SEHS4653 Control System Analysis
Tutorial Questions (Part 1) Solution

1. (a) Taking Laplace transform,

$$sY(s) - y(0) + Y(s) = (2)\left(\frac{1}{s-1}\right)$$

Putting initial values,

$$(s+1)Y(s) - 4 = \frac{2}{s-1}$$

$$Y(s) = \frac{2}{(s+1)(s-1)} + \frac{4}{s+1}$$

Partial fraction expansion, we have

$$\frac{2}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1}$$

Equating the terms in the numerator, $A(s-1) + B(s+1) = 2$

Put $s = 1$, $A(1-1) + B(1+1) = 2 \Rightarrow 2B = 2 \Rightarrow B = 1$
Put $s = -1$, $A(-1-1) + B(+1) = 2 \Rightarrow -2A = 2 \Rightarrow A = -1$

Hence,

$$Y(s) = \frac{-1}{s+1} + \frac{1}{s-1} + \frac{4}{s+1} = \frac{3}{s+1} + \frac{1}{s-1}$$

Taking inverse Laplace transform, $y(t) = e^{t} + 3e^{-t}$

(b) Taking Laplace transform,

$$[s^2Y(s) - sy(0) - \dot{y}(0)] + 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s}$$

Putting initial values,

$$[s^2Y(s) - s(-1) - 0] + 3[sY(s) - (-1)] + 2Y(s) = \frac{1}{s}$$

$$Y(s)[s^2 + 3s + 2] + s + 3 = \frac{1}{s}$$

$$Y(s) = \frac{\frac{1}{s} - s - 3}{s^2 + 3s + 2} = \frac{1 - 3s - s^2}{s(s^2 + 3s + 2)}$$

Partial fraction expansion, we have

$$\frac{1 - 3s - s^2}{s(s^2 + 3s + 2)} = \frac{1 - 3s - s^2}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Equating the terms in the numerator,

$$A(s+1)(s+2) + Bs(s+2) + Cs(s+1) = 1 - 3s - s^2$$

Put $s = 0$, we have

$$A(0+1)(0+2)+B(0)(0+2)+C(0)(0+1)=1-3(0)-(0)^2 \Rightarrow A=\frac{1}{2}$$

Put $s = -1$, we have

$$A(-1+1)(-1+2)+B(-1)(-1+2)+C(-1)(-1+1)=1-3(-1)-(-1)^2 \Rightarrow B=-3$$

Put $s = -2$, we have

$$A(-2+1)(-2+2)+B(-2)(-2+2)+C(-2)(-2+1)=1-3(-2)-(-2)^2 \Rightarrow C=\frac{3}{2}$$

Hence,

$$Y(s)=\frac{1}{2}\frac{1}{s}+(-3)\frac{1}{s+1}+\frac{3}{2}\frac{1}{s+2}$$

Taking inverse Laplace transform,

$$y(t)=\frac{1}{2}-3e^{-t}+\frac{3}{2}e^{-2t}$$

2. (a) Taking Laplace Transform with zero initial condition,

$$s^2Y(s)+2sY(s)+Y(s)=R(s)$$
$$Y(s)=\frac{1}{s^2+2s+1}R(s)$$

<u>Unit-impulse response</u>

$$Y(s)=\frac{1}{s^2+2s+1}(1)=\frac{1}{(s+1)^2}$$

Taking inverse Laplace transform,

$$y(t)=h(t)=te^{-t}$$

<u>Unit-step response</u>

$$Y(s)=\frac{1}{s^2+2s+1}\left(\frac{1}{s}\right)=\frac{1}{s}\frac{1}{(s+1)^2}$$

Partial fraction expansion, we have

$$\frac{1}{s}\frac{1}{(s+1)^2}=\frac{A}{s}+\frac{B}{(s+1)^2}+\frac{C}{s+1}$$

Equating the terms in the numerator, $A(s+1)^2+Bs+Cs(s+1)=1$

Put $s = 0$, we have

$$A(0+1)^2+B(0)+C(0)(0+1)=1 \Rightarrow A=1$$

Put $s = -1$, we have

$$A(-1+1)^2+B(-1)+C(-1)(-1+1)=1 \Rightarrow B=-1$$

Put $s = 1$, we have

$$A(1+1)^2 + B(1) + C(1)(1+1) = 1 \Rightarrow C = -1$$

Hence,

$$Y(s) = \frac{1}{s} + \frac{-1}{(s+1)^2} + \frac{-1}{s+1}$$

Taking inverse Laplace transform,

$$y(t) = u(t) = 1 - te^{-t} - e^{-t}$$

(b) Taking Laplace Transform with zero initial condition,

$$s^2Y(s) + sY(s) = R(s)$$
$$Y(s) = \frac{1}{s(s+1)}R(s)$$

<u>Unit-impulse response</u>

$$Y(s) = \frac{1}{s(s+1)}(1)$$

Partial fraction expansion, we have

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

Equating the terms in the numerator, $A(s+1) + Bs = 1$

Put $s = 0$, we have $A(0+1) + B(0) = 1 \Rightarrow A = 1$
Put $s = -1$, we have $A(-1+1) + B(-1) = 1 \Rightarrow B = -1$

Hence,

$$Y(s) = \frac{1}{s} + \frac{-1}{s+1}$$

Taking inverse Laplace transform,

$$y(t) = h(t) = 1 - e^{-t}$$

<u>Unit-step response</u>

$$Y(s) = \frac{1}{s(s+1)}\left(\frac{1}{s}\right) = \frac{1}{s^2}\frac{1}{s+1}$$

Partial fraction expansion, we have

$$\frac{1}{s^2}\frac{1}{s+1} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

Equating the terms in the numerator, $A(s+1) + Bs(s+1) + Cs^2 = 1$

Put $s = 0$, we have

$$A(0+1) + B(0)(0+1) + C(0)^2 = 1 \Rightarrow A = 1$$

Put $s = -1$, we have

$$A(-1+1) + B(-1)(-1+1) + C(-1)^2 = 1 \Rightarrow C = 1$$

Put $s = 1$, we have

$$A(1+1) + B(1)(1+1) + C(1)^2 = 1 \Rightarrow B = -1$$

Hence,

$$Y(s) = \frac{1}{s^2} + \frac{-1}{s} + \frac{1}{s+1}$$

Taking inverse Laplace transform,

$$y(t) = u(t) = t - 1 + e^{-t}$$

(c) Taking Laplace Transform with zero initial condition,

$$s^2Y(s) + 2sY(s) + 2Y(s) = R(s)$$
$$Y(s) = \frac{1}{s^2+2s+2}R(s)$$

Unit-impulse response

$$Y(s) = \frac{1}{s^2+2s+2}(1) = \frac{1}{(s+1)^2+1^2}$$

(The above method is called "completing square")

Taking inverse Laplace transform,

$$y(t) = h(t) = e^{-t}\sin t$$

Unit-step response

$$Y(s) = \frac{1}{s^2+2s+2}\left(\frac{1}{s}\right)$$

Partial fraction expansion, we have

$$\frac{1}{s^2+2s+2}\left(\frac{1}{s}\right) = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

Equating the terms in the numerator, $A(s^2+2s+2) + (Bs+C)(s) = 1$

Put $s = 0$, we have

$$A(0^2+2(0)+2)+(B(0)+C)(0)=1 \Rightarrow A=1/2$$

Put s =–1, we have

$$A((-1)^2+2(-1)+2)+(B(-1)+C)(-1)=1 \Rightarrow B-C=1/2$$

Put s = 1, we have

$$A(1^2+2(1)+2)+(B(1)+C)(1)=1 \Rightarrow B+C=-3/2$$

Solving the above two equations, we have

$$\therefore B=-1/2 \quad C=-1$$

Hence,

$$Y(s)=\frac{1/2}{s}+\frac{-\frac{1}{2s}-1}{s^2+2s+2}=\frac{1}{2}\frac{1}{s}-\frac{1}{2}\left[\frac{s+2}{(s+1)^2+1^2}\right]=\frac{1}{2}\frac{1}{s}-\frac{1}{2}\left[\frac{s+1+1}{(s+1)^2+1^2}\right]$$

$$=\frac{1}{2}\frac{1}{s}-\frac{1}{2}\left[\frac{s+1}{(s+1)^2+1^2}\right]-\frac{1}{2}\left[\frac{1}{(s+1)^2+1^2}\right]$$

Taking inverse Laplace transform,

$$y(t)=u(t)=\frac{1}{2}-\frac{1}{2}e^{-t}\cos t-\frac{1}{2}e^{-t}\sin t$$

3. (a)

$$\frac{C(s)}{R(s)}=\frac{\frac{1}{T^2s^2+2\delta Ts}}{1+\frac{1}{T^2s^2+2\delta Ts}}=\frac{1}{T^2s^2+2\delta Ts+1}$$

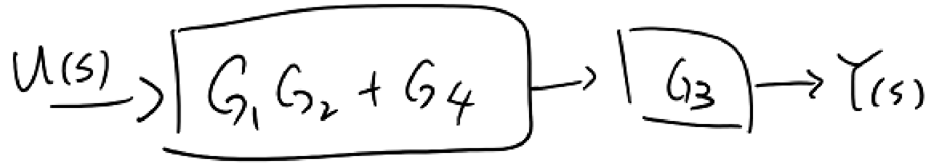
(b)

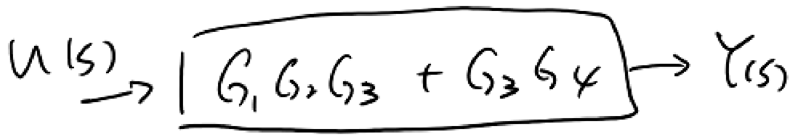
$$\frac{C(s)}{R(s)}=\frac{\frac{1}{s+4}}{1+(2)\left(\frac{1}{s+4}\right)}=\frac{1}{s+6}$$

(c)

$$\frac{C(s)}{R(s)}=\frac{\frac{1}{(s+4)(s+2)}}{1-\frac{1}{(s+4)(s+2)}(2s)}=\frac{\frac{1}{(s+4)(s+2)}}{\frac{(s+4)(s+2)-2s}{(s+4)(s+2)}}=\frac{1}{s^2+4s+8}$$

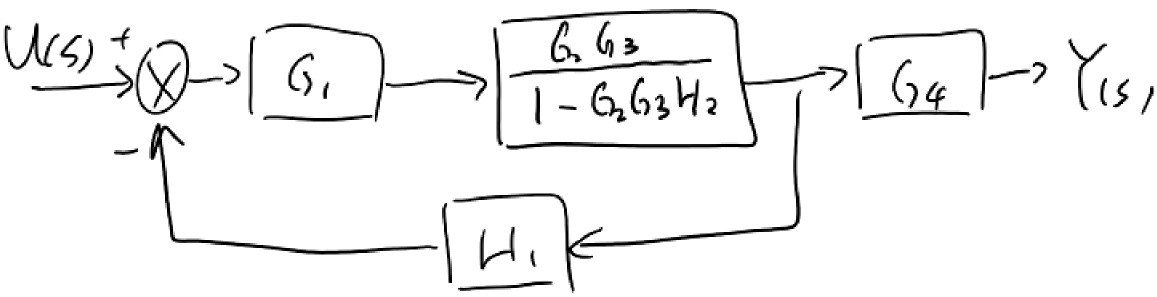
4. (a)

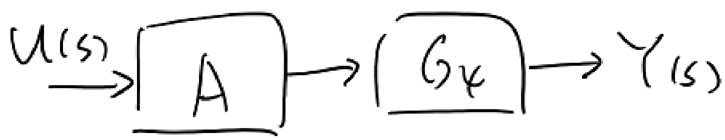




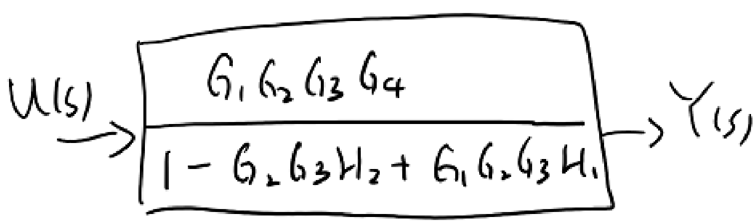
$$\frac{Y(s)}{U(s)} = G_1G_2G_3 + G_4G_3$$

(b)



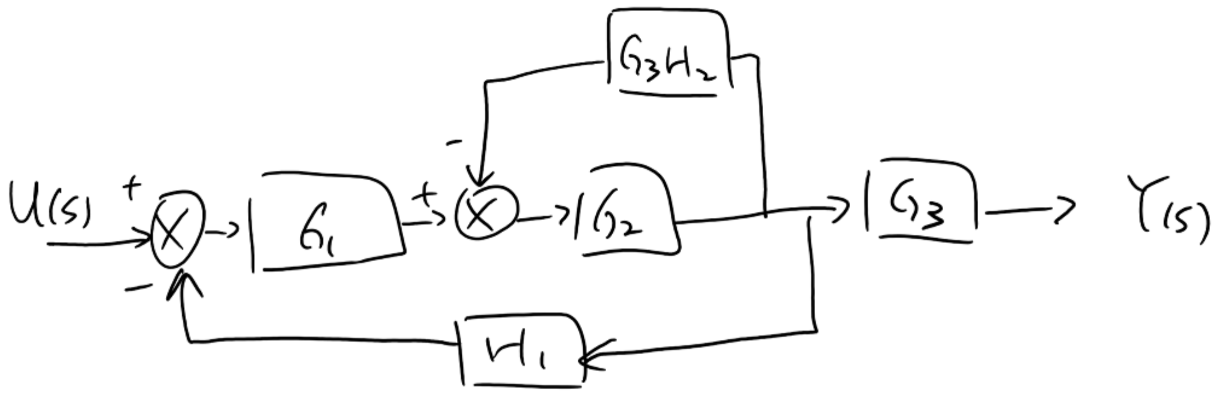


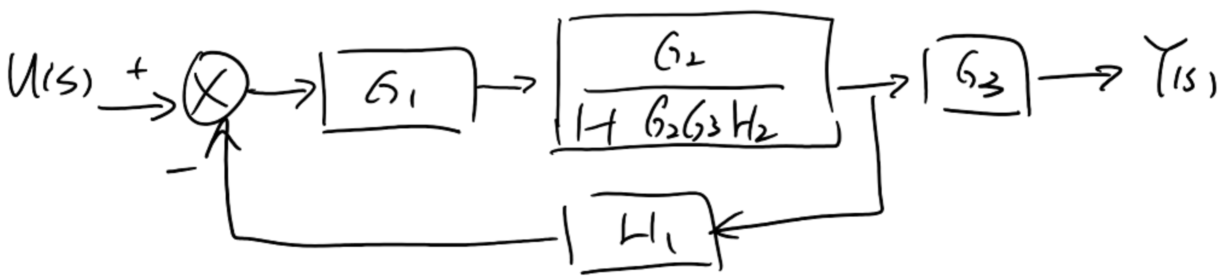
$$A = \frac{\frac{G_1G_2G_3}{1 - G_2G_3H_2}}{1 + \left(\frac{G_1G_2G_3}{1 - G_2G_3H_2}\right)(H_1)} = \frac{G_1G_2G_3}{1 - G_2G_3H_2 + G_1G_2G_3H_1}$$

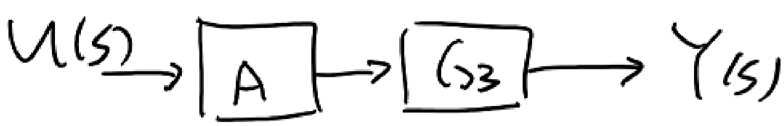


$$\frac{Y(s)}{U(s)} = \frac{G_1G_2G_3G_4}{1 - G_2G_3H_2 + G_1G_2G_3H_1}$$

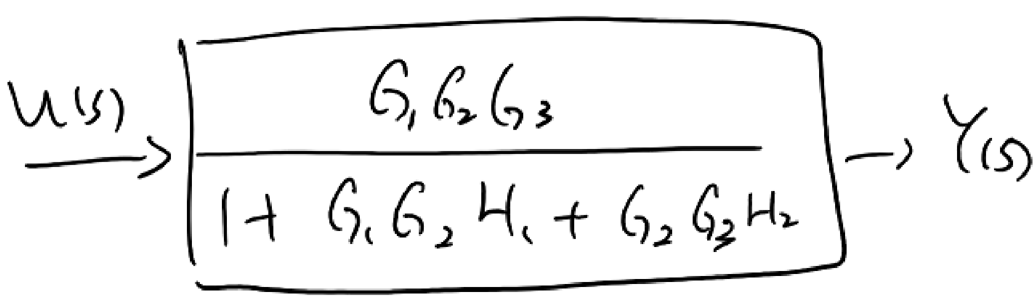
(c)





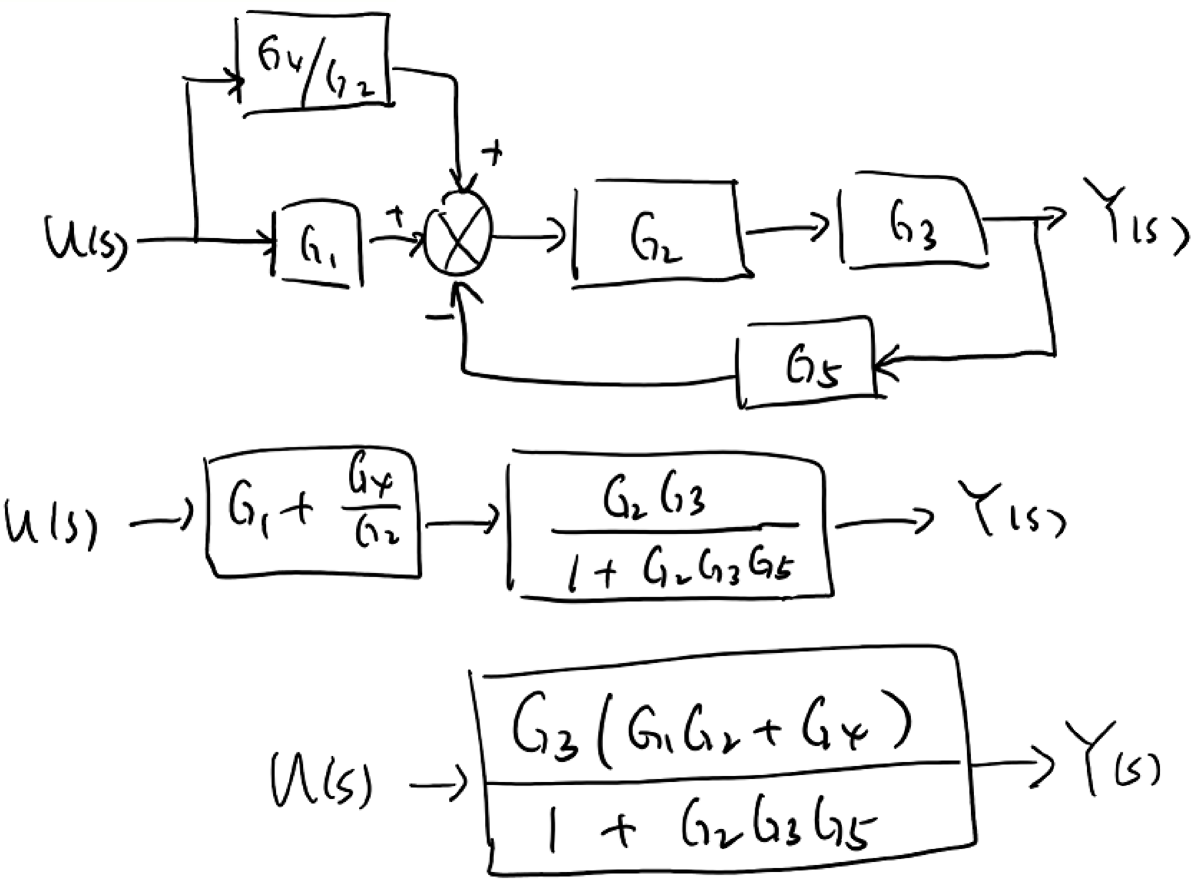


$$A = \frac{\dfrac{G_1G_2}{1 + G_2G_3H_2}}{1 + \dfrac{G_1G_2H_1}{1 + G_2G_3H_2}} = \frac{G_1G_2}{1 + G_2G_3H_2 + G_1G_2H_1}$$



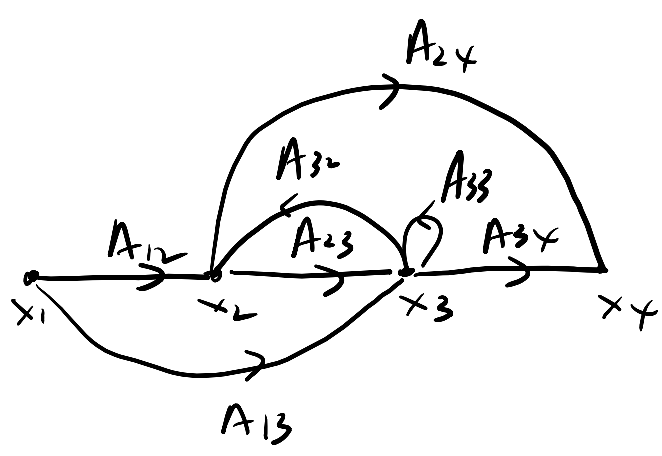
$$\frac{Y(s)}{U(s)} = \frac{G_1G_2G_3}{1 + G_2G_3H_2 + G_1G_2H_1}$$

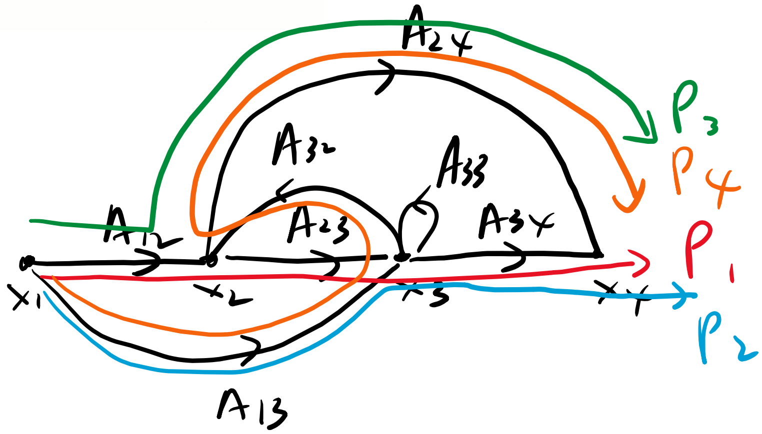
(d)



$$\frac{Y(s)}{U(s)} = \frac{G_3(G_1G_2 + G_4)}{1 + G_2G_3G_5}$$

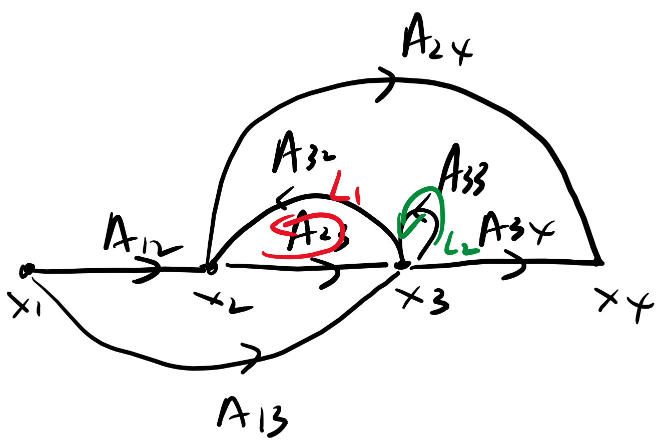
5.





Forward path:

$$P_1 = A_{21}A_{32}A_{43} \quad P_2 = A_{31}A_{43} \quad P_3 = A_{21}A_{42} \qquad P_4 = A_{31}A_{23}A_{42}$$



Loop:

$$L_1 = A_{32}A_{23} \quad L_2 = A_{33}$$

Hence,

$$\frac{x_4}{x_1} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4}{\Delta} = \frac{A_{21}A_{32}A_{43} + A_{31}A_{43} + (A_{21}A_{42})(1 - A_{33}) + A_{31}A_{23}A_{42}}{1 - A_{32}A_{23} - A_{33}}$$

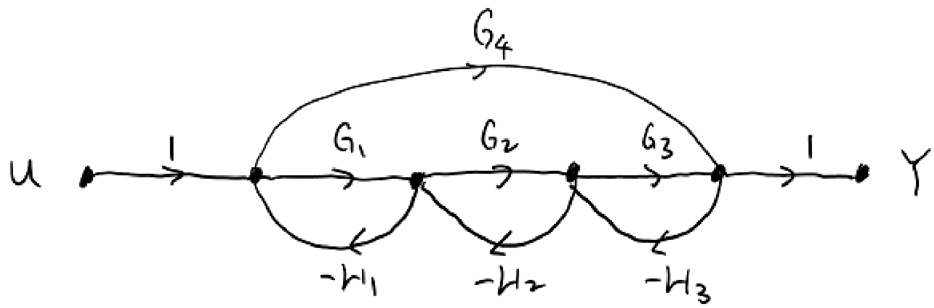
6. Forward path: $P_1 = ae \quad P_2 = abc$

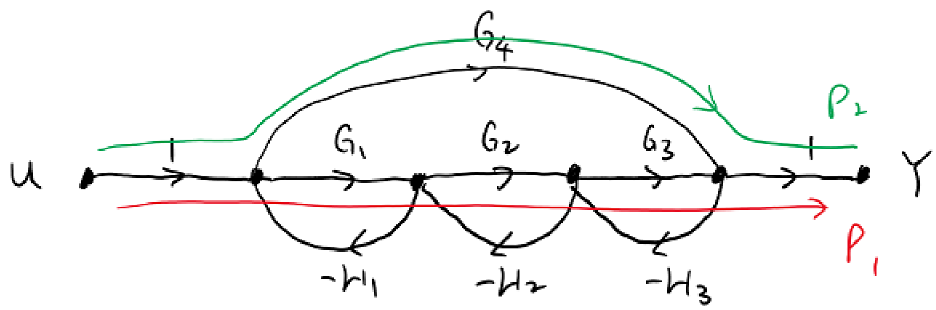
Loops: $L_1 = -eg \quad L_2 = -d \quad L_3 = -bcg$

Hence,

$$\frac{y_4}{y_1} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{P_1(1 - L_2) + P_2(1)}{1 - (L_1 + L_2 + L_3) + (L_1L_2)} = \frac{ae(1 + d) + abc}{1 + d + eg + bcg + edg}$$

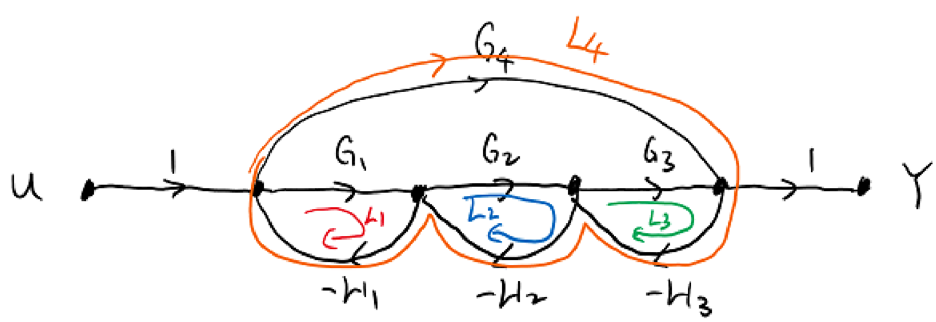
7.





Forward path:

$$P_1 = G_1G_2G_3 \quad P_2 = G_4$$



Loops:

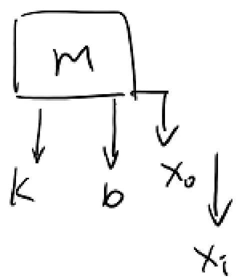
$$L_1 = -G_1H_1 \quad L_2 = -G_2H_2 \quad L_3 = -G_3H_3 \quad L_4 = -G_4H_1H_2H_3$$

Hence,

$$\frac{Y}{U} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{P_1\Delta_1 + P_2(1 - L_2)}{1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_3)}$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{G_1G_2G_3 + G_4(1 + G_2H_2)}{1 + G_1H_1 + G_2H_2 + G_3H_3 + G_4H_1H_2H_3 + G_1G_3H_1H_3}$$

8.



$$m\ddot{x}_o = k(x_i - x_o) + b(\dot{x}_i - \dot{x}_o)$$

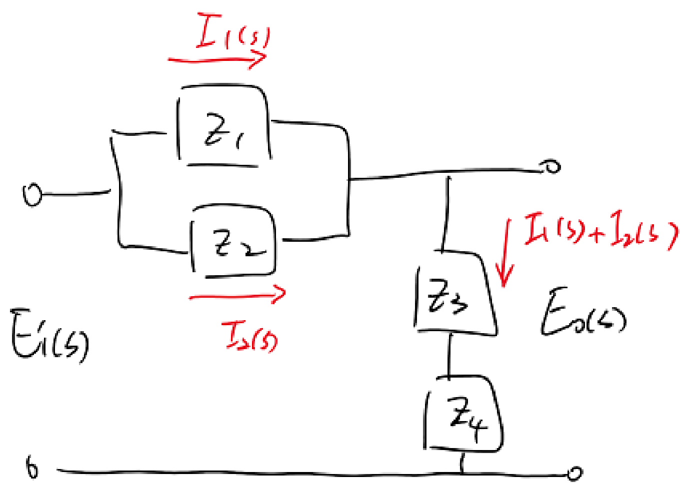
$$m\ddot{x}_o + b\dot{x}_o + kx_o = b\dot{x}_i + kx_i$$

Taking Laplace Transform, we have

$$ms^2X_o(s) + bsX_o(s) + kX_o(s) = bsX_i(s) + kX_i(s)$$

$$\therefore \frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

9.



The impedance after taking Laplace transform, we have

$$Z_1 = R_2 \quad Z_2 = \frac{1}{sC_2} \quad Z_3 = R_1 \quad Z_4 = \frac{1}{sC_1}$$

$$E_o(s) = \big(I_1(s) + I_2(s)\big)(Z_3 + Z_4)$$

$$I_1(s) = \frac{E_i(s) - E_o(s)}{Z_1}, \quad I_2(s) = \frac{E_i(s) - E_o(s)}{Z_2}$$

$$\therefore E_o(s) = \left(\frac{E_i(s) - E_o(s)}{Z_1} + \frac{E_i(s) - E_o(s)}{Z_2}\right)(Z_3 + Z_4)$$

$$Z_1Z_2E_o(s) = [(Z_1 + Z_2)E_i(s) - (Z_1 + Z_2)E_o(s)](Z_3 + Z_4)$$

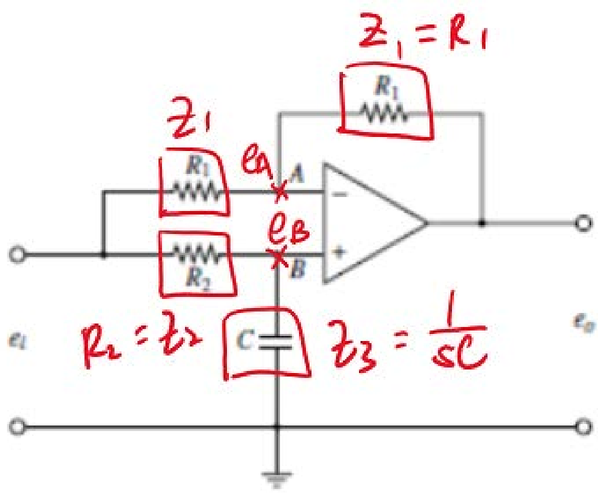
$$Z_1Z_2E_o(s) + (Z_1 + Z_2)(Z_3 + Z_4)E_o(s) = (Z_1 + Z_2)(Z_3 + Z_4)E_i(s)$$

$$\frac{E_o(s)}{E_i(s)} = \frac{(Z_1 + Z_2)(Z_3 + Z_4)}{(Z_1 + Z_2)(Z_3 + Z_4) + Z_1Z_2}$$

Substituting the corresponding impedances into the equation, we have

$$\frac{E_o(s)}{E_i(s)} = \frac{\left(R_2 + \frac{1}{sC_2}\right)\left(R_1 + \frac{1}{sC_1}\right)}{\left(R_2 + \frac{1}{sC_2}\right)\left(R_1 + \frac{1}{sC_1}\right) + R_2\left(\frac{1}{sC_2}\right)}$$

$$\therefore \frac{E_o(s)}{E_i(s)} = \frac{(sR_1C_1 + 1)(sR_2C_2 + 1)}{sR_2C_1 + (sR_1C_1 + 1)(sR_2C_2 + 1)}$$



From the above circuit, we have the following equations,

$$e_B = \frac{Z_3}{Z_2 + Z_3}E_i(s) = \frac{\frac{1}{sC}}{R_2 + \frac{1}{sC}}E_i(s) = \frac{1}{R_2Cs + 1}E_i(s)$$

$$\frac{E_i(s) - e_A}{R_1} = \frac{e_A - E_o(s)}{R_1} \Rightarrow e_A = \frac{1}{2}[E_i(s) + E_o(s]$$

Since $e_A \approx e_B$, we have

$$\frac{1}{R_2Cs + 1}E_i(s) = \frac{1}{2}[E_i(s) + E_o(s]$$

$$\frac{1}{R_2Cs + 1}E_i(s) - \frac{1}{2}E_i(s) = \frac{1}{2}E_o(s)$$

$$\therefore \frac{E_o(s)}{E_i(s)} = \frac{1 - sR_2C}{sR_2C + 1}$$

End of Tutorial Questions (Part 1) Solution