THE HONG KONG POLYTECHNIC UNIVERSITY
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SEHS4653
Control System Analysis

Unit 1

Introduction to Control Systems and Elementary Mathematics

(Reference: [1] chapter 1, Appendices A and B)

Content

- Introduction
- Definitions
- Examples of Control Systems
- Closed-loop Control Versus Open-loop Control
- Design and Compensation of Control Systems
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  - Step and Ramp Functions
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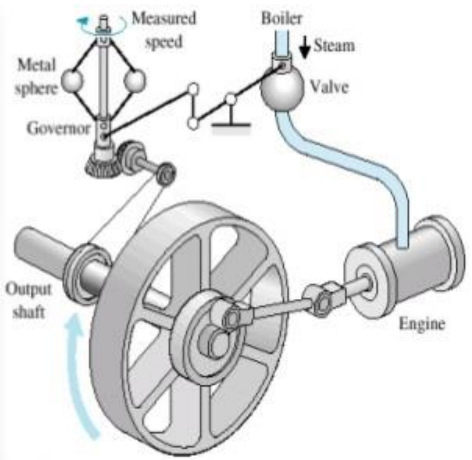
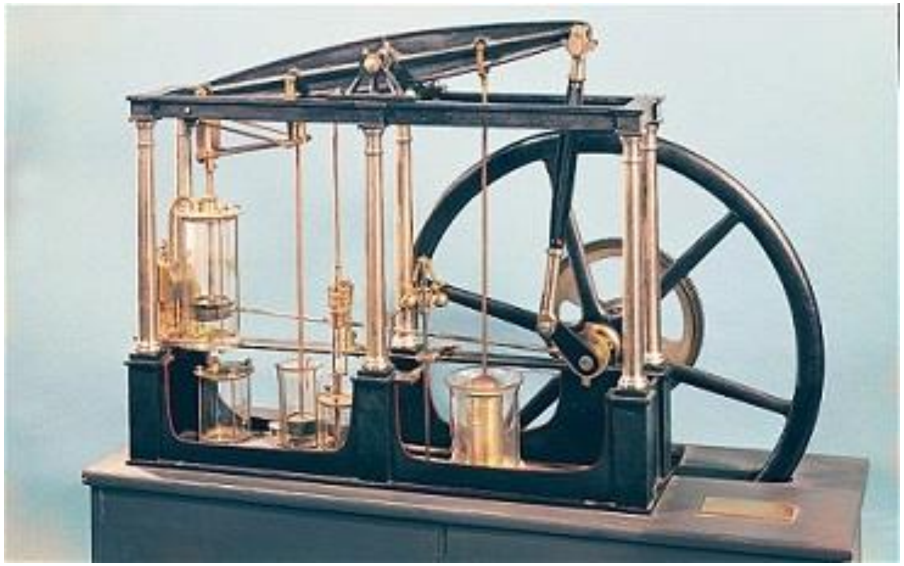
Introduction

- Historical Review
  - James Watt (1781)
    - First significant work in automatic control
    - Centrifugal governor speed controller of a steam engine





Introduction

- Historical Review
  - Nyquist (1932)
    - developed a relatively simple procedure for determining the stability of closed-loop systems on the basis of open-loop response to steady-state sinusoidal inputs.
  - H. W. Bode (1945)
    - Bode-diagram method

      (frequency-response method)
  - W. R. Evans (1948)
    - Root-locus method

Core of classical control theory

Can design control systems that are in stable and acceptable, but not optimal in any meaningful sense.

Introduction

- Historical Review
  - Late 1950s: focus on designing optimal systems
  - 1960s: digital computers help the development of modern control theory to cope with the increased complexity of modern plants
  - 1960 to 1980: optimal control of both deterministic and stochastic systems
  - 1980 to present: focus on robust control and H_∞ control
  - Recent applications to non-engineering: biological, biomedical, economics, …

Modern control theory

Definitions

- Controlled Variable and Control Signal or Manipulated Variable
  - ***Controlled variable***: quantity or condition that is measured and controlled. Normally, it is the *output* of the system
  - ***Control signal (or manipulated variable)***: quantity or condition that is varied by the controller so as to affect the value of the controlled variable
  - Control means measuring the value of the *controlled variable* of the system and applying the control signal to the system to correct or limit deviation of *controlled variable*
- Plants
  - Any physical object to be controlled, e.g. a mechanical device, a heating furnace, a chemical reactor, or a spacecraft
- Processes
  - Any operation to be controlled, e.g. chemical, economic, and biological processes

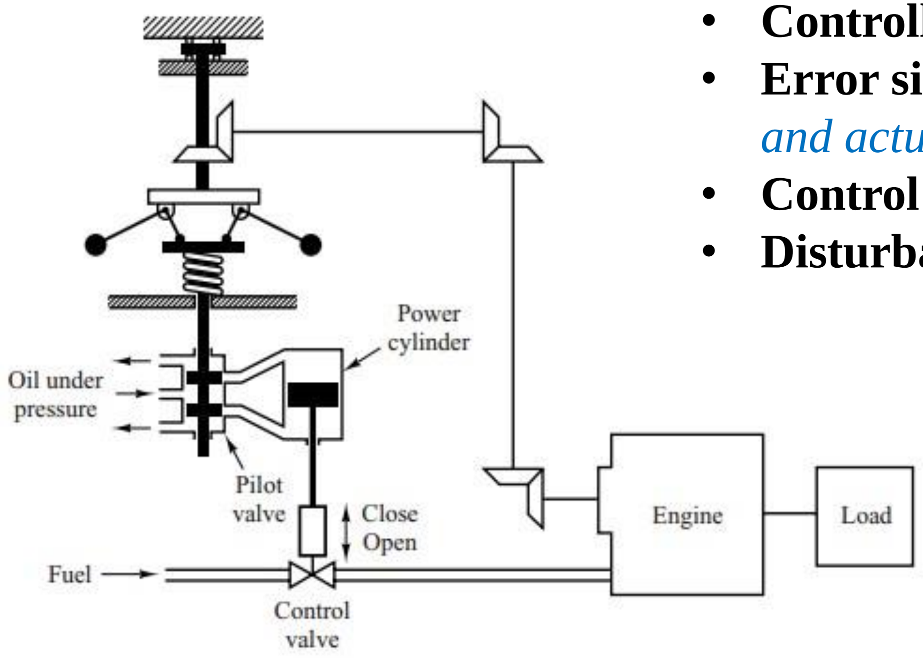
Definitions

- Disturbances
  - A signal that tends to adversely affect the value of the output of a system, can be generated *internally* or *externally*

- Feedback Control
  - An operation that, in the presence of disturbances, tends to reduce the *difference* between the *output* of a system and *reference input*
  - Here only unpredictable disturbances are so specified, since predictable or known disturbances can always be compensated for within the system.

Examples of Control Systems

- Speed Control System

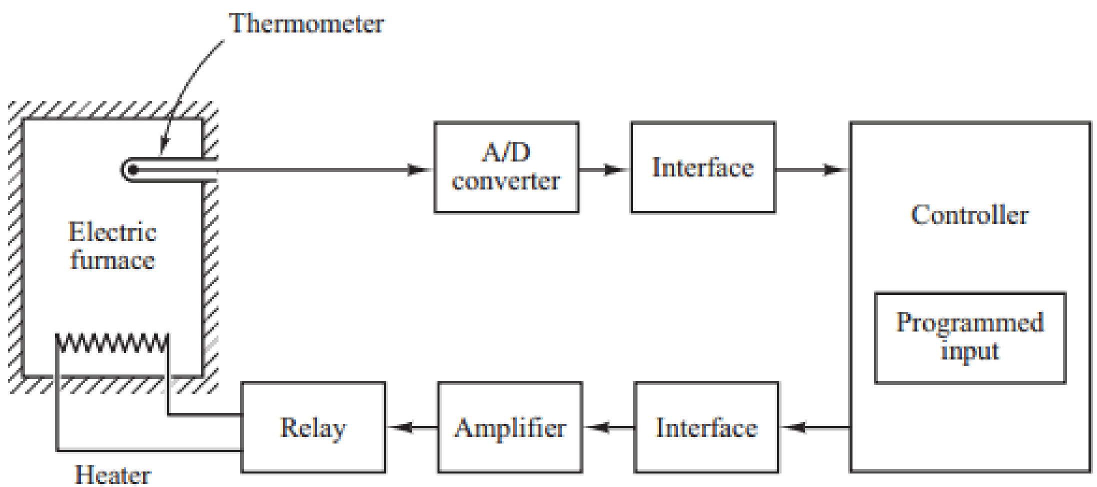


Watt's speed governor for an engine.

- **Plant (controlled system):** *Engine*
- **Controlled variable:** *speed of the engine*
- **Error signal:** *difference between desired speed and actual speed*
- **Control signal:** *the amount of fuel*
- **Disturbance:** *unexpected change in the load*

Examples of Control Systems

- Temperature Control System



- **Plant (controlled system):** *electric furnace*
- **Controlled variable:** *temperature of the furnace*
- **Error signal:** *difference between desired and actual temperature*
- **Control signal:** *the current of the heater*
- **Disturbance:** *heat loss in the electric furnace*

Closed-loop Control Versus Open-loop Control

- Feedback Control Systems
  - A system that maintains a *prescribed relationship* between the *output* and the *reference input* by comparing them and using the *difference* as a means of control
  - Examples: room temperature control system, human body

- Closed-loop Control System
  - The *actuating error signal* (the difference between the reference input signal and the feedback (or output) signal) is *fed* to the *controller* so as to reduce the error and bring the output of the system to a desired value

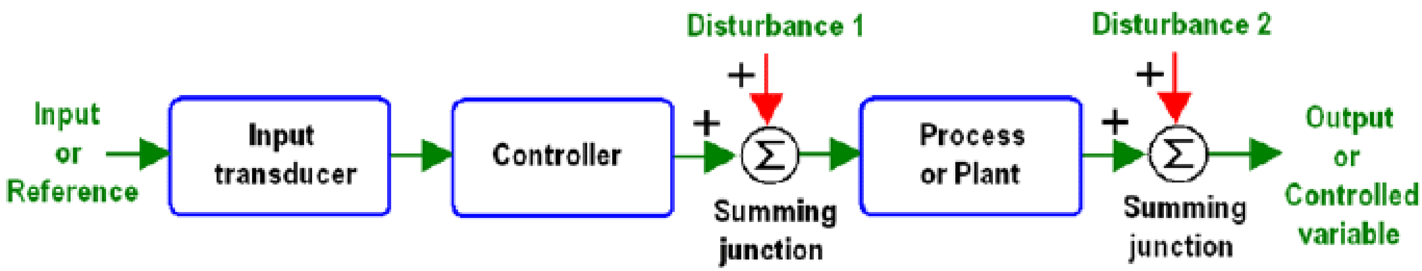
The terms feedback control and closed-loop control are used interchangeably.

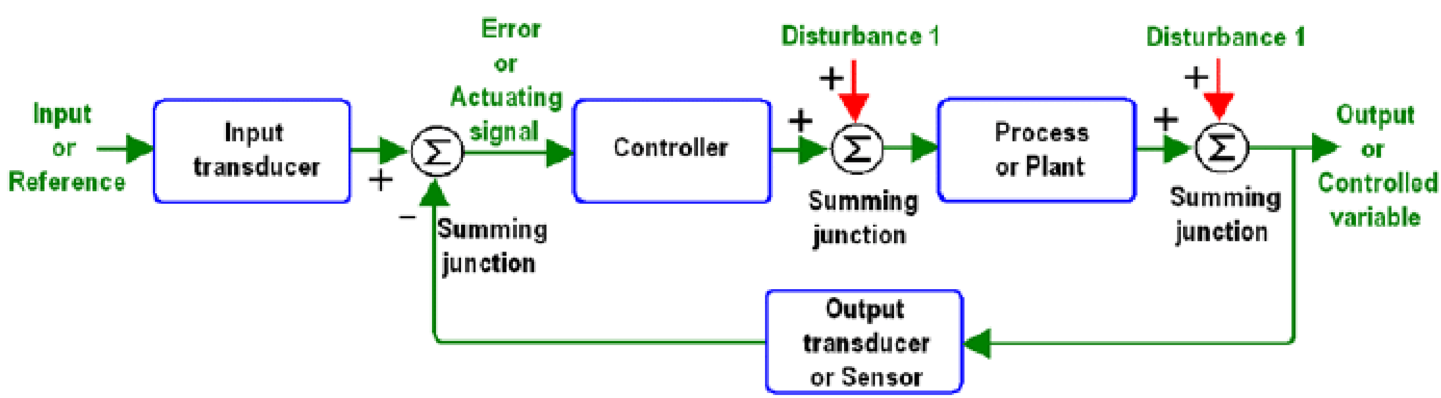
Closed-loop Control Versus Open-loop Control

- Open-loop Control System
  - The *output* has *no effect* on the *control* action, i.e. neither measured nor fed back for comparison with the *input*
  - The accuracy of the system depends on *calibration*
  - only if the *relationship* between the *input* and *output* is known and if there are neither *internal* nor *external* disturbances
  - Example: washing machine, traffic control by means of signals (*operate on a time basis*)

Closed-loop Control Versus Open-loop Control

- Closed-loop and Open-loop Control Systems





Design and Compensation of Control System

- Compensation
  - modification of the system dynamics to satisfy the given *specifications*
  - Examples: root-locus [Unit 4], and frequency-response (Bode diagram) [Unit 5]
- Performance Specifications
  - The requirements imposed on the control system
  - May be given in terms of transient response and steady-state requirements [Unit 3], or frequency-response requirements [Unit 5]
  - May be given in terms of precise numerical or qualitative statements
  - Examples: accuracy, relative stability, speed of response

Design and Compensation of Control System

- System Compensation
  - Adjusting the system gain value will improve the steady-state behavior but will result in poor stability or even instability.
  - Modifying the structure (redesign) or by incorporating additional devices or components to alter the overall behavior
  - A device inserted into the system for the purpose of satisfying the specifications is called a ***compensator*** [Unit 6]

Design and Compensation of Control System

- Design Procedures
  1. Set up a mathematical model of the control system and adjust the parameters of a compensator
  2. Checking of the system performance by analysis with each adjustment of the parameters (use available computer software to avoid much of the numerical drudgery necessary for this checking)
  3. Construct a prototype and test the open-loop system after obtaining a satisfactory mathematical models
  4. Close the loop and test the performance of the resulting closed-loop system in case that absolute stability of the closed loop is assured
  5. Adjust system parameters and make changes in the prototype until the system meets the specifications by analyzing each trial, and the results of the analysis must be incorporated into the next trial

*The final system meets the **performance specifications**; and is **reliable** and **economical***

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Review on Calculus

Solution of Quadratic Equations

YouTube video

- An equation of the form $ax^2 + bx + c = 0$ $(a \neq 0)$, is said to be a **quadratic equation**
- The equation can have at most 2 solutions (or **roots**)
- The solutions can be obtained by factorization or **Quadratic Formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The term, $b^2 - 4ac$, is called the **discriminant**

Discriminant	Nature of Roots	Roots
Case 1: > 0	2 distinct real roots	$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$
Case 2: = 0	Double real roots	$x_1 = x_2 = -\frac{b}{2a}$
Case 3: < 0	No real roots ⇒ 2 **complex roots**	$x_1 = \frac{-b+j\sqrt{4ac-b^2}}{2a}$ and $x_2 = \frac{-b-j\sqrt{4ac-b^2}}{2a}$

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Example 1

Solve the equations, (a) $2x^2 + 3x + 1 = 0$, (b) $x^2 + 6x + 9 = 0$, and (c) $x^2 + 2x + 5 = 0$.

Answer:

(a) $a = 2, b = 3, c = 1,$ $\quad x = \dfrac{-3 \pm \sqrt{3^2 - 4(2)(1)}}{2(2)}$

$$x_1 = \frac{-3 + \sqrt{9 - 8}}{4} = -0.5$$

$$x_2 = \frac{-3 - \sqrt{9 - 8}}{4} = -1$$

Case 1: two real roots

(b) $a = 1, b = 6, c = 9,$ $\quad x = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(9)}}{2(1)} = -3$

Case 2: double real roots

(c) $a = 1, b = 2, c = 5,$ $\quad x = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = \dfrac{-2 \pm \sqrt{-16}}{2}$

$$x_1 = -1 + j2$$

$$x_2 = -1 - j2$$

Case 3: no real roots (**2 complex roots**)

Complex conjugate

Review on Calculus

Complex Number

- The following equation is not solvable in $\mathbb{R}$
- Then, we introduce a new number (it is not real), denoted by i

$$i^2 + 1 = 0, \quad i = \sqrt{-1}$$

- This number is quite useful in studying electricity and sometimes it is denoted by j
- Hence, we have

Imaginary unit $(j^2 = -1)$

$$z = a + jb$$

Real part (Re): a

Imaginary part (Im): b

Review on Calculus

Complex Number

- A **complex conjugate** is found by changing the sign of the imaginary part from *positive to negative* (or *negative to positive*) of a complex number (Example 1)

$$z = a + jb, \quad \bar{z} = a - jb$$

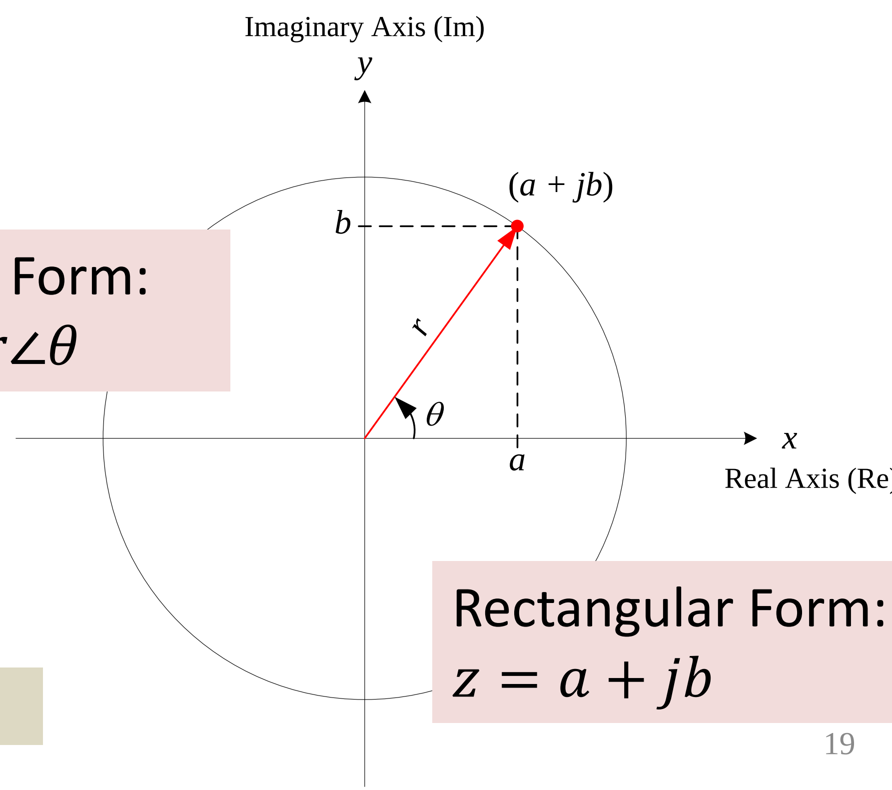
- Complex Plane

Complex conjugate in polar form?

Equations for changing rectangular and polar forms?

Polar form of real number, 1 and -2 ?

Polar Form:
$z = r\angle\theta$



Rectangular Form:
$z = a + jb$

Review on Calculus

Ordinary Differential Equations

YouTube video

- Focus on 2nd order homogenous equations with constant coefficients
$$\ddot{y} + a\dot{y} + by = 0$$
- Consider the **characteristic equation** (or *auxiliary equation*), we have
$$\lambda^2 + a\lambda + b = 0$$
- It's now like a quadratic equation!
- Hence, the general solution of the 2nd order differential equation will be

Case	Roots	General Solution
1 (> 0)	Distinct real: λ_1 , λ_2	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
2 (= 0)	Double real: $\lambda = -\frac{a}{2}$	$y = (c_1 + c_2 x)e^{\frac{-ax}{2}}$
3 (< 0)	Complex conjugate: $\lambda_1 = \frac{-a+j\omega}{2}, \lambda_2 = \frac{-a-j\omega}{2}$	$y = e^{\frac{-ax}{2}}(A\cos\omega x + B\sin\omega x)$

Review on Calculus

Partial Fraction Decomposition

YouTube video

- Use to find the inverse of Laplace transform
- In control systems analysis, $F(s)$, usually occurs in the form $F(s) = \frac{A(s)}{B(s)}$, where $A(s)$ and $B(s)$ are polynomials

For example, $$\frac{2s+5}{s^2+3s+2} = \frac{3}{s+1} - \frac{1}{s+2}$$

- There are 4 types of partial fraction decomposition

(1) Non-repeated linear factors in denominator

$$\frac{F(s)}{(s+a)(s+b)(s+c)} = \frac{A}{s+a} + \frac{B}{s+b} + \frac{C}{s+c}$$

(2) Repeated linear factor in denominator

$$\frac{F(s)}{(s+a)^n} = \frac{A}{(s+a)^n} + \cdots + \frac{X}{(s+a)^2} + \frac{Y}{s+a}$$

Review on Calculus

Partial Fraction Decomposition

(3) Non-repeated quadratic factors in denominator

$$\frac{F(s)}{(s^2+as+b)(s^2+cs+d)} = \frac{As+B}{s^2+as+b} + \frac{Cs+D}{s^2+cs+d}$$

(4) Repeated quadratic factor in denominator

$$\frac{F(s)}{(s^2+as+b)^n} = \frac{As+B}{(s^2+as+b)^n} + \cdots + \frac{Ws+X}{(s^2+as+b)^2} + \frac{Ys+Z}{s^2+as+b}$$

Example 2

Find the partial fraction of

$$\frac{1}{(s^2-9)}$$

$$s=\frac{0\pm\sqrt{0^2-4(1)(-9)}}{2(1)}=\pm 3$$

<u>Answer:</u>

Write

$$\frac{1}{s^2-9}=\frac{A}{s+3}+\frac{B}{s-3}=\frac{A(s-3)+B(s+3)}{s^2-9}$$

So that

$$A(s-3)+B(s+3)=1$$

Put $s=3$, and $s=-3$ respectively on both sides of the above equality. We have,

$$A(3-3)+B(3+3)=1\Rightarrow B=\frac{1}{6}$$

$$A(-3-3)+B(-3+3)=1\Rightarrow A=-\frac{1}{6}$$

$$\therefore \frac{1}{s^2-9}=-\frac{1}{6}\frac{1}{s+3}+\frac{1}{6}\frac{1}{s-3}$$

Example 3

Find the partial fraction of $\dfrac{1}{s(s+2)}$.

<u>Answer:</u>

Write $$\frac{1}{s(s+2)}=\frac{A}{s}+\frac{B}{s+2} \qquad =\frac{A(s+2)+Bs}{s(s+2)}$$

$$A(s+2)+Bs=1$$

$$s=-2,\quad -2B=1 \quad \Rightarrow B=-\frac{1}{2}$$

$$s=0,\quad 2A=1 \quad \Rightarrow A=\frac{1}{2}$$

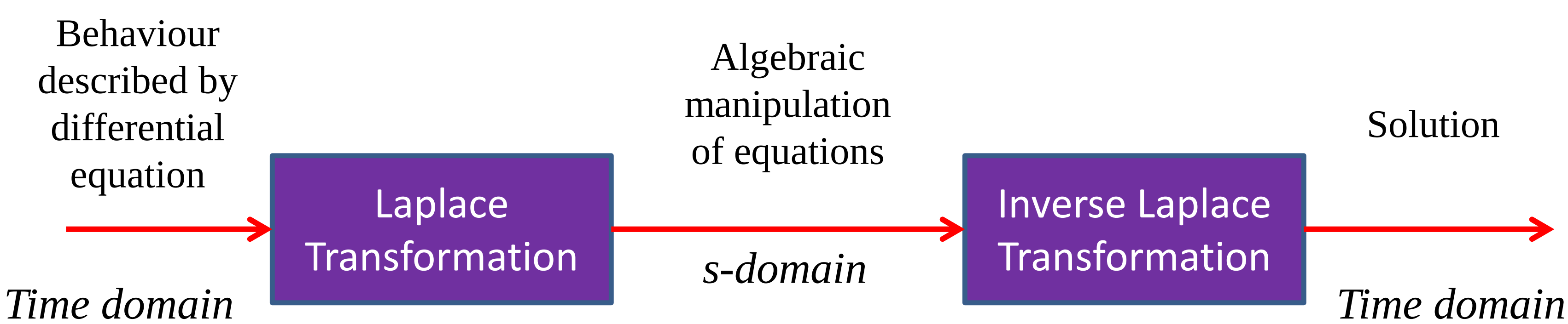
$$\therefore \frac{1}{s(s+2)}=\frac{1}{2}\frac{1}{s}-\frac{1}{2}\frac{1}{s+2}$$

Laplace Transform: Introduction

- Time domain and transform domain
  - The study of control systems, linear systems and signal processing will usually analyse the systems or signals either in *time domain* or other *transform domain*
  - Transform domain: Laplace, Fourier and z-transforms
- Fourier, Laplace and *z*-transforms
  - Fourier transform (FT) decomposes a function of time (a signal) into its constituent frequencies $\Rightarrow$ *Frequency domain*
  - Laplace transform (LT) transforms a function of a real variable t (often time) to a function of a complex variable s $\Rightarrow$ s-domain
  - Z-transform is considered as a discrete-time equivalent of the Laplace transform $\Rightarrow$ z-domain

The Laplace Transform

- Convert sinusoidal, exponential functions into *algebraic functions*
- Use to solve linear differential equations ⇒ algebraic equations in a complex variable *s*
- Simultaneously obtain both *transient* component and *steady-state* components



The Laplace Transform

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)dt$$

$f(t)$	=	a function of time t such that $f(t) = 0$ for $t < 0$
s	=	a complex variable $(= \sigma + j\omega)$
$\mathcal{L}$	=	Laplace transform operator
$F(s)$	=	Laplace transform of $f(t)$

Inverse Laplace Transform

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds\,, \text{ for } t > 0$$

$\mathcal{L}^{-1}$	=	Inverse Laplace transform operator
c	=	the abscissa of convergences, a real constant and is chosen larger than the real parts of all singular points of $F(s)$

Properties and Theorem

Addition and Subtraction

$$\mathcal{L}[f_1(t) \pm f_2(t)] = \mathcal{L}[f_1(t)] \pm \mathcal{L}[f_2(t)] = F_1(s) \pm F_2(s)$$

Multiplication

$$\mathcal{L}[Af(t)] = A\mathcal{L}[f(t)] = AF(s)$$

Properties and Theorem

- Differentiation

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0)$$

where $f(0)$ is the initial value of $f(t)$ evaluated at $t = 0$

- Integration

$$\mathcal{L}\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$$

where $f^{-1}(0) = \int f(t)dt$ evaluated at $t = 0$

Properties and Theorem

- Final Value Theorem

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

where $\lim_{t \to \infty} f(t)$ exists

It relates to the steady-state behaviour of $f(t)$ to the behaviour of $sF(s)$

- Initial Value Theorem

$$f(0^+) = \lim_{s \to \infty} sF(s)$$

where $t > 0$

It is the counterpart of the final value theorem

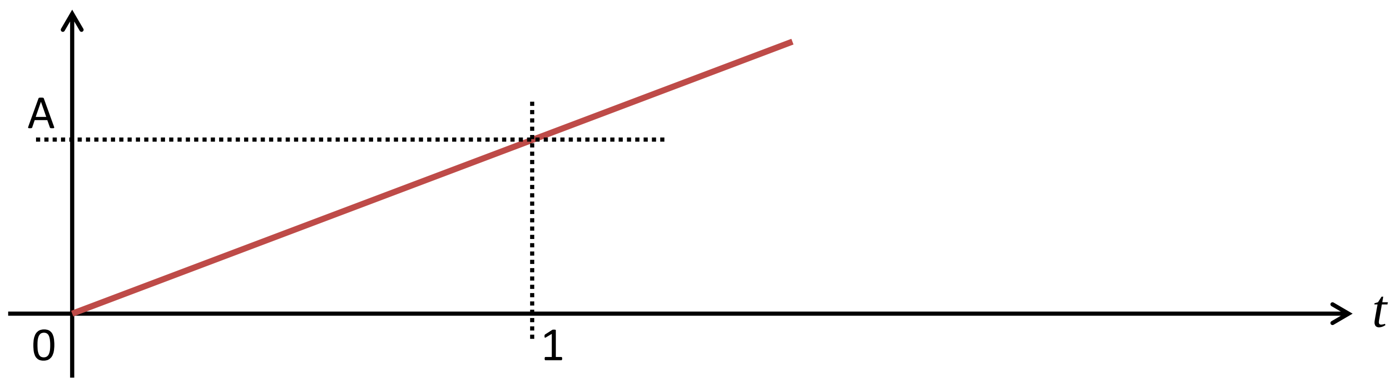
Step Function

$$f(t)\begin{cases}= 0, & \text{for } t < 0 \\ = A, & \text{for } t > 0\end{cases}$$

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st}(A)dt = -\frac{A}{s}[e^{-st}]_0^{\infty} = \frac{A}{s}$$

Ramp Function

$$f(t)\begin{cases} = 0, & \text{for } t < 0 \\ = At, & \text{for } t \geq 0 \end{cases}$$



$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st}(At)dt = A\left[\frac{-te^{-st}}{s}\right]_0^{\infty} - \int_0^{\infty} \frac{Ae^{-st}}{-s}dt$$

$$= \frac{A}{s}\int_0^{\infty} e^{-st}dt = \frac{A}{s^2}$$

Example 4

Use the **Laplace Transform Table** to determine the Laplace Transform of the following functions:

(a) te^{4t} and (b) $e^{-5t}\sin 377t$

Answer:

Example 5

Find the solution of $x(t)$ of the differential equation with zero initial condition, $x''(t) + 2x'(t) - 3x(t) = 3$, using Laplace Transform

Answer:

From the Laplace transform table,

$$s^2X(s) - sx(0) - x'(0) + 2[sX(s) - x(0)] - 3X(s) = \frac{3}{s}$$

With zero initial condition, $x(0) = 0, x'(0) = 0$. We have,

$$s^2X(s) + 2sX(s) - 3X(s) = \frac{3}{s}$$

$$(s^2 + 2s - 3)X(s) = \frac{3}{s}$$

$$\therefore X(s) = \frac{3}{s(s^2 + 2s - 3)}$$

Example 5

<u>Answer:</u>

$$X(s) = \frac{3}{s(s^2 + 2s - 3)} = \frac{3}{s(s+3)(s-1)}$$

By partial fraction decomposition,

$$X(s) = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-1} = \frac{A(s-1)(s+3) + Bs(s-1) + Cs(s+3)}{s(s+3)(s-1)}$$

So that,

$$A(s+3)(s-1) + Bs(s-1) + Cs(s+3) = 3$$

Put $s = 0$, $s = -3$, and $s = 1$ respectively on both sides of the above equality. We have

$$A(0+3)(0-1) + B(0)(0-1) + C(0)(0+3) = 3 \Longrightarrow A = -1$$

$$A(-3+3)(-3-1) + B(-3)(-3-1) + C(-3)(-3+3) = 3 \Longrightarrow B = \frac{1}{4}$$

$$A(1+3)(1-1) + B(1)(1-1) + C(1)(1+3) = 3 \Longrightarrow C = \frac{3}{4}$$

Example 5

<u>Answer:</u>

By partial fraction expansion,

$$X(s) = \frac{-1}{s} + \frac{\frac{1}{4}}{s+3} + \frac{\frac{3}{4}}{s-1}$$

Hence, the inverse Laplace transform becomes,

$$\therefore x(t) = -1 + \frac{1}{4}e^{-3t} + \frac{3}{4}e^{t}$$

Example 6

Find the solution of $v(t)$ of the differential equation with zero initial condition,

$$2v(t) + \frac{1}{2}v'(t) = u_s(t)$$

<u>Answer:</u>

Taking Laplace Transform, we have

$$2V(s) + \frac{1}{2}[sV(s) - v(0)] = \frac{1}{s}$$

With zero initial condition and rearranging the terms, we have

$$2V(s) + \frac{1}{2}sV(s) = \frac{1}{s} \rightarrow V(s)\left(2 + \frac{1}{2}s\right) = \frac{1}{s} \rightarrow V(s) = \frac{2}{s(s+4)}$$

Taking Inverse Laplace Transform from the Table, we have

$$v(t) = (2)\left(\frac{1}{4}(1 - e^{-4t})\right) = \frac{1}{2}(1 - e^{-4t})$$