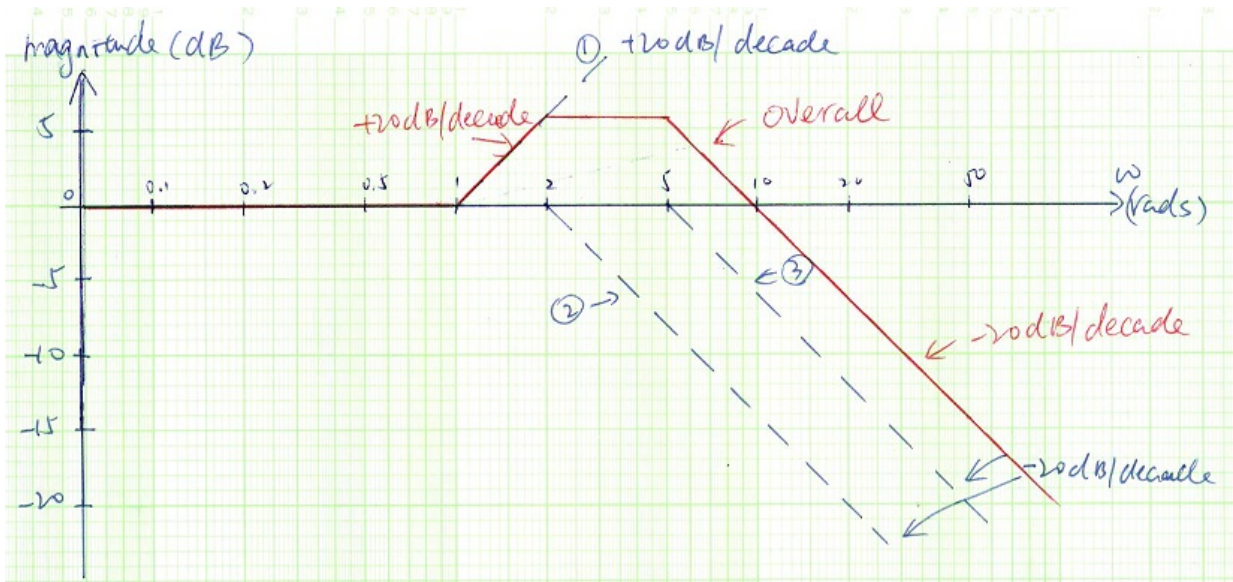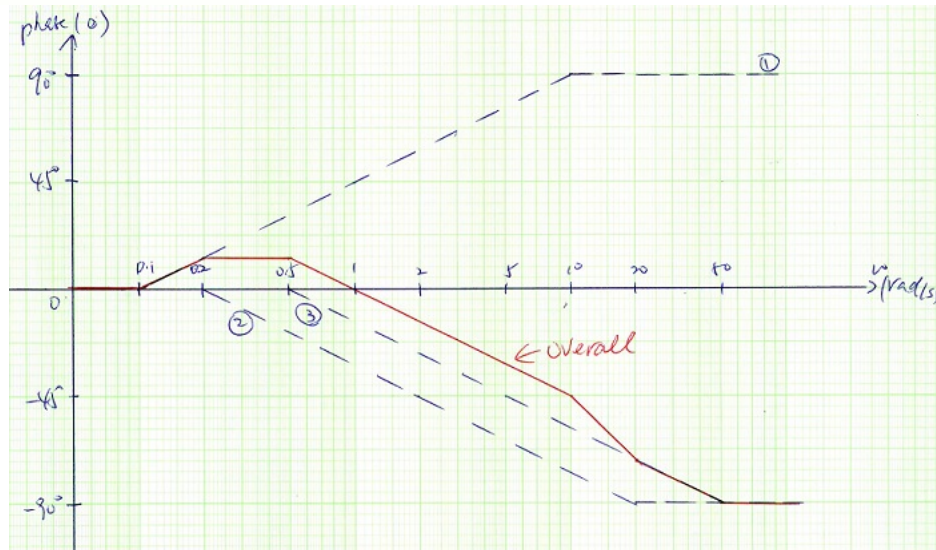1. (a) $$G(s) = \frac{10(s + 1)}{(s + 2)(s + 5)}$$

Asymptotic Bode Diagram:

$$G(j\omega) = \frac{10(j\omega + 1)}{(j\omega + 2)(j\omega + 5)} = \frac{10(j\omega + 1)}{\left(1 + \frac{j\omega}{2}\right)(2)\left(1 + \frac{j\omega}{5}\right)(5)} = \frac{j\omega + 1}{\left(1 + \frac{j\omega}{2}\right)\left(1 + \frac{j\omega}{5}\right)}$$

From the standard form, we have identified 3 basic factors:
1. Zero: $j\omega + 1$; Corner frequency: $\omega = 1$ rad/s
2. Pole: $1 + \frac{j\omega}{2}$; Corner frequency: $\omega = 2$ rad/s
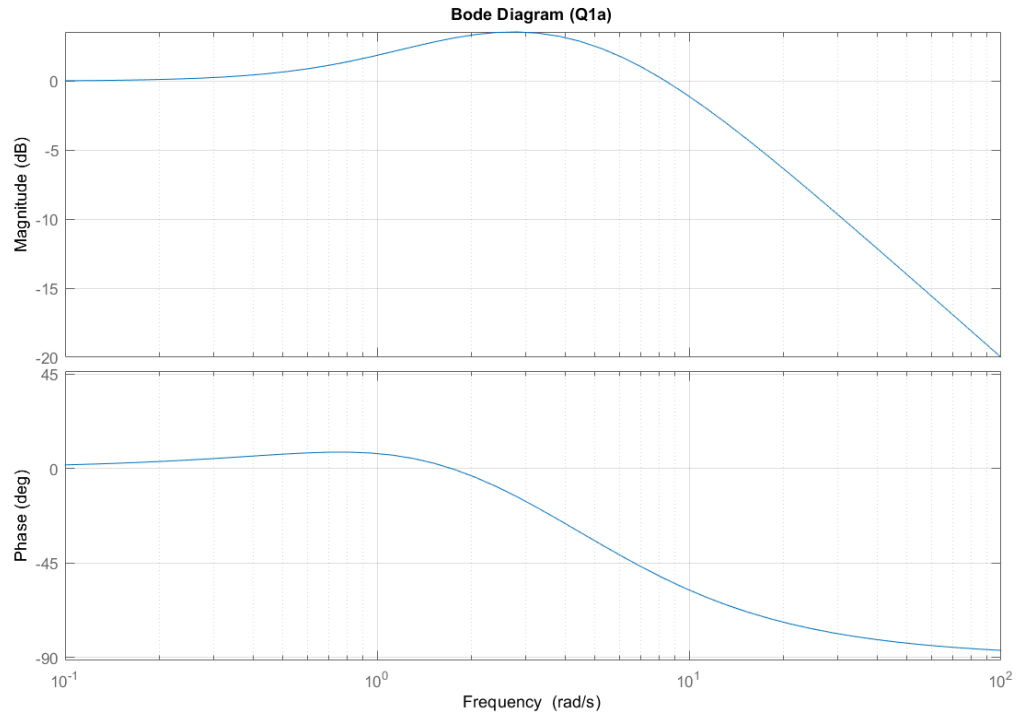3. Pole: $1 + \frac{j\omega}{5}$; Corner frequency: $\omega = 5$ rad/s

Exact Bode Diagram:

$$G(j\omega) = \frac{10(j\omega + 1)}{(j\omega + 2)(j\omega + 5)}$$

$$|G(j\omega)| = 20\log\frac{10\sqrt{1^2 + \omega^2}}{\sqrt{2^2 + \omega^2}\sqrt{5^2 + \omega^2}} \quad (dB)$$

$$\angle G(j\omega) = \tan^{-1}\omega - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{5} \quad (°)$$

ω (rad/s)	Magnitude (dB)	Phase (°)
0.1	0.0306	1.702
0.5	0.663	6.818
1	1.871	7.125
2	3.335	−3.366
5	2.536	−34.508
8	0.310	−51.083
10	−1.096	−57.836
15	−4.037	−67.784
20	−6.316	−73.116
30	−9.676	−78.633
50	−14.028	−83.145
80	−20.012	−85.708

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Bode Diagram (Q1a)

2.
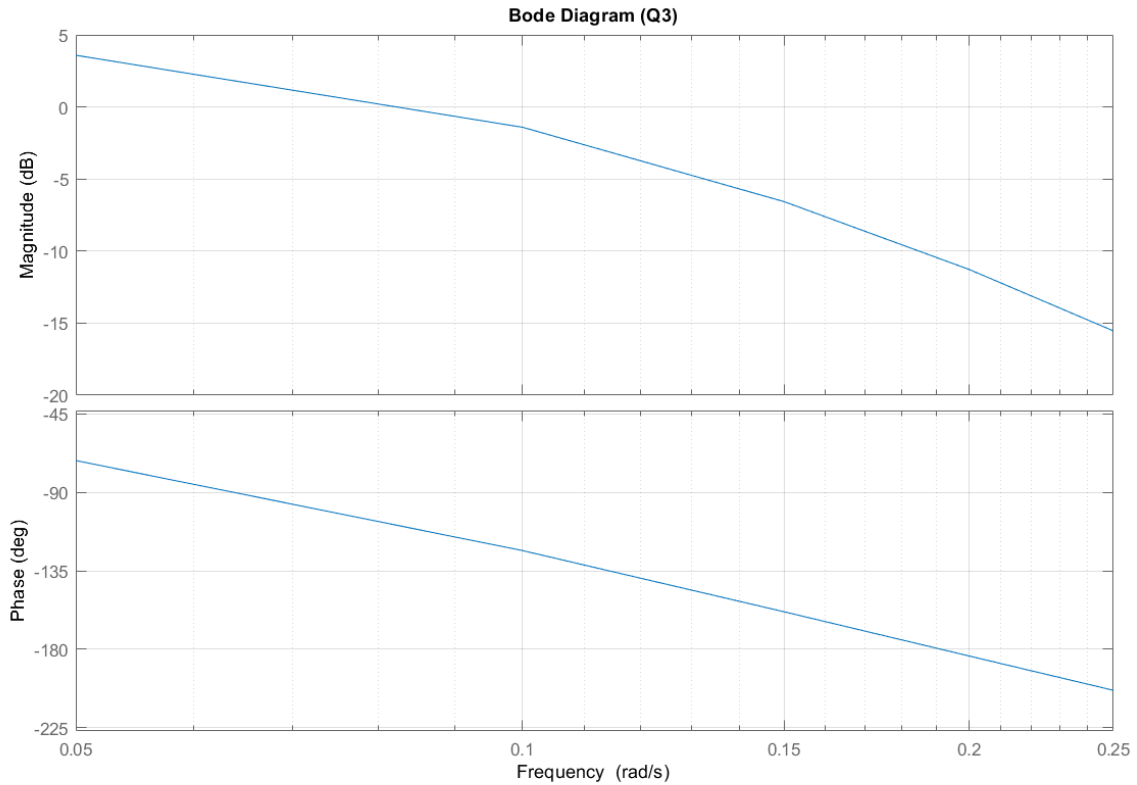$$G(s) = \frac{2}{(1 + 2s)(1 + 3s)(1 + 11s)^2}$$

$$G(j\omega) = \frac{2}{(1 + 2j\omega)(1 + 3j\omega)(1 + 11j\omega)^2}$$

$$|G(j\omega)| = 20 \log \frac{2}{\sqrt{1^2 + (2\omega)^2}\sqrt{1^2 + (3\omega)^2}\left(\sqrt{1^2 + (11\omega)^2}\right)^2} \quad (dB)$$

$$\angle G(j\omega) = -\tan^{-1} 2\omega - \tan^{-1} 3\omega - \tan^{-1} 11\omega - \tan^{-1} 11\omega \quad (°)$$

ω (rad/s)	Magnitude (dB)	Phase (°)
0.05	3.585	−71.86
0.1	−1.412	−123.46
0.15	−6.571	−158.49
0.2	−11.288	−183.88
0.25	−15.539	−203.47

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Bode Diagram (Q3)

3. (a)

$$G(s) = \frac{K}{s(s+2)(s+10)}$$

$$G(j\omega) = \frac{K}{j\omega(j\omega+2)(j\omega+10)}$$

$$|G(j\omega)| = 20\log\frac{K}{\omega\sqrt{\omega^2+2^2}\sqrt{\omega^2+10^2}} \quad \text{(dB)}$$

$$\angle G(j\omega) = -90° - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10} \quad (°)$$

When system phase $\phi = -180°$, and put into the phase equation, we can find the phase crossover frequency, ω_{pc}:

$$-180° = -90° - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10} \quad \Rightarrow \quad \tan^{-1}\frac{\omega}{2} + \tan^{-1}\frac{\omega}{10} = 90°$$

$$\frac{\frac{\omega}{2} + \frac{\omega}{10}}{1 - \frac{\omega^2}{20}} = \infty \quad \Rightarrow \quad 1 - \frac{\omega^2}{20} = 0$$

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$\therefore \omega = \sqrt{20}$ rad/s or 4.47 rad/s

Substitute into this phase crossover frequency into the magnitude equation, we have

$$|G(j\omega)| = 20\log\frac{K}{\sqrt{20}\sqrt{\left(\sqrt{20}\right)^2 + 2^2}\sqrt{\left(\sqrt{20}\right)^2 + 10^2}} = 20\log\frac{K}{\sqrt{20}\sqrt{24}\sqrt{120}}$$

If gain margin = 6 dB, we have

$$20\log\frac{K}{\sqrt{20}\sqrt{24}\sqrt{120}} = -6$$

$\therefore K = 120.28$

3. (b)   With the phase margin = 45°, the corresponding open-loop phase = −135°

$$-135° = -90° - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10} \quad\Rightarrow\quad \tan^{-1}\frac{\omega}{2} + \tan^{-1}\frac{\omega}{10} = 45°$$

$$\frac{\frac{\omega}{2} + \frac{\omega}{10}}{1 - \frac{\omega^2}{20}} = 1 \quad\Rightarrow\quad 1 - \frac{\omega^2}{20} = \frac{\omega}{2} + \frac{\omega}{10} \quad\Rightarrow\quad \frac{\omega^2}{20} + \frac{6\omega}{10} - 1 = 0 \quad\Rightarrow\quad \frac{\omega^2 + 12\omega - 20}{20} = 0$$

$\therefore \omega = 1.48$ rad/s or -13.48 rad/s (rejetced)

Noting that the magnitude of the system ($|G(j\omega)|$) must be equal to 0 dB at $\omega = 1.48$ rad/s, we have

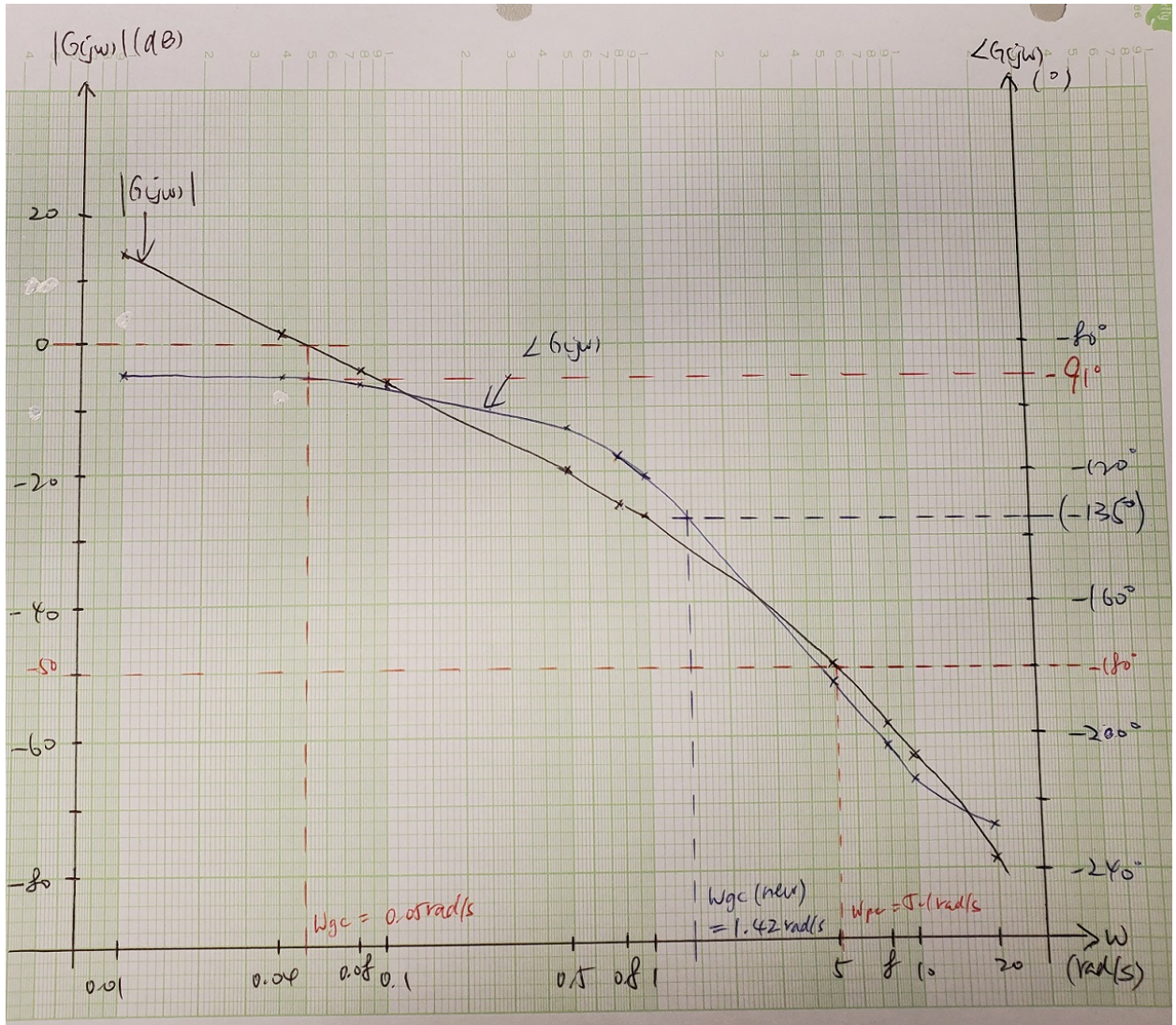$$|G(j\omega)| = 20\log\frac{K}{1.48\sqrt{(1.48)^2 + 2^2}\sqrt{(1.48)^2 + 10^2}} = 0$$

$\therefore K = 37.22$

3. (c)   From the system magnitude and phase equations obtained in part (a), we can draw the exact Bode diagrams

ω (rad/s)	$|G(j\omega)|$ (dB)	$\angle G(j\omega)$ (°)
0.01	14.0	−90.3
0.04	1.9	−91.4
0.08	−4.1	−92.7
0.1	−6.0	−93.4
0.5	−20.2	−106.9
0.8	−24.8	−116.4
1	−27.0	−122.3
5	−49.6	−184.8
8	−58.5	−204.6

| 10 | −63.2 | −213.7 |
| 20 | −79.1 | −237.7 |

From the exact bode diagrams, the gain margin = 50 dB (G.M. = 0 − $|G(j\omega)|$) at 5.1 rad/s. Phase margin = 89° at 0.05 rad/s.

G.M. = 6 = 0 − $|G(j\omega)|$ (phase crossover frequency will not change)

$$\therefore |G(j\omega)| = 20\log\frac{K}{5.1\sqrt{(5.1)^2 + 2^2}\sqrt{(5.1)^2 + 10^2}} = -6$$

$$\therefore K = 157.18$$

P.M. = 180° + $\angle G(j\omega)$ = 45°, $\angle G(j\omega) = -135°$
From the exact bode diagrams, the gain crossover frequency at $\angle G(j\omega) = -135°$ is 1.42 rad/s

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$$\therefore |G(j\omega)| = 20\log\frac{K}{1.42\sqrt{(1.42)^2+2^2}\sqrt{(1.42)^2+10^2}} = 0$$

$$\therefore K = 35.18$$

4. From the block diagram, the open-loop transfer function is,

$$G(s) = \frac{K}{s(s+1)(s+5)}$$

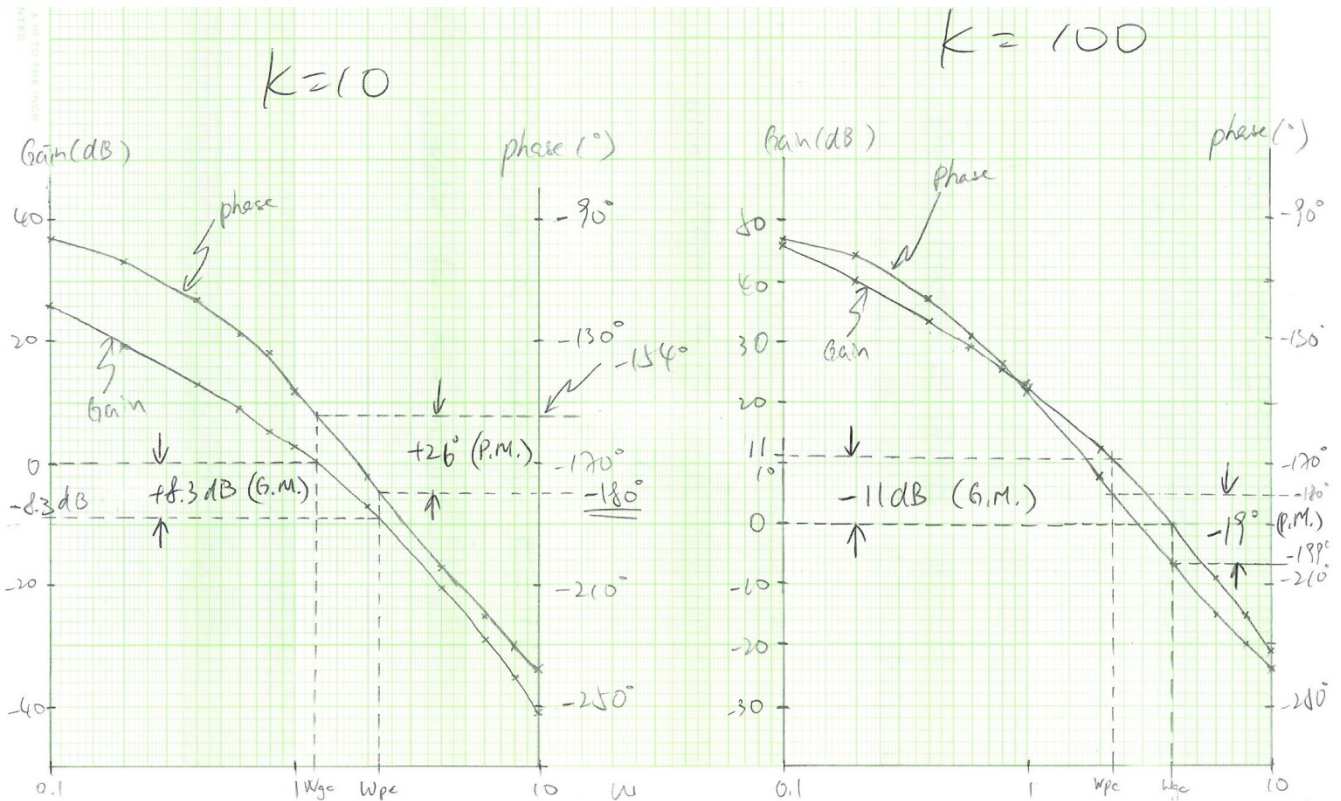Plot the exact Bode diagrams for finding gain margin and phase margin with $K = 10$ and $K = 100$.

$$G(j\omega) = \frac{K}{j\omega(j\omega+1)(j\omega+5)}$$

$$|G(j\omega)| = 20\log\frac{K}{\omega\sqrt{\omega^2+1^2}\sqrt{\omega^2+5^2}} \quad (dB)$$

$$\angle G(j\omega) = -90° - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{5} \quad (°)$$

Change of gain K will Not alter the system phase, we have

ω (rad/s)	Magnitude (dB)		Phase (°)
	$K = 10$	$K = 100$	
0.1	25.98	45.98	−96.86
0.2	19.82	39.82	−103.13
0.4	13.31	33.31	−116.38
0.6	9.06	29.06	−127.81
0.8	5.70	25.70	−137.75
1	2.84	22.84	−146.31
2	−7.63	12.37	−175.24
4	−20.47	−0.47	−204.62
6	−29.10	−9.1	−220.73
8	−35.68	−15.68	−230.87
10	−41.01	−21.01	−237.72

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K	Gain margin	Phase margin
10	+8.3 dB	+26°
100	−11 dB	−19°

Hence, the system is stable for $K = 10$, but unstable for $K = 100$.

5. From the block diagram, the open-loop transfer function, we have

$$G(s) = \frac{K(s + 2)}{s^2}$$

$$G(j\omega) = \frac{K(j\omega + 2)}{(j\omega)^2}$$

$$|G(j\omega)| = 20 \log \frac{K\sqrt{\omega^2 + 2^2}}{\omega^2} \quad \text{(dB)}$$

$$\angle G(j\omega) = \tan^{-1}\frac{\omega}{2} - 90° - 90° \quad \text{(°)}$$

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The required phase margin of 50°means that the system phase be equal to −130°. Hence, the gain crossover frequency can be found from the system phase equation as follow,

$$-130° = \tan^{-1}\frac{\omega}{2} - 90° - 90° \quad \Rightarrow \quad \tan^{-1}\frac{\omega}{2} = 50°$$

$\therefore \omega_{gc} = 2.3835$ rad/s

Noting that the magnitude of the system ($|G(j\omega)|$) must be equal to 0 dB at $\omega = 2.3835$ rad/s, we have

$$20\log\frac{K\sqrt{2.3835^2 + 2^2}}{2.3835^2} = 0$$

$\therefore K = 1.826$

<center>End of Tutorial Questions (Part 3) Solution</center>