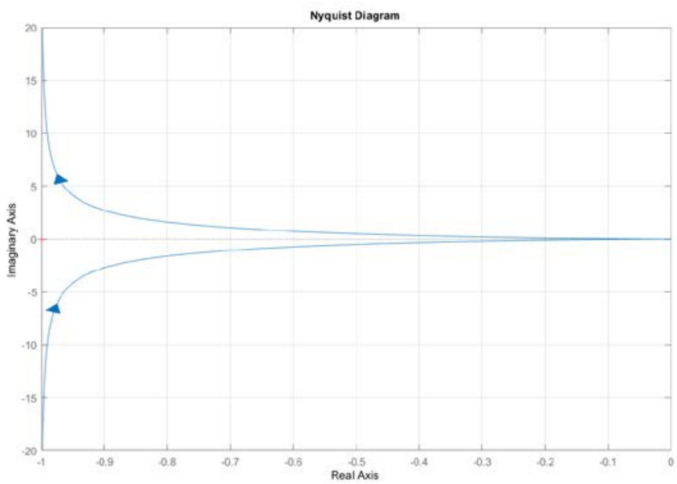
SEHS4653 Control System Analysis
Tutorial Questions (Part 4) Solution

1. $|G(j\omega)| = \dfrac{1}{\omega\sqrt{\omega^2+1}}$ and $\angle G(j\omega) = -90° - \tan^{-1}\omega$

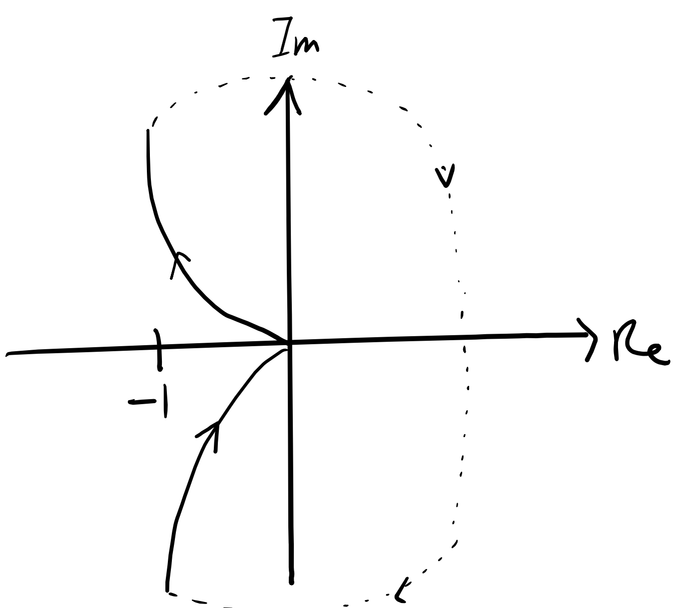
From Matlab:



Sketch:

$\omega = 0^+; |G(j\omega)| = \infty$ and $\angle G(j\omega) = -90°$

$\omega = +\infty; |G(j\omega)| = 0$ and $\angle G(j\omega) = -180°$



2. (a)

$$|G(j\omega)H(j\omega)| = \frac{2}{\omega\sqrt{\omega^2+1}\sqrt{(2\omega)^2+1}}$$

$$\angle G(j\omega)H(j\omega) = -90° - \tan^{-1}\omega - \tan^{-1}2\omega$$

$$-90° - \tan^{-1}\omega - \tan^{-1}2\omega = -180°, \quad -\tan^{-1}\omega - \tan^{-1}2\omega = -90°$$

$$\because \tan^{-1}X + \tan^{-1}Y = \tan^{-1}\left(\frac{X+Y}{1-XY}\right)$$

$$\therefore \tan^{-1}\left(\frac{\omega+2\omega}{1-(\omega)(2\omega)}\right) = 90°$$

$$\frac{3\omega}{1-2\omega^2} = \infty$$

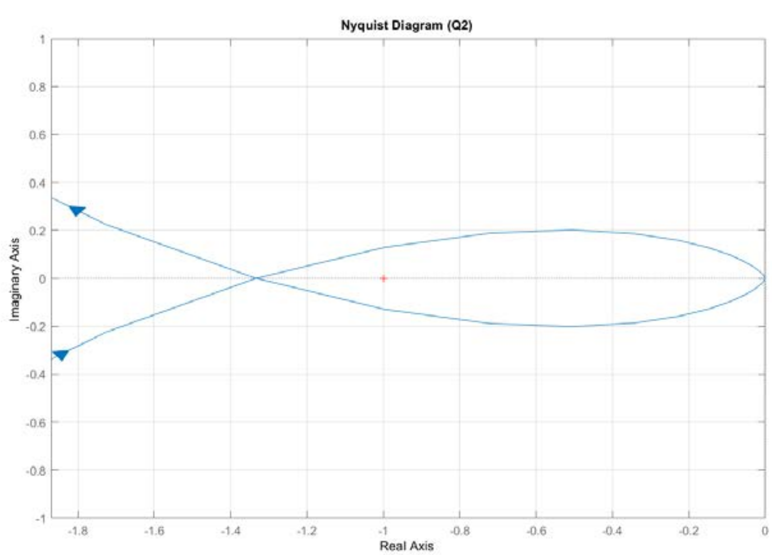
$$\therefore 1-2\omega^2 = 0, \quad \omega = \sqrt{\frac{1}{2}} = 0.707 \text{ rad/s}$$

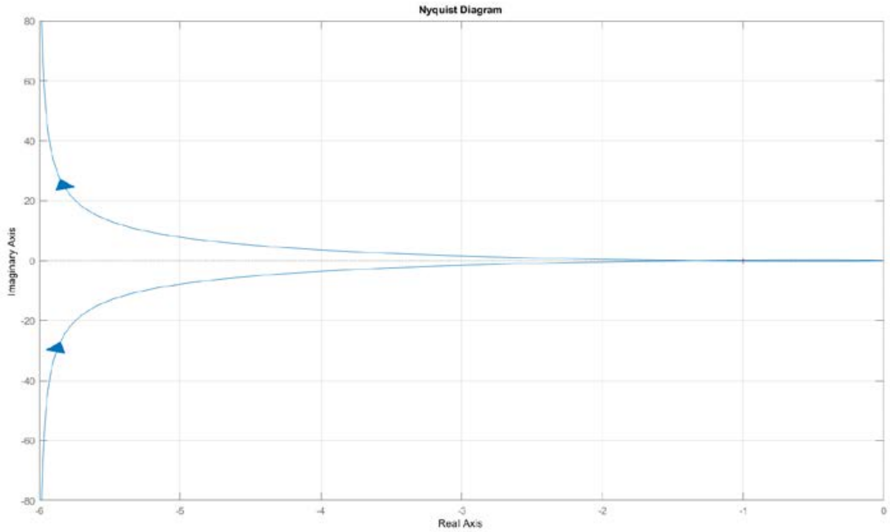
At ω = 0.707 rad/s, the system magnitude will be

$$|G(j\omega)H(j\omega)| = \frac{2}{\sqrt{1/2}\sqrt{\left(\sqrt{1/2}\right)^2+1}\sqrt{\left(2\sqrt{1/2}\right)^2+1}} = 1.333$$

Hence, the Nyquist plot will cross in the x-axis at –1.333.

From Matlab:

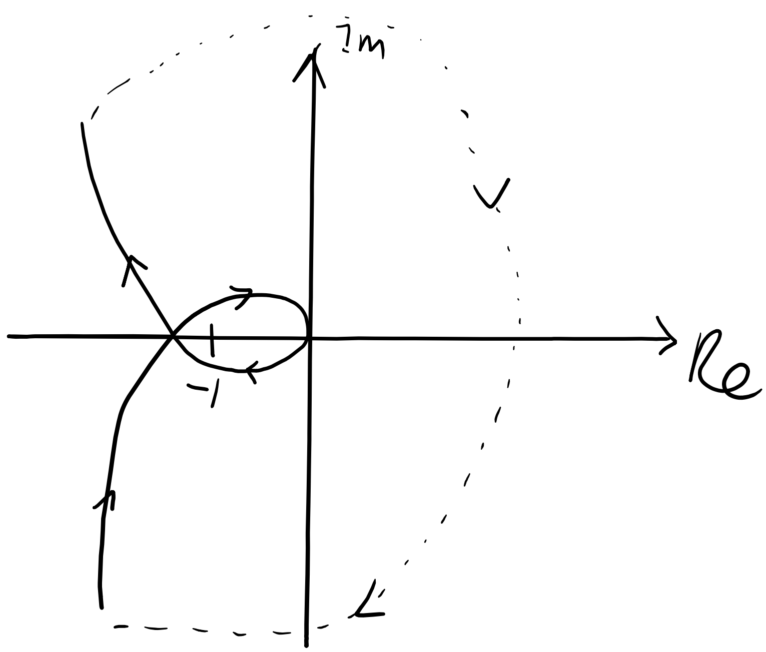




Sketch:

$\omega = 0^+; |G(j\omega)| = \infty$ and $\angle G(j\omega) = -90°$

$\omega = +\infty; |G(j\omega)| = 0$ and $\angle G(j\omega) = -270°$



2. (b) Nyquist stability criterion:

$P = 0$

$N = 2$

$\therefore Z = 2$

Hence, the system is unstable since there are 2 closed-loop poles in the right-half *s* plane.

2. (c) Let find the point where the Nyquist plot crosses the negative real axis ⇒ the imaginary part of $G(j\omega)H(j\omega) = 0$, which

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j\omega+1)(2j\omega+1)} = \frac{K}{-3\omega^2 + j(\omega - 2\omega^3)}$$

$$\omega - 2\omega^3 = 0 \Rightarrow \omega(1 - 2\omega^2) = 0$$

$$\therefore \omega = 0, \quad 1 - 2\omega^2 = 0 \Rightarrow \omega = \pm\frac{1}{\sqrt{2}}$$

Substituting $\omega = 1/\sqrt{2}$,

$$\frac{K}{-3\left(\frac{1}{\sqrt{2}}\right)^2 + j(0)} = -\frac{2K}{3}$$

The critical value of the gain K is obtained by equating $-\frac{2K}{3} = -1 \Rightarrow K = \frac{3}{2}$.
The system is stable if

$$0 < K < \frac{3}{2}$$

OR using Routh Array

The characteristic equation, $\Delta(s) = s(s+1)(2s+1) + K = 2s^3 + 3s^2 + s + K = 0$

s^3	2	1
s^2	3	K
s^1	$\frac{3-2K}{2}$	
s^0	K	

$$\frac{3-2K}{2} > 0, \quad K < \frac{3}{2}$$

Hence the system is stable if

$$0 < K < \frac{3}{2}$$

3. (a) & (b) First-order lag system

$$G_1(j\omega) = \frac{1}{j\omega + 1}$$

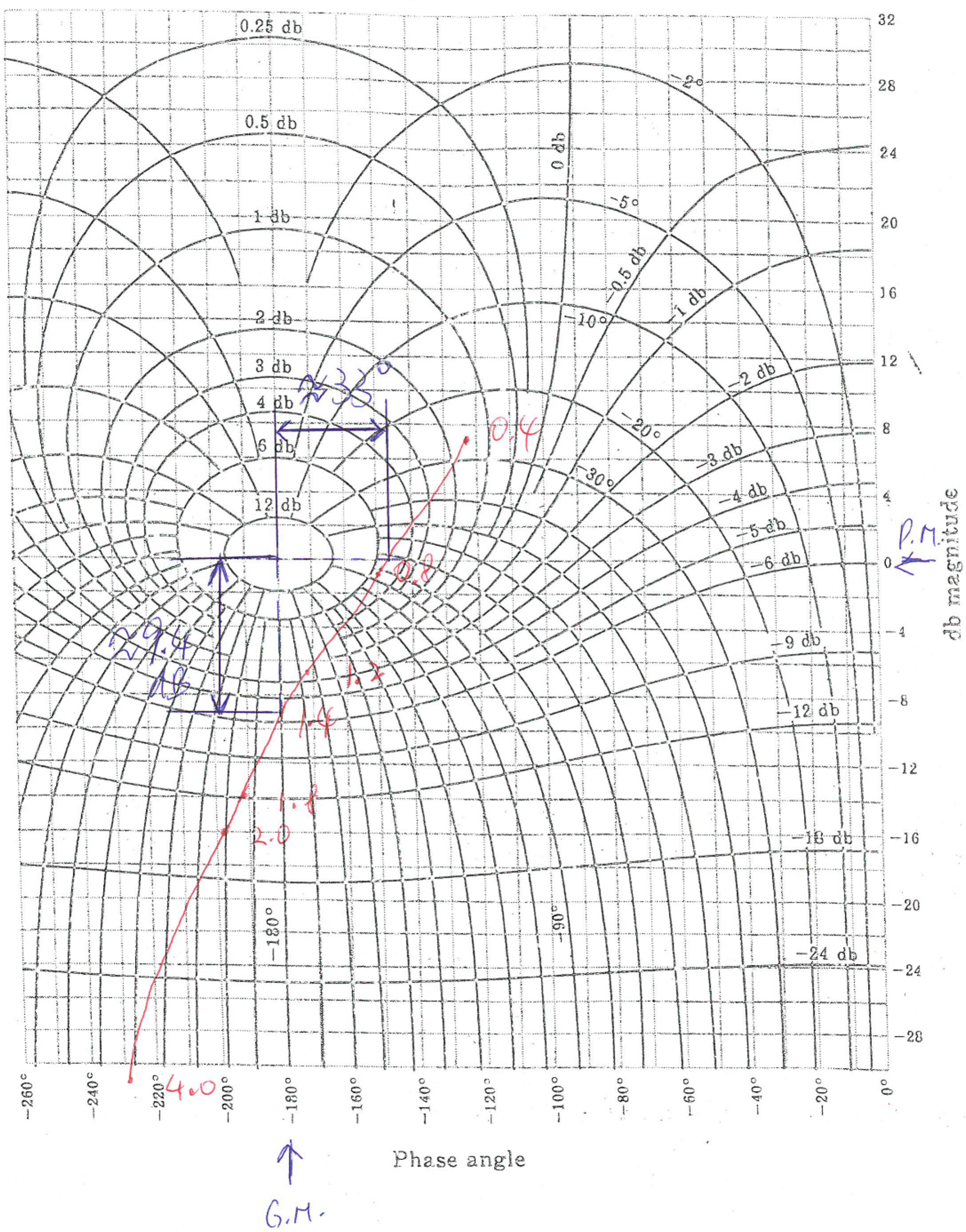
$$|G(j\omega)| = 20\log\left(\frac{1}{\sqrt{\omega^2+1}}\right) = -10\log(\omega^2 + 1)$$

$$\angle G(j\omega) = -\tan^{-1}\omega$$

Change the gain value to dB for the given open-loop frequency response.

ω (rad/s)	Open-loop system: $G_p(j\omega)$		$G_1(j\omega)$		Combined system: $G_1(j\omega)G_p(j\omega)$	
	Gain (dB)	Phase (°)	Gain (dB)	Phase (°)	Gain (dB)	Phase (°)
0.4	7.79	–101.31	–0.64	–21.80	7.14	–123.11
0.8	1.29	–111.80	–2.15	–38.66	–0.85	–150.46
1.2	–2.92	–120.96	–3.87	–50.19	–6.79	–171.16
1.4	–4.65	–124.99	–4.71	–54.46	–9.37	–179.45
1.8	–7.68	–131.99	–6.27	–60.95	–13.96	–192.94
2.0	–9.03	–135	–6.99	–63.43	–16.02	–198.43
4.0	–19.03	–153.43	–12.30	–75.96	–31.33	–229.40
8.0	–30.37	–165.96	–18.13	–82.88	–41.50	–248.84

Use the data of the combined system for drawing on the Nichlos chart.



From the Nichlos chart, gain margin = 9.4 dB and phase margin = 33°

4. (a) Similar to Q3

$$G_B(j\omega) = \frac{1}{1 + 0.2j\omega}$$

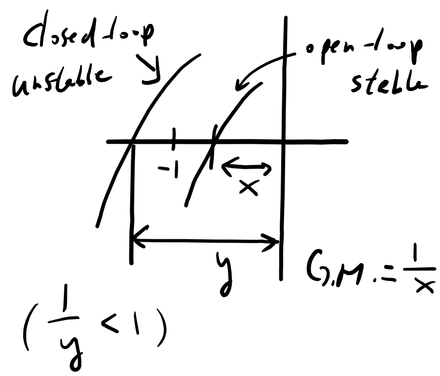
$$|G(j\omega)| = \frac{1}{\sqrt{(0.2\omega)^2 + 1}} \quad \text{and} \quad \angle G(j\omega) = -\tan^{-1} 0.2\omega$$

ω	$G_A(j\omega)$		$G_B(j\omega)$		$G_C(j\omega)$		$G_A(j\omega)G_B(j\omega)G_C(j\omega)$	
	Gain	Phase (°)	Gain	Phase (°)	Gain	Phase (°)	Gain	Phase (°)
5	G	0	0.707	–45	0.894	–63.26	0.632G	–108.26
7	G	0	0.581	–54.46	0.714	–89.12	0.415G	–143.58
10	G	0	0.447	–63.43	0.447	–116.34	0.2G	–179.77
10.8	G	0	0.4201	–65.16	0.394	–121.42	0.1655G	–186.58
20	G	0	0.243	–75.96	0.124	–152.39	0.03G	–228.35

Gain margin: G.M = 1 / 0.2G
For stability,

$$\frac{1}{0.2G} > 1 \quad or\ G < 5$$

Hence, the system is unstable if G > 5.



4. (b) The transfer function of the compensation element: $G_c(j\omega) = 1 + Kj\omega$, then
$|G(j\omega)| = \sqrt{(\omega K)^2 + 1}$ and $\angle G(j\omega) = \tan^{-1} K\omega$

ω	$G_A(j\omega)G_B(j\omega)G_C(j\omega)$		$G_c(j\omega)$	
	Gain	Phase (°)	Gain	Phase (°)
5	3.792	–108.26	$\sqrt{25K^2+1}$	$\tan^{-1} 5K$
7	2.49	–143.58	$\sqrt{49K^2+1}$	$\tan^{-1} 7K$
10	1.2	–179.77	$\sqrt{100K^2+1}$	$\tan^{-1} 10K$
10.8	0.993	–186.58	$\sqrt{116.64K^2+1}$	$\tan^{-1} 10.8K$
20	0.18	–228.35	$\sqrt{400K^2+1}$	$\tan^{-1} 20K$

Consider the design point at $\omega = 10.8$ rad/s, for neutral stability,

$$\left(\sqrt{116.64K^2+1}\right)(0.993) = 1 \Rightarrow K = 0.0110$$

(It will be more accurate if you take more decimal places in calculating the Gain of various elements.)

5. (a) The transfer function of the phase-lead compensator is,

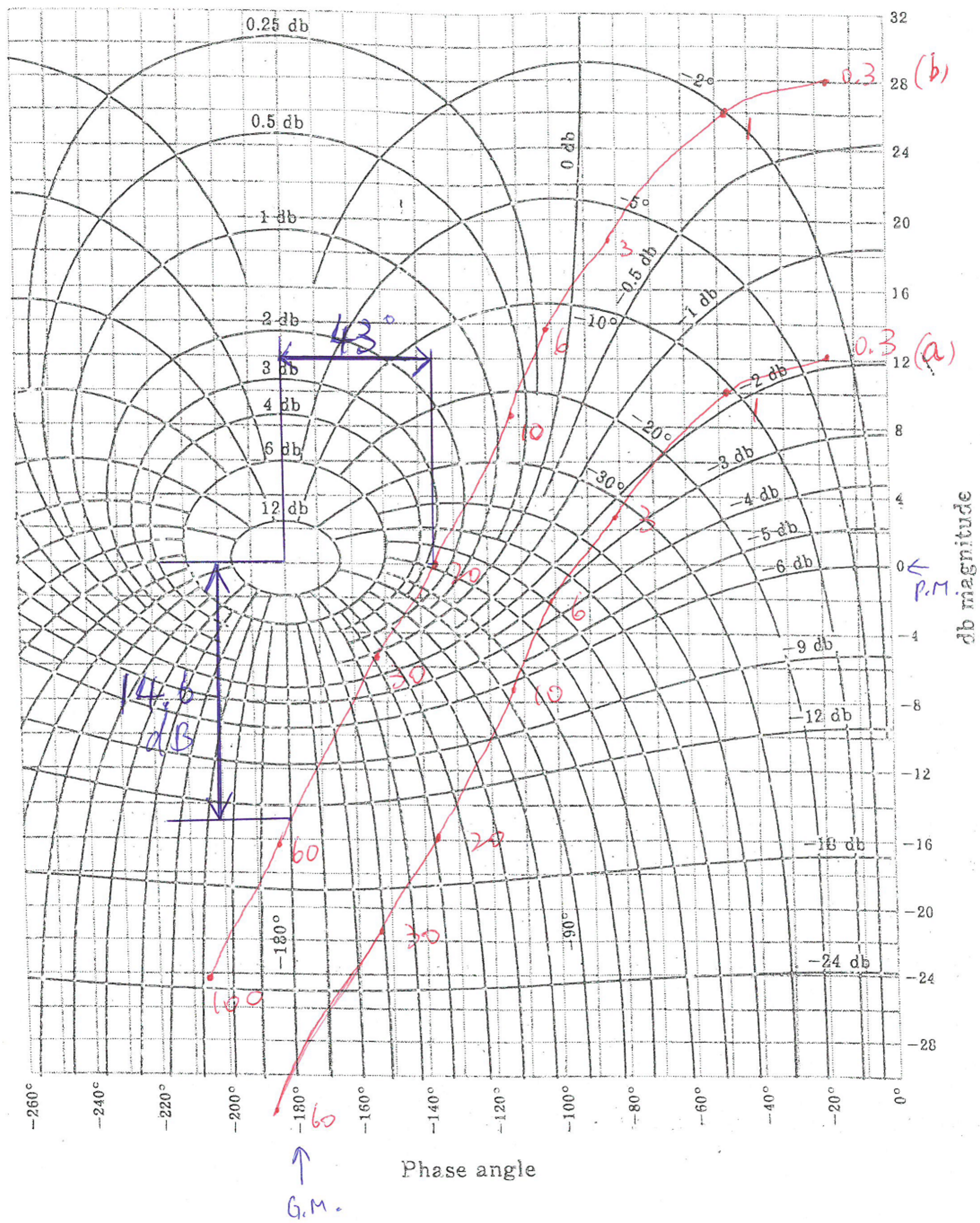
$$G_c(j\omega) = \frac{0.4(1 + 0.08j\omega)}{1 + 0.032j\omega}$$

$$|G_c(j\omega)| = 20\log\frac{0.4\sqrt{(0.08\omega)^2+1}}{\sqrt{(0.032\omega)^2+1}} \quad \text{and} \quad \angle G_c(j\omega) = \tan^{-1} 0.08\omega - \tan^{-1} 0.032\omega$$

ω	$G_p(j\omega)$		$G_c(j\omega)$		$G_c(j\omega)G_p(j\omega)$	
	Gain (dB)	Phase (°)	Gain (dB)	Phase (°)	Gain (dB)	Phase (°)
0.3	20	–19	–7.96	0.82	12.04	–18.19

1	18	−51	−7.94	2.74	10.06	−48.26
3	10.5	−91	−7.76	8.01	2.74	−82.99
6	5	−116	−7.22	14.77	−2.22	−101.23
10	−1	−135	−6.23	20.92	−7.23	−114.08
20	−12	−163	−3.94	25.38	−15.94	−137.62
30	−19	−177	−2.50	23.55	−21.50	−153.45
60	−31.5	−201	−0.86	15.74	−32.36	−185.26
100	−40	−218	−0.34	10.23	−40.34	−207.77

The Nichols chart,



5. (b) (i) & (ii) If the system gain is increased by 16 dB, the curve in the Nichols chart in part (a) will shift up by 16 dB which is shown in the Nichols chart in part (a) as well.

From the new curve, Gain margin = 14.6 dB and Phase margin = 43°

6. By setting $T_i = \infty$ and $T_d = 0$ (i.e. only K_p) , we obtain the closed-loop transfer function as follow,

$$\frac{C(s)}{R(s)} = \frac{K_p}{s(s+1)(s+5)+K_p}$$

The value of K_p that makes the system marginally stable (so that sustained oscillation occurs) by using the Routh's stability criterion, $\Delta(s) = s^3 + 6s^2 + 5s + K_p = 0$

s^3	1	5
s^2	6	K_p
s^1	$\frac{(6)(5)-(1)(K_p)}{6}$	
s^0	K_p	

$$\therefore \frac{30-K_p}{6} = 0 \Rightarrow K_p = 30$$

With gain K_p sets equal to K_c (= 30), the characteristic equation becomes,

$$\Delta(s) = s^3 + 6s^2 + 5s + 30 = 0$$

We substitute $s = j\omega$ into the $\Delta(s)$, we have

$$(j\omega)^3 + 6(j\omega)^2 + 5j\omega + 30 = 0$$
$$-j\omega^3 - 6\omega^2 + 5j\omega + 30 = 0$$

Rearranging the terms, we have

$$j\omega(5-\omega^2) + 6(5-\omega^2) = 0$$

Hence, we have

$$5-\omega^2 = 0 \Rightarrow \omega = \pm\sqrt{5} \quad \text{and} \quad \omega = 0$$

The period of sustained oscillation is,

$$T_c = \frac{2\pi}{\omega} = 2.8099 \text{ sec}$$

Referring to Ziegler and Nichols tuning rule,

$K_p = 0.6K_c = (0.6)(30) = 1.8$

$T_i = 0.5T_c = (0.5)(2.8099) = 1.405$

$T_d = 0.125T_c = (0.125)(2.8099) = 0.351$

7. $\frac{Y(s)}{X(s)} = C(sI-A)^{-1}B = [1 \quad 2]\left(s\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}-5 & -1\\3 & -1\end{bmatrix}\right)^{-1}\begin{bmatrix}2\\5\end{bmatrix}$

$$\frac{Y(s)}{X(s)} = [1 \quad 2]\begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1} = \frac{1}{(s+5)(s+1)-(-3)(1)}\begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}$$

$$\frac{Y(s)}{X(s)} = [1 \quad 2]\frac{1}{(s+5)(s+1)-(-3)(1)}\begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \frac{1}{s^2+6s+8}[s+7 \quad 2s+9]\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \frac{12s+59}{s^2+6s+8}$$

8. $$X_1(s) = \frac{1}{s+4}X_2(s) \;\Rightarrow\; sX_1(s) = -4X_1(s) + X_2(s)$$

$$X_2(s) = \frac{5}{s+2}[V(s) - X_1(s)] \;\Rightarrow sX_2(s) = 5V(s) - 5X_1(s) - 2X_2(s)$$

Taking inverse Laplace Transform, we have

$\dot{x}_1(t) = -4x_1(t) + x_2(t)$ and $\dot{x}_2(t) = -5x_1(t) - 2x_2(t) + 5v(t)$

Hence, the state space equations are,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} v(t)$$

$$y(t) = [1 \quad 0]\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The state transmition matrix in s-domain, $\Phi(s) = (sI - A)^{-1}$

$$\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 1 \\ -5 & -2 \end{bmatrix}\right)^{-1} = \begin{bmatrix} s+4 & -1 \\ 5 & s+2 \end{bmatrix}^{-1} = \frac{1}{(s+2)(s+4)-(5)(-1)}\begin{bmatrix} s+2 & 1 \\ -5 & s+4 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+2}{s^2+6s+13} & \frac{1}{s^2+6s+13} \\ \frac{-5}{s^2+6s+13} & \frac{s+4}{s^2+6s+13} \end{bmatrix} = \begin{bmatrix} \frac{(s+3)-1}{(s+3)^2+2^2} & \frac{1}{(s+3)^2+2^2} \\ \frac{-5}{(s+3)^2+2^2} & \frac{(s+3)+1}{(s+3)^2+2^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+3}{(s+3)^2+2^2} - \frac{1}{2}\frac{2}{(s+3)^2+2^2} & \frac{1}{2}\frac{2}{(s+3)^2+2^2} \\ -\frac{5}{2}\frac{2}{(s+3)^2+2^2} & \frac{s+3}{(s+3)^2+2^2} + \frac{1}{2}\frac{2}{(s+3)^2+2^2} \end{bmatrix}$$

Then, taking inverse Laplace transform (rules 15 and 16), we have

$$\phi(t) = \begin{bmatrix} e^{-3t}\cos(2t) - \frac{1}{2}e^{-3t}\sin(2t) & \frac{1}{2}e^{-3t}\sin(2t) \\ -\frac{5}{2}e^{-3t}\sin(2t) & e^{-3t}\cos(2t) + \frac{1}{2}e^{-3t}\sin(2t) \end{bmatrix}$$

9. We first write out the differential equations of the system,

$$V_i(t) = L_1\frac{d}{dt}i_1(t) + [i_1(t) - i_2(t)]R$$

$$[i_1(t) - i_2(t)]R = L_2\frac{d}{dt}i_2(t) + \frac{1}{C}\int i_2(t)\,dt$$

$$V_o(t) = \frac{1}{C}\int i_2(t)\,dt$$

Since state space equations related differntial functions instead of integrals, hence, we need to transform integral into differentiation as follow,

$$V_o(t) = \frac{1}{C}\int i_2(t)\,dt \;\Rightarrow i_2(t) = C\frac{d}{dt}V_o(t)$$

Let $x_1(t) = V_o(t) = y(t)$, $x_2(t) = i_1(t)$, $x_3(t) = i_2(t)$ and $u(t) = V_i(t)$, then we have,

$$u(t) = L_1\dot{x}_2(t) + x_2(t)R - x_3(t)R$$

$$x_2(t)R - x_3(t)R = L_2\dot{x}_3(t) + x_1(t)$$

$$x_3(t) = C\dot{x}_1(t)$$

Rearranging the terms,

$$\dot{x}_2(t) = -\frac{R}{L_1}x_2(t) + \frac{R}{L_1}x_3(t) + \frac{1}{L_1}u(t)$$

$$\dot{x}_3(t) = -\frac{1}{L_2}x_1(t) + \frac{R}{L_2}x_2(t) - \frac{R}{L_2}x_3(t)$$

$$\dot{x}_1(t) = \frac{1}{C}x_3(t)$$

Hence, the state space equations of the sytsem is,

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}0 & 0 & \frac{1}{C}\\ 0 & -\frac{R}{L_1} & \frac{R}{L_1}\\ -\frac{1}{L_2} & \frac{R}{L_2} & -\frac{R}{L_2}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{L_1}\\ 0\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

End of Tutorial Questions (Part 4) Solution