SEHS4653 Control System Analysis
Tutorial Questions (Part 2) Solution

1. (a) The closed-loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{\frac{25}{s(s+2)}}{1+\frac{25}{s(s+2)}} = \frac{25}{s(s+2)+25} = \frac{25}{s^2+2s+25}$$

Compared the above transfer function with the 2nd order system, we have

$$\frac{C(s)}{R(s)} = \frac{25}{s^2+2s+25} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Equating terms, we have $2\zeta\omega_n = 2, \quad \omega_n^2 = 25$

$\therefore \omega_n = 5$ rad/s, $(2\zeta)(5) = 2 \Rightarrow \zeta = 0.2$

Since $\omega_d = \omega_n\sqrt{1-\zeta^2} = (5)\left(\sqrt{1-0.2^2}\right) = 4.899$ rad/s

(b) The unit-step response,

$$C(s) = \frac{25}{s^2+2s+25}\frac{1}{s}$$

Taking inverse Laplace transform, we have

$$c(t) = 1 - \frac{1}{\sqrt{1-0.2^2}}e^{-(0.2)(5)t}\sin\left(5\sqrt{1-0.2^2}t+\phi\right), \phi = \cos^{-1}0.2$$

$$\therefore c(t) = 1 - 1.021e^{-t}\sin(4.899t+1.369)$$

(c) Rise time:

$$t_r = \frac{\pi-\beta}{\omega_d} = \frac{\pi-1.3694}{4.899} = 0.362 \text{ sec}$$

$$\beta = \tan^{-1}\frac{\omega_d}{\zeta\omega_n} = \tan^{-1}\frac{4.899}{(0.2)(5)} = 1.3694 \text{ rad}$$

Peak time:

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4.899} = 0.641 \text{ sec}$$

2% Settling time:

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{(0.2)(5)} = 4 \text{ sec}$$

% of overshoot:

$$M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} = e^{-\frac{0.2}{\sqrt{1-0.2^2}}\pi} = 0.5266 \text{ (or } 52.66\%)$$

2. (a) The closed-loop transfer function of the system,

$$\frac{C(s)}{R(s)}=\frac{(1+ks)\frac{25}{s(s+2)}}{1+(1+ks)\frac{25}{s(s+2)}}=\frac{25(1+ks)}{s(s+2)+25(1+ks)}=\frac{25(1+ks)}{s^2+(2+25k)s+25}$$

Equating the terms with the 2nd order equation, we have

$$\omega_n^2=25\Rightarrow\omega_n=5\text{ rad/s}$$

$$2\zeta\omega_n=2+25k\Rightarrow(2)(0.5)(5)=2+25k\Rightarrow k=0.12$$

(b) The unit-step response of the system,

$$C(s)=\left(\frac{25(1+0.12s)}{s^2+\big(2+25(0.12)\big)s+25}\right)\left(\frac{1}{s}\right)=\left(\frac{3s+25}{s^2+5s+25}\right)\left(\frac{1}{s}\right)$$

Rearrange the above equation, we have

$$C(s)=\frac{3s+25}{s(s^2+5s+25)}=\frac{3s}{s(s^2+5s+25)}+\frac{25}{s(s^2+5s+25)}$$
$$=\frac{3}{25}\frac{25}{s^2+5s+25}+\frac{25}{s(s^2+5s+25)}$$

Taking inverse Laplace transform, we have

$$c(t)=\left(\frac{3}{25}\right)\left(\frac{5}{\sqrt{1-0.5^2}}e^{-(0.5)(5)t}\sin\left(5\sqrt{1-0.5^2}t\right)\right)+1$$
$$-\frac{1}{\sqrt{1-0.5^2}}e^{-(0.5)(5)t}\sin\left(5\sqrt{1-0.5^2}t+\phi\right),\phi=\cos^{-1}0.5$$

$$c(t)=1+0.693e^{-2.5t}\sin 4.33t-1.15e^{-2.5t}\sin(4.33t+1.047)$$

3. (a)

Position error constant: $K_p=\lim_{s\to 0}G(s)H(s)=\lim_{s\to 0}\frac{4(s+1)}{s^2(s+2)}=\frac{4}{0}=\infty$

Velocity error constant: $K_v=\lim_{s\to 0}sG(s)H(s)=\lim_{s\to 0}s\frac{4(s+1)}{s^2(s+2)}=\frac{4}{0}=\infty$

Acceleration error constant: $K_a=\lim_{s\to 0}s^2G(s)H(s)=\lim_{s\to 0}s^2\frac{4(s+1)}{s^2(s+2)}=\frac{4}{2}=2$

(b)

State-error (Position): $e_{p,ss}(\infty)=\frac{1}{1+K_p}=\frac{1}{1+\infty}=0$

State-error (Velocity): $e_{v,ss}(\infty)=\frac{1}{K_v}=\frac{1}{\infty}=0$

State-error (Acceleration): $e_{a,ss}(\infty) = \frac{1}{K_a} = \frac{1}{2} = 0.5$

The input signal, $R(s)$, consists of 3 elements, step $\left(\frac{3}{s}\right)$ + ramp $\left(\frac{1}{s^2}\right)$ + parabolic $\left(\frac{1}{2s^3}\right)$ inputs, because the system is LTI, the principle of superposition holds, so the steady-state error will be equal to,

$$e_{ss}(\infty) = 3e_{p,ss}(\infty) - e_{v,ss}(\infty) + \frac{1}{2}e_{a,ss}(\infty) = 3(0) - 0 + \frac{1}{2}(0.5) = 0.25$$

4. (a) The Routh's array,

$$\begin{array}{c|ll}
s^3 & 1 & 8 \\
s^2 & 4 & 12 \\
s^1 & \dfrac{(4)(8)-(1)(12)}{4} = 5 & \\
s^0 & \dfrac{(5)(12)-(4)(0)}{5} = 12 &
\end{array}$$

Since there is no sign change on the 1st column of the Routh's array, the system is **Stable.**

(b) The Routh's array,

$$\begin{array}{c|ll}
s^3 & 2 & 4 \\
s^2 & 4 & 12 \\
s^1 & \dfrac{(4)(4)-(2)(12)}{4} = -2 & \\
s^0 & \dfrac{(-2)(12)-(4)(0)}{-2} = 12 &
\end{array}$$

Since there are 2 sign changes on the 1st column of the Routh's array, the system is **Unstable.**

5. From the Routh's array,

$$\begin{array}{c|lll}
s^4 & 1 & 11 & K \\
s^3 & 6 & 6 & \\
s^2 & \dfrac{(6)(11)-(1)(6)}{6} = 10 & \dfrac{(6)(K)-(1)(0)}{6} = K & \\
s^1 & \dfrac{(10)(6)-(6)(K)}{10} = 6 - 0.6K & & \\
s^0 & \dfrac{(6-0.6K)(K)-(10)(0)}{6-0.6K} = K & &
\end{array}$$

The system is stable if there is no sign change on the 1st column of the Routh's array. Hence, we have $6 - 0.6K > 0, \quad K < 10$ and $K > 0 \Rightarrow 0 < K < 10$.

6. (a) The open-loop transfer function is,

$$G(s)H(s) = \frac{K}{s(s+3)(s+8)}$$

1. **Locate the open-loop poles and zeros of $G(s)H(s)$ on the complex plane (or *s*-plane)**
   Poles: $s = 0, \quad s = -3, \quad s = -8$

2. **Determine the root loci on the real axis**
   Root loci: $(-\infty, -8]$ and $[-3, 0]$

3. **Determine the asymptotes of root loci**
   $$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{n-m} = \frac{\pm 180°(2k+1)}{3-0} = +60°, -60°, \pm 180°$$

   The intersection of the asymptotes and the real axis is found from,
   $$s = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{(0)+(-3)+(-8)}{3-0} = -3.667$$

4. **Find the breakaway point and/or break-in points**
   The characteristic equation for the system is,
   $$\Delta(s) = s(s+3)(s+8) + K = s^3 + 11s^2 + 24s + K = 0$$
   We have
   $$K = -s^3 - 11s^2 - 24s$$

   The breakaway and/or break-in points are found from,
   $$\frac{dK}{ds} = -3s^2 - 22s - 24 = 0$$
   from which we get,
   $$s = -1.333, \quad -6 \text{ (rejected)}$$
   Only $s = -1.333$ lies on the root loci.

5. **Determine the angle of departure (angle of arrival) of the root locus from a complex pole/zero**
   There are no angle of departure (angle of arrival) since the system has no complex pole/zero.

6. **Find the points where the root loci may cross the imaginary axis**
   The characteristic equation for the system is,
   $$\Delta(s) = s(s+3)(s+8) + K = s^3 + 11s^2 + 24s + K = 0$$
   <u>Methods 1</u>
   The Routh's array,

s^3	1	24
s^2	11	K
s^1	$\frac{(11)(24)-(1)(K)}{11} = 24 - \frac{1}{11}K$	
s^0	$\frac{\left(24 - \frac{1}{11}K\right)(K) - (11)(0)}{24 - \frac{1}{11}K} = K$	

   For stable system, $K > 0, \quad 24 - \frac{1}{11}K > 0 \Rightarrow 24 > \frac{1}{11}K$
   $\therefore 0 < K < 264$

Refer to the 2nd row of the Routh's array, we have $11s^2 + 264 = 0$, yielding,

$$s = \pm j4.899$$

<u>Methods 2</u>

Substitute $s = j\omega$ into the characteristic equation to find the points where root-locus branches may cross the imaginary axis, yielding,

$$(j\omega)^3 + 11(j\omega)^2 + 24(j\omega) + K = 0$$

or

$$(K - 11\omega^2) + j\omega(24 - \omega^2) = 0$$

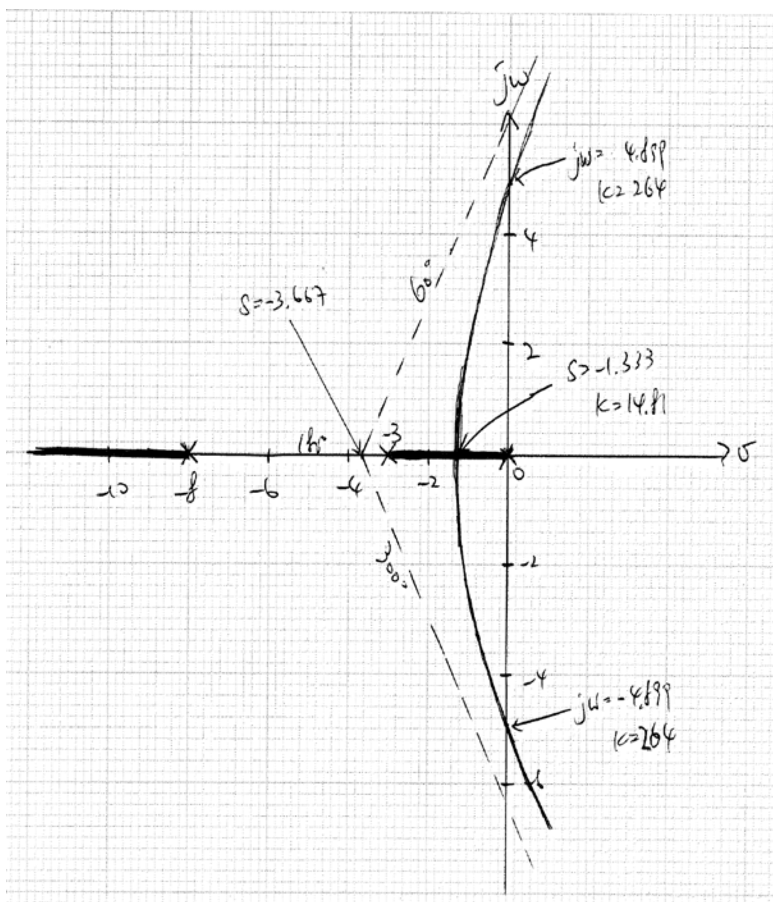
In order to satisfy the equation,

$$K - 11\omega^2 = 0 \text{ and } j\omega(24 - \omega^2) = 0$$

$$\therefore j\omega = \pm\sqrt{24} = \pm j4.899$$

With $\omega = 4.899, K = 11\omega^2 \Rightarrow K = 264$

7. Draw the root locus on the graph paper



(b) The open-loop transfer function is,

$$G(s)H(s) = \frac{K(s+2)(s+3)}{s(s+1)}$$

1. **Locate the open-loop poles and zeros of $G(s)H(s)$ on the complex plane (or s-plane)**
   Poles: $s = 0, \quad s = -1$
   Zeros: $s = -2, \quad s = -3$

2. **Determine the root loci on the real axis**
   Root loci: $[-3, -2]$ and $[-1, 0]$

3. **Determine the asymptotes of root loci**
   Since the number of open-loop poles and zeros are the same. There are NO asymptotes in the complex region of the *s*-plane.

4. **Find the breakaway point and/or break-in points**
   The characteristic equation for the system is,

$$\Delta(s) = s(s+1) + K(s+2)(s+3) = 1 + \frac{K(s+2)(s+3)}{s(s+1)} = 0$$

or

$$K = -\frac{s(s+1)}{(s+2)(s+3)}$$

The breakaway and/or break-in points are found from,

$$\frac{dK}{ds} = -\frac{(s+2)(s+3)\frac{d}{ds}s(s+1) - s(s+1)\frac{d}{ds}(s+2)(s+3)}{[(s+2)(s+3)]^2}$$

$$\frac{dK}{ds} = -\frac{(s+2)(s+3)(2s+1) - s(s+1)(2s+5)}{[(s+2)(s+3)]^2} = -\frac{4s^2+12s+6}{[(s+2)(s+3)]^2} = 0$$

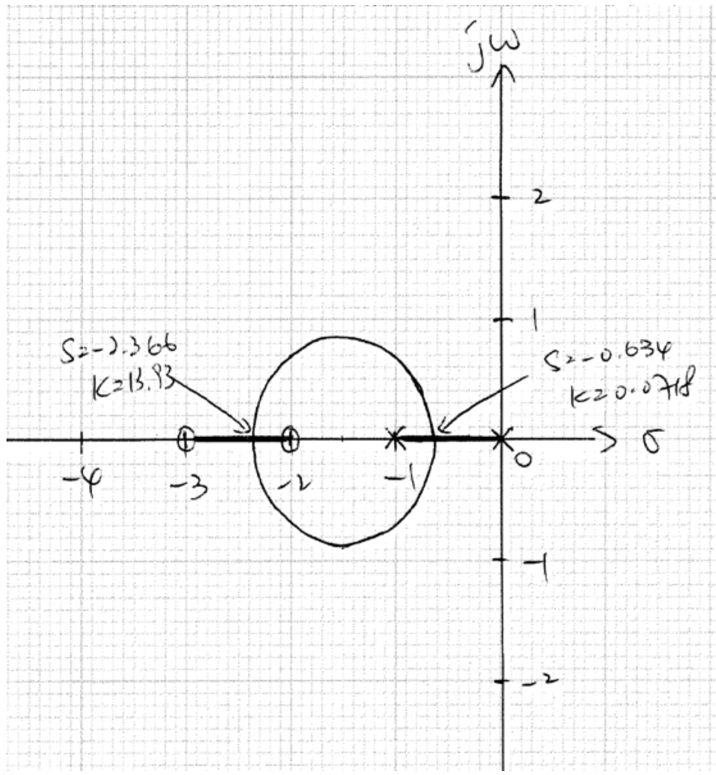
Hence, we have $4s^2 + 12s + 6 = 0$, which yielding,

$$s = -0.634, \quad s = -2.366$$

Both points are on the root loci. Because point lies s = –0.634 between two poles, it is a breakaway point, and because point
s = –2.366 lies between two zeros, it is a break-in point.

5. **Determine the angle of departure (angle of arrival) of the root locus from a complex pole/zero**
   There are no angle of departure (angle of arrival) since the system has no complex pole/zero.

6. **Draw the root locus on the graph paper**
   Determine a sufficient number of points that satisfy the angle condition. (It can be found that the root loci involve a circle with center at –1.5 that passes through the breakaway and break-in points.)



7. The open-loop transfer function is,

$$G(s)H(s) = \frac{K}{s(s+4)}$$

1. Locate the open-loop poles and zeros of $G(s)H(s)$ on the complex plane (or s-plane)
Poles: $s = 0, \quad s = -4$

2. Determine the root loci on the real axis
Root loci: $[-4, 0]$

3. Determine the asymptotes of root loci

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{n-m} = \frac{\pm 180°(2k+1)}{2-0} = +90°, -90°$$

The intersection of the asymptotes and the real axis is found from,

$$s = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{(0)+(-4)}{2-0} = -2$$

4. Find the breakaway point and/or break-in points
The characteristic equation for the system is,

$$\Delta(s) = s(s+4) + K = s^2 + 4s + K = 0$$

We have

$$K = -s^2 - 4s$$

The breakaway and/or break-in points are found from,

$$\frac{dK}{ds} = -2s - 4 = 0$$

from which we get,

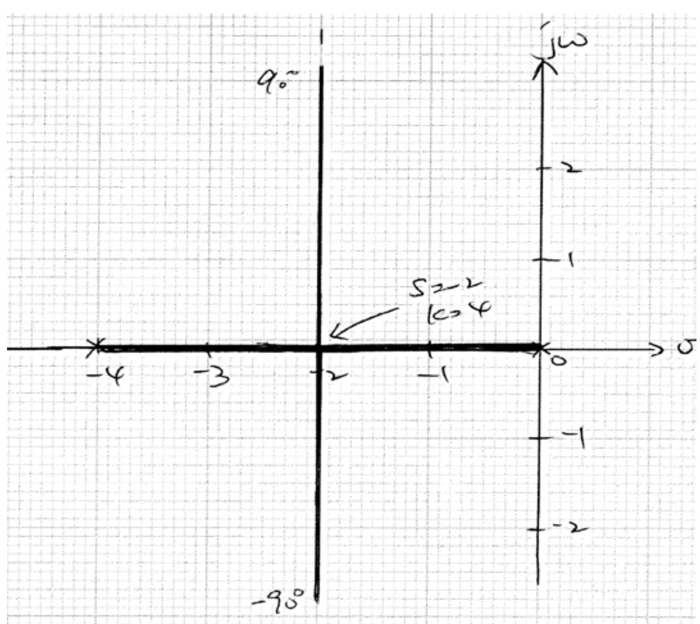
$$s = -2$$

The point $s = -2$ lies on the root loci.

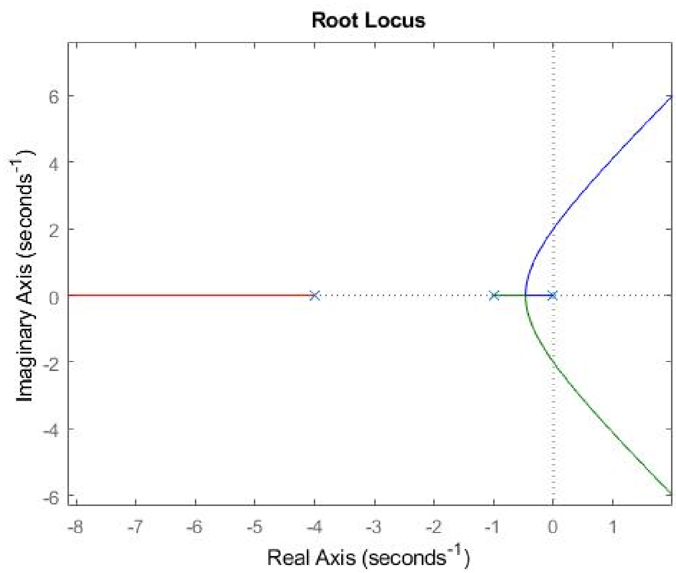
5. **Determine the angle of departure (angle of arrival) of the root locus from a complex pole/zero**
   There is no angle of departure (angle of arrival) since the system has no complex pole/zero.

6. **Find the points where the root loci may cross the imaginary axis**
   The characteristic equation for the system is,
   $$\Delta(s) = s^2 + 4s + K$$
   Substitute $s = j\omega$ into the characteristic equation to find the points where root-locus branches may cross the imaginary axis, yielding,
   $$(j\omega)^2 + 4(j\omega) + K = 0$$
   or
   $$(K - \omega^2) + 4j\omega = 0$$
   Notice that this equation can be satisfied only if $\omega = 0, \quad K = 0$. The root-locus branches do not cross the $j\omega$ axis.

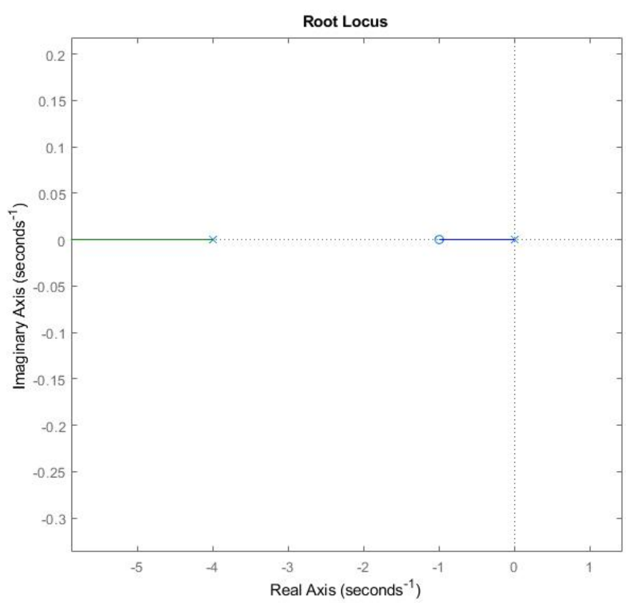
7. **Draw the root locus on the graph paper**



(a) The addition of a pole has the effect of pulling the root locus to the right, tending to lower the system's relative stability. The root locus plot is shown below.



(b) The addition of zeros has the effect of pulling the root locus to the left, tending to make the system more stable. The root locus plot is shown below.



End of Tutorial Questions (Part 2) Solution