The Hong Kong Polytechnic University

School of Professional Education and Executive Development

Mid-term Test (Solution)

Student Name : ______________________

Student Number : ______________________

Subject Title	:	Control System Analysis	**Subject Code**	:	SEHS4653
Session	:	Semester Two 2023/24	**Date**	:	9 April 2024
Time	:	20:00 – 21:00	**Time Allowed**	:	60 mins

This question paper has a total of **NINE** pages (including this covering page).

Instructions to Candidates:

1. There are THREE questions in this paper. Answer **ALL** questions on the blank space provided in this paper.
2. Show all your workings clearly. Reasonable steps must be provided.
3. Total marks of this paper is 70 marks.
4. All answers must be written in ink. Pencils may only be used for drawing, sketching, or graphical work.
5. Laplace Transform table and useful formulae are provided on pages 7 - 9.

Authorised Materials:

	YES	NO
CALCULATOR	[√]	[ ]
SPECIFICALLY PERMITTED ITEMS	[ ]	[√]

DO NOT TURN OVER THE PAGE UNTIL YOU ARE TOLD TO DO SO

Answer <u>ALL</u> questions in the blank space provided.

<u>Question 1 (20 marks)</u>

The characteristic equation of a control system is, $\Delta(s) = s^4 + 4s^3 + 3s^2 + 5s + 2 + K.$ Find the range of K such that the system is stable.

<u>Solution</u>

$$\begin{array}{c|ccc} s^4 & 1 & 3 & 2+K \\ s^3 & 4 & 5 & \\ s^2 & \dfrac{(4)(3)-(1)(5)}{4} = \dfrac{7}{4} = 1.75 & 2+K & \\ s^1 & \dfrac{(1.75)(5)-(4)(2+K)}{1.75} & & \\ s^0 & 2+K & & \end{array}$$

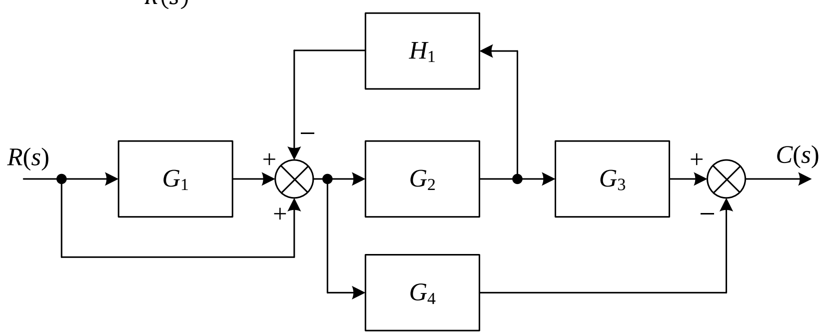
$2 + K > 0 \rightarrow K > -2$

$$\frac{(1.75)(5)-(4)(2+K)}{1.75} > 0 \rightarrow K < 0.1875$$

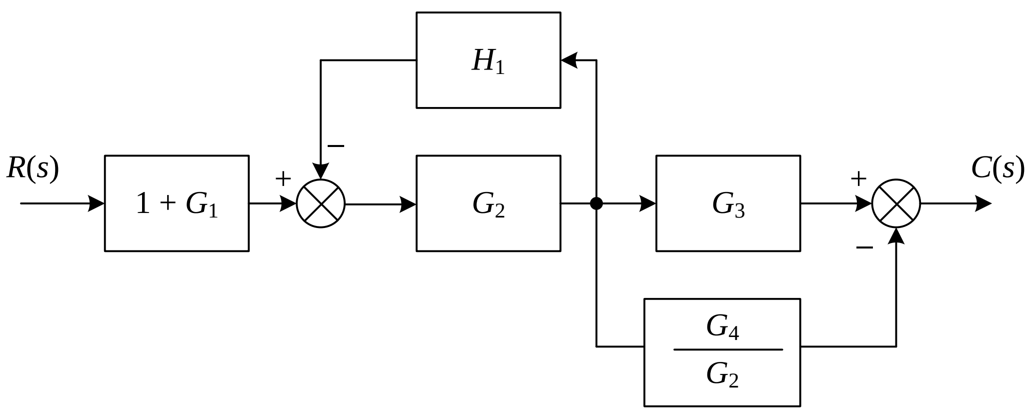
$$\therefore -2 < K < 0.1875$$

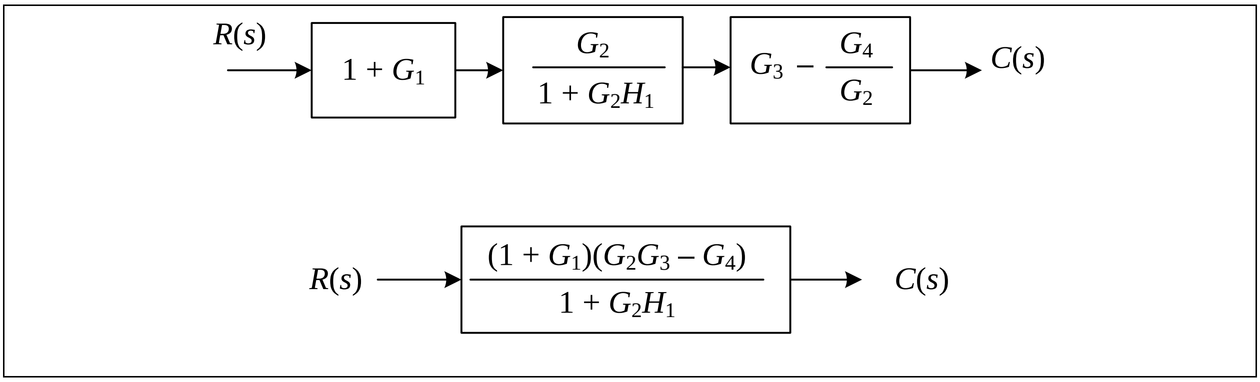
<u>Question 2 (20 marks)</u>

Find the transfer function, $\frac{C(s)}{R(s)}$, of block diagram below.



<u>Your Answer</u>





<u>Question 3 (30 marks)</u>

A negative unity feedback system with the following open-loop transfer function,

$$G(s) = \frac{K}{s^2 + 5s + 4}.$$

(a) Draw the root locus of the system on the graph paper provided. (14 marks)
(b) Locate the closed-loop pole if a damping ratio of 0.5 is required. (8 marks)
(c) Find the maximum overshoot (percentage), peak time, and rise time based on the result obtained in part (b). (6 marks)
(d) Comment on the stability of the system. (2 marks)

<u>Your Answer</u>

(a)
Open-loop poles: $s = -1, \quad s = -4$
Root locus on the *x*-axis: $(-4, -1)$
Angle of asymptotes:

$$\phi_A = \frac{180°(2k \pm 1)}{2 - 0} = \pm 90° \ (K = 0)$$

Intersection point of asymptotes: $\sigma_A = \frac{(-1)+(-4)}{2-0} = -2.5$

Breakaway point:
$\Delta(s) = s^2 + 5s + 4 + K = 0$

$K = -s^2 - 5s - 4$

$$\frac{dK}{ds} = -2s - 5 = 0 \rightarrow s = -2.5$$

There will be no point intersecting the *jw*-axis and angle of departure (angle of arrival).

(b) Under the damping ratio of 0.5, $\theta = \cos^{-1}(0.5) = 60°$
Draw a line from origin with this angle to the left. Then, the intersection point between this line and root locus is the closed-loop pole location, i.e. $s = -2.5 \pm j4.3$

(c)
The maximum overshoot (percentage):

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\% = e^{-\frac{(0.5)\pi}{\sqrt{1-(0.5)^2}}} \times 100\% = 16.3\%$$

From the closed-loop, we have $s = -2.5 \pm j4.3 = -\zeta\omega_n \pm j\omega_d$

Peak Time:

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4.3} = 0.731 \text{ s}$$

Rise Time:

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - \cos^{-1} 0.5}{4.3} = 0.487 \text{ s}$$

(d)
Since the entire root locus of the system lies on the left half *s*-plane, the system will be stable regarding the value of *K*.

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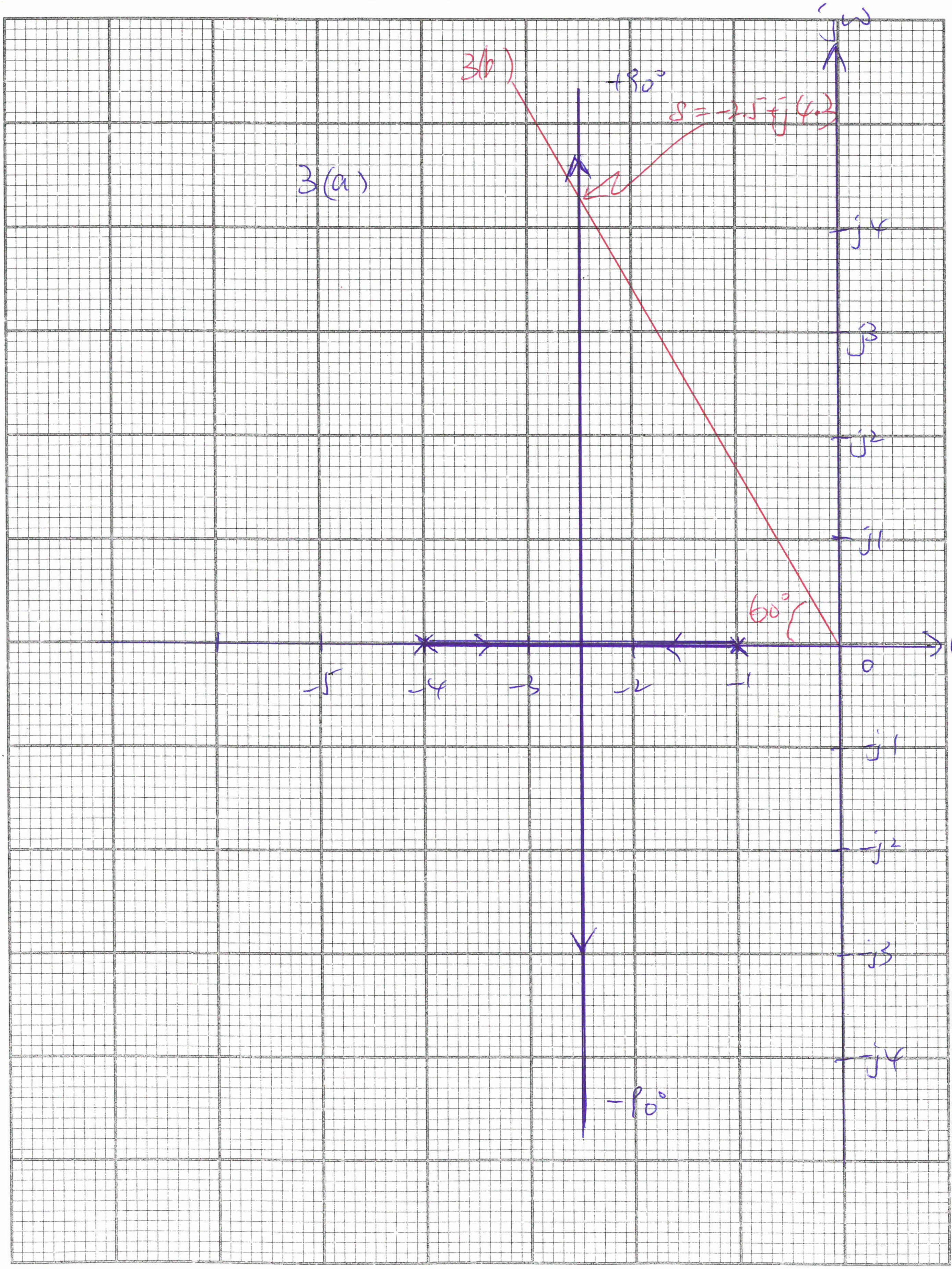

Appendix I: Laplace Transform Table

	Time Function $f(t)$	Laplace Transform $F(s)$
1	Unit-impulse function $\delta(t)$	1
2	Unit-step function $u_s(t)$	$\frac{1}{s}$
3	Unit-ramp function t	$\frac{1}{s^2}$
4	t^n ($n =$ positive integer)	$\frac{n!}{s^{n+1}}$
5	e^{-at}	$\frac{1}{s+a}$
6	te^{-at}	$\frac{1}{(s+a)^2}$
7	$t^n e^{-at}$ ($n =$ positive integer)	$\frac{n!}{(s+a)^{n+1}}$
8	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
9	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
10	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
11	$\frac{1}{b-a}(e^{-at}-e^{-bt})$ $(a \neq b)$	$\frac{1}{(s+a)(s+b)}$
12	$\frac{1}{b-a}(be^{-bt}-ae^{-at})$ $(a \neq b)$	$\frac{s}{(s+a)(s+b)}$
13	$\frac{1}{a^2}(1-e^{-at}-ate^{-at})$	$\frac{1}{s(s+a)^2}$
14	$\frac{1}{a^2}(at-1+e^{-at})$	$\frac{1}{s^2(s+a)}$
15	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
16	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
17	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

18	$-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t-\phi)$ where $\phi = \cos^{-1}\zeta$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
19	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t+\phi)$ where $\phi = \cos^{-1}\zeta$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
20	$1-\cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
21	$\frac{d}{dt}f(t)$	$sF(s)-f(0)$
22	$\frac{d^2}{dt^2}f(t)$	$s^2F(s)-sf(0)-f'(0)$
23	$\frac{d^n}{dt^n}f(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\ldots$ $-sf^{(n-2)}(0)$ $-f^{(n-1)}(0)$
24	$\int f(t)dt$	$\frac{F(s)}{s}+\frac{1}{s}\left[\int f(t)dt\right]$
25	$f(t-T)$	$e^{-Ts}F(s)$
26	$f(\infty)=\lim_{t\to\infty}f(t)$	$=\lim_{s\to 0}sF(s)$
27	$f(0^+)=\lim_{t\to 0^+}f(t)$	$=\lim_{s\to\infty}sF(s)$

Appendix II: Useful Formulae

Rise Time: $$t_r = \frac{\pi - \beta}{\omega_d}$$

Peak Time: $$t_p = \frac{\pi}{\omega_d}$$

Maximum Overshoot: $$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Settling Time (2%): $$t_s = \frac{4}{\zeta\omega_n}$$

Settling Time (5%): $$t_s = \frac{3}{\zeta\omega_n}$$

Damped Natural Frequency: $$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

<u>Series-lead Compensator:</u>

Transfer Function: $$G_c(s) = K_c\alpha\frac{Ts+1}{\alpha Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$$

Maximum Phase Lead: $$\sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

New Gain Crossover Frequency and its Corresponding Gain: $$\omega_m = \frac{1}{\sqrt{\alpha}T} \quad \text{and} \quad |G_1(j\omega)| = -20\log\frac{1}{\sqrt{\alpha}}$$

<u>Series-lag Compensator:</u>

Transfer Function: $$G_c(s) = K_c\beta\frac{Ts+1}{\beta Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

At New Gain Crossover Frequency: $$|G_c(j\omega)| = 20\log\beta$$

– End of Paper –