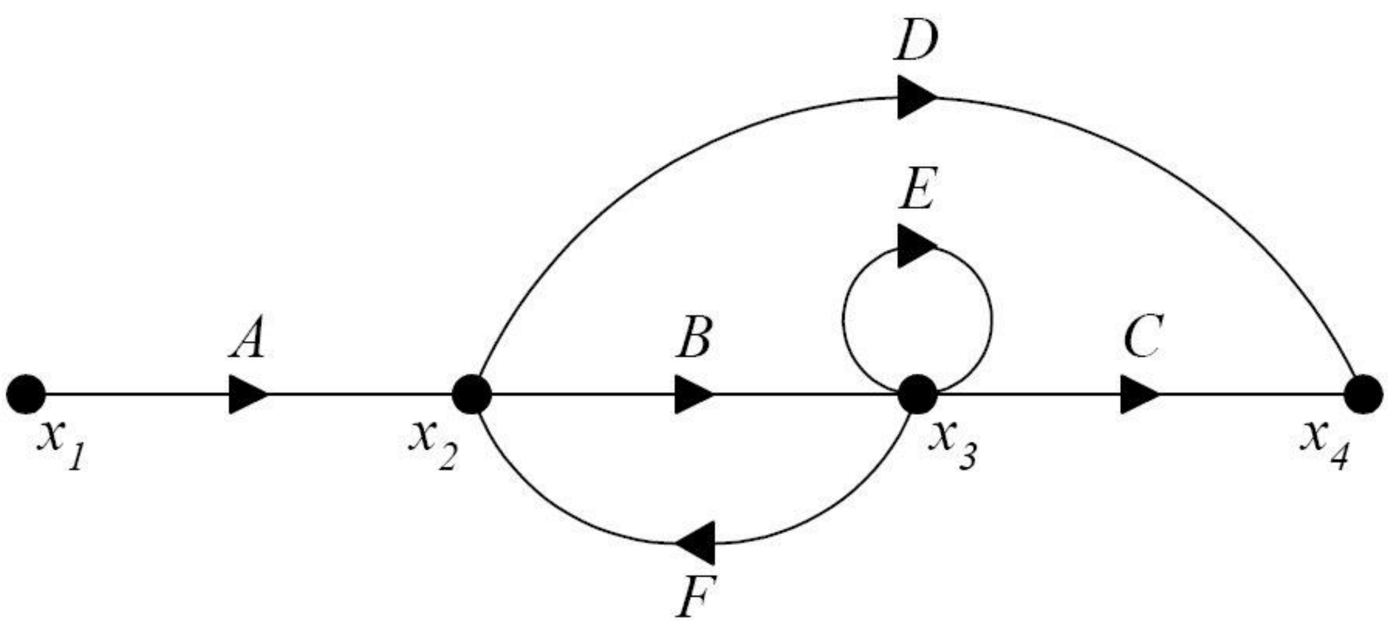
- Input Node (Source): An input node is a node that has only outgoing branches
- Output Node (Sink): An output node is a node that has only incoming branches. However, this condition is not always readily met by an output node
- Path: A path is any collection of a continuous succession of branches traversed in the same direction
- Forward Path: A path of an input node to an output node, no node is traversed more than once
- Feedback Path or Loop: Originates and ends at the same node, no node is traversed more than once
- Self Loop: A feedback loop consisting of one branch
- Path Gain: Product of the branch gains encountered in traversing a path
- Loop Gain: Path gain of a loop
- Non-touching Loops: Two parts of an SFG are non-touching if they do not share a common node

Signal Flow Graphs

Forward path? Feedback path? Self loop?

Gain? Path gain? Loop gain?

Input node? Output node?



Signal Flow Graphs

Mason's rule

$$M = \frac{Y}{U} = \frac{1}{\Delta}\sum_{k=1}^{N}(P_k \Delta_k)$$

Y = Output-node variable

U = Input-node variable

N = Total number of forward paths between Y and U

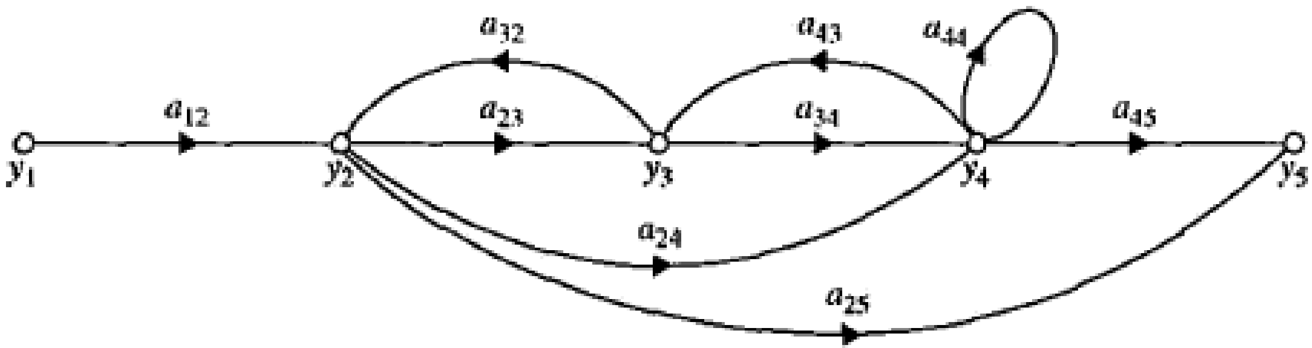
P_k = Gain of the kth forward paths between Y and U

Δ = 1 – (sum of all individual loop gains) + (sum of gain products of 2 non-touching loops) – (sum of gain products of 3 non-touching loops) + …

Δ_k = Δ evaluated with all loops touching P_k eliminated (i.e. set equal to zero)

Example 10

Consider the signal flow graph constructed in Example 9. Determine the gain by using the Mason's rule.



<u>Answer:</u>

Forward Path (2):

Loop (4):

Non-touching Loop (1):

Example 10

Answer:

Try it yourself !

$$\therefore \frac{y_5}{y_1} = \frac{a_{12}a_{23}a_{34}a_{45} + a_{12}a_{24}a_{45} + a_{12}a_{25}(1 - a_{34}a_{43} - a_{44})}{1 - a_{23}a_{32} - a_{34}a_{43} - a_{44} - a_{24}a_{43}a_{32} + a_{23}a_{32}a_{44}}$$

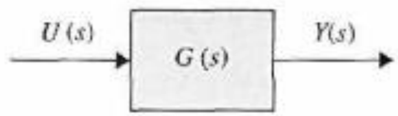
Signal Flow Graphs

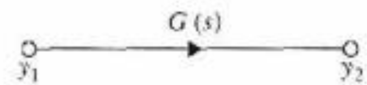
Block diagrams and their SFG equivalent representations

	Block Diagram	Signal Flow Diagram

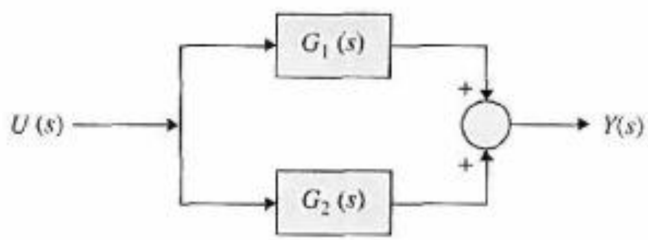
Simple Transfer Function

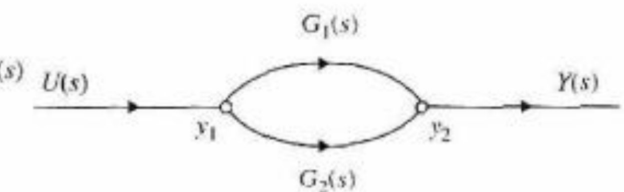
$$\frac{Y(s)}{U(s)} = G(s)$$



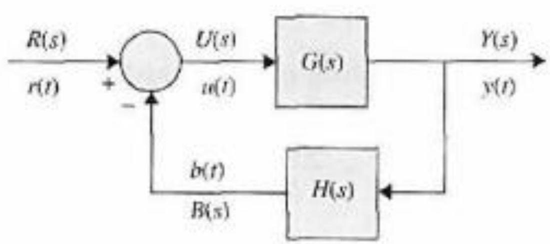


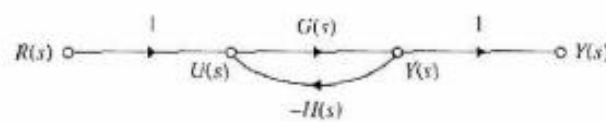
Parallel Feedback





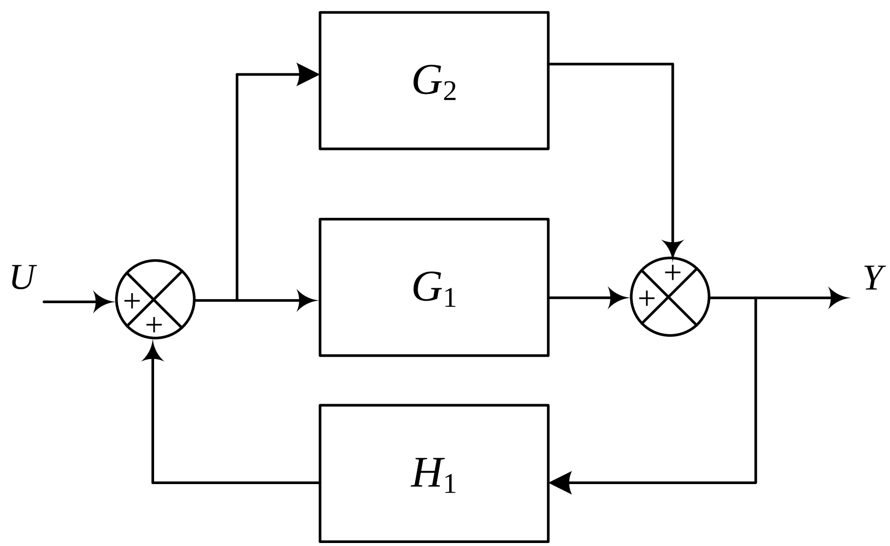
$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$





Example 11

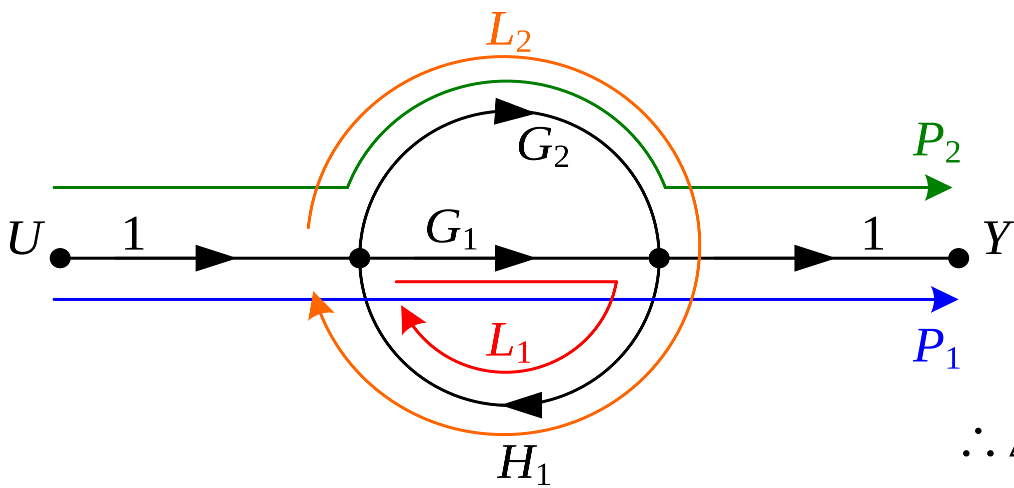
Construct a signal flow graph for the following block diagram and hence determine the transfer function (*Y* / *U*)



Example 11

Answer:

The signal flow graph of the block diagram



Forward paths: $P_1 = G_1$

$P_2 = G_2$

Loops: $L_1 = G_1 H_1$

$L_2 = G_2 H_1$

$$\therefore \Delta = 1 - (L_1 + L_2) = 1 - G_1 H_1 - G_2 H_1$$

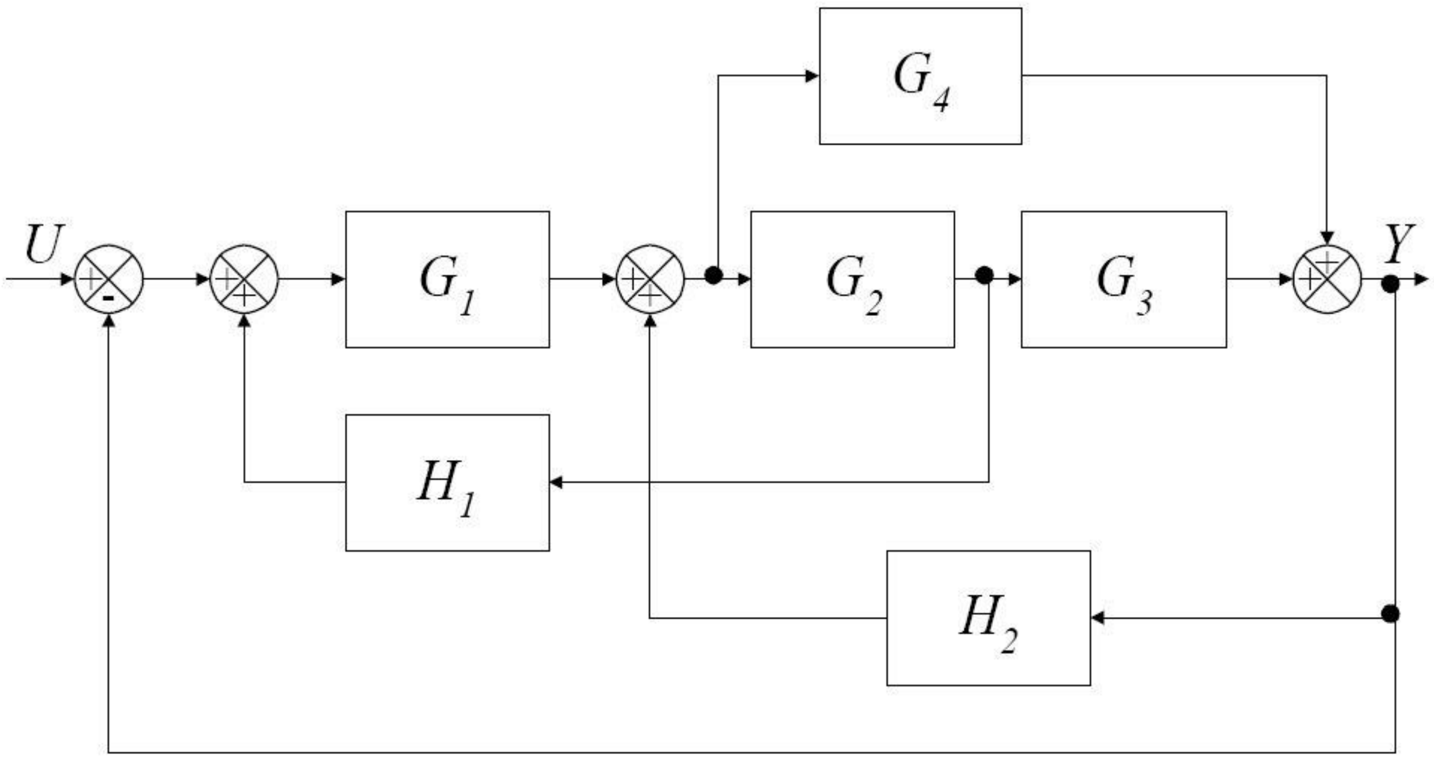
Since both loops touch P_1, hence set $L_1 = L_2 = 0$ in Δ to give $\Delta_1 = 1$

Similarly, since both loops touch P_2, hence set $L_1 = L_2 = 0$ in Δ to give $\Delta_2 = 1$

$$\therefore \frac{Y}{U} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta} = \frac{G_1 + G_2}{1 - G_1 H_1 - G_2 H_1}$$

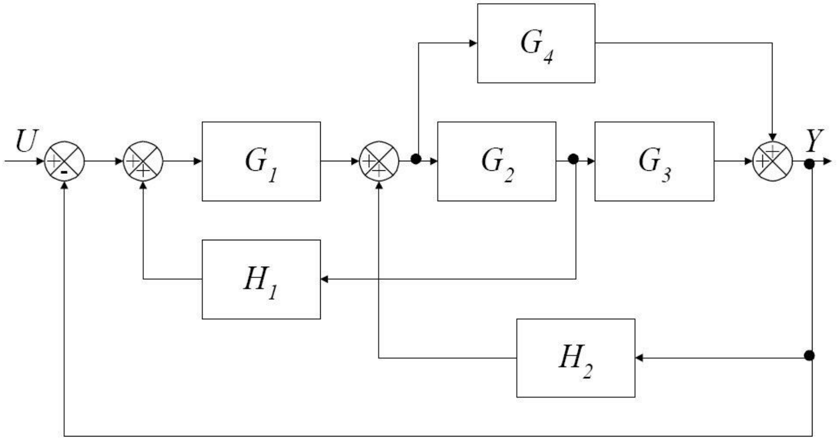
Example 12

Construct a signal flow graph for the following block diagram and hence determine the transfer function (*Y* / *U*).



Example 12

Answer:



Example 12

Answer:

Try it yourself !

$$\therefore \frac{Y}{U} = \frac{G_1G_2G_3 + G_1G_4}{1 - (G_1G_2H_1 + G_2G_3H_2 - G_1G_2G_3 - G_1G_4 + G_4H_2)}$$