The Hong Kong Polytechnic University
School of Professional Education and Executive Development
Mid-term Test (Version C) Solution

Student Name : ____________________

Student Number : ____________________

Tutorial Group : ☐ A01 (Dr Norbert CHEUNG, Thu, 19:00-22:00)
☐ A02 (Dr Kenneth LO, Sat, 19:00-22:00)
☐ A03 (Dr Norbert CHEUNG, Thu, 12:00-15:00)

Subject Title	: Control System Analysis	**Subject Code**	: SEHS4653
Session	: Semester One 2024/25	**Date**	: 60 mins

This question paper has a total of **EIGHT** pages (including this covering page).

Instructions to Candidates:

1. There are THREE questions in this paper. Answer **ALL** questions on the blank space provided in this paper.
2. Show all your workings clearly. Reasonable steps must be provided.
3. Total marks of this paper is 40 marks.
4. All answers must be written in ink. Pencils may only be used for drawing, sketching, or graphical work.
5. Laplace Transform table and useful formulae are provided on pages 6 - 8.

Authorised Materials:

	YES	NO
CALCULATOR	[√]	[ ]
SPECIFICALLY PERMITTED ITEMS	[ ]	[√]

DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO

Answer ALL questions in the blank space provided.

Question 1 (10 marks)

Find the time function of the equation below by Laplace Transform.

$$G(s) = \frac{1}{s^2 + 4s + 3}$$

Solution

$$G(s) = \frac{1}{(s+1)(s+3)}$$

Taking inverse Laplace Transform, we have

$$g(t) = \frac{1}{3-1}(e^{-t} - e^{-3t}) = \frac{1}{2}(e^{-t} - e^{-3t})$$

Question 2 (10 marks)

Determine the range of K such that the system with the characteristic equation, $\Delta(s) = s^4 + 3s^3 + 3s^2 + 6s + K = 0$, is stable.

Solution

s^4	1	3	K
s^3	3	6	
s^2	$\frac{(3)(3)-(1)(6)}{3} = 1$	$\frac{(3)(K)-(1)(0)}{3} = K$	
s^1	$\frac{(1)(6)-(3)(K)}{1}$		
s^0	K		

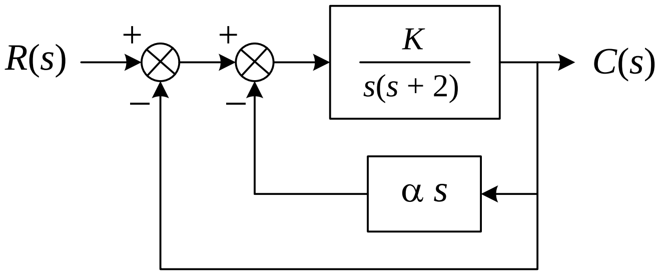
$K > 0$

$6 - 3K > 0, K < 2$

$\therefore 0 < K < 2$

Question 3 (20 marks)

Determine the values of K and α if the damping ratio is 0.643 and steady-state error for unit-ramp input is 0.3.



Solution

The open-loop transfer function will be,

$$G(s)H(s) = \frac{\frac{K}{s(s+2)}}{1 + \left(\frac{K}{s(s+2)}\right)(\alpha s)} = \frac{K}{s(s+2) + K\alpha s}$$

The static velocity error constant is,

$$K_v = \lim_{s\to 0}(s)\left(\frac{K}{s(s+2) + K\alpha s}\right) = \frac{K}{2 + K\alpha}$$

The steady-state error will then be,

$$e_{ss} = \frac{1}{K_v} = 0.3 = \frac{2 + K\alpha}{K} \ldots (1)$$

The closed-loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+2) + K\alpha s}}{1 + \frac{K}{s(s+2) + K\alpha s}} = \frac{K}{s(s+2) + K\alpha s + K}$$

Equating with standard 2nd order equation, we have

$$\Delta(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + (2 + K\alpha)s + K$$

$$2 + K\alpha = 2\zeta\omega_n \ldots (2)$$

$$K = \omega_n^2 \ldots (3)$$

From (1) and (3), we have $K\alpha = 0.3K - 2 \ldots (4)$ and $\omega_n = \sqrt{K} \ldots (5)$

Substitute (4) into (2), we have $2 + (0.3K - 2) = 2\zeta\sqrt{K}$

$$K = \left(\frac{(2)(0.643)}{0.3}\right)^2 = 18.376$$

$$\alpha = \frac{(0.3)(18.376) - 2}{18.376} = 0.191$$

Appendix I: Laplace Transform Table

	Time Function $f(t)$	Laplace Transform $F(s)$
1	Unit-impulse function $\delta(t)$	1
2	Unit-step function $u_s(t)$	$\frac{1}{s}$
3	Unit-ramp function t	$\frac{1}{s^2}$
4	t^n ($n =$ positive integer)	$\frac{n!}{s^{n+1}}$
5	e^{-at}	$\frac{1}{s+a}$
6	te^{-at}	$\frac{1}{(s+a)^2}$
7	$t^n e^{-at}$ ($n =$ positive integer)	$\frac{n!}{(s+a)^{n+1}}$
8	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
9	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
10	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
11	$\frac{1}{b-a}(e^{-at}-e^{-bt})$ $(a \neq b)$	$\frac{1}{(s+a)(s+b)}$
12	$\frac{1}{b-a}(be^{-bt}-ae^{-at})$ $(a \neq b)$	$\frac{s}{(s+a)(s+b)}$
13	$\frac{1}{a^2}(1-e^{-at}-ate^{-at})$	$\frac{1}{s(s+a)^2}$
14	$\frac{1}{a^2}(at-1+e^{-at})$	$\frac{1}{s^2(s+a)}$
15	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
16	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
17	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n\sqrt{1-\zeta^2}t$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

18	$-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t-\phi)$ where $\phi=\cos^{-1}\zeta$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
19	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t+\phi)$ where $\phi=\cos^{-1}\zeta$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
20	$1-\cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
21	$\frac{d}{dt}f(t)$	$sF(s)-f(0)$
22	$\frac{d^2}{dt^2}f(t)$	$s^2F(s)-sf(0)-f'(0)$
23	$\frac{d^n}{dt^n}f(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\ldots$ $-sf^{(n-2)}(0)$ $-f^{(n-1)}(0)$
24	$\int f(t)dt$	$\frac{F(s)}{s}+\frac{1}{s}\left[\int f(t)dt\right]$
25	$f(t-T)$	$e^{-Ts}F(s)$
26	$f(\infty)=\lim_{t\to\infty}f(t)$	$=\lim_{s\to 0}sF(s)$
27	$f(0^+)=\lim_{t\to 0^+}f(t)$	$=\lim_{s\to\infty}sF(s)$

Appendix II: Useful Formulae

Rise Time: $t_r = \dfrac{\pi - \beta}{\omega_d}$

Peak Time: $t_p = \dfrac{\pi}{\omega_d}$

Maximum Overshoot: $M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$

Settling Time (2%): $t_s = \dfrac{4}{\zeta\omega_n}$

Settling Time (5%): $t_s = \dfrac{3}{\zeta\omega_n}$

Damped Natural Frequency: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

<u>Series-lead Compensator:</u>

Transfer Function:

$$G_c(s) = K_c\alpha\frac{Ts+1}{\alpha Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$$

Maximum Phase Lead: $\sin\phi_m = \dfrac{1-\alpha}{1+\alpha}$

New Gain Crossover Frequency and its Corresponding Gain: $\omega_m = \dfrac{1}{\sqrt{\alpha}T}$ and $|G_1(j\omega)| = -20\log\dfrac{1}{\sqrt{\alpha}}$

<u>Series-lag Compensator:</u>

Transfer Function:

$$G_c(s) = K_c\beta\frac{Ts+1}{\beta Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

At New Gain Crossover Frequency: $|G_c(j\omega)| = 20\log\beta$

– End of Paper (Solution) –