Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 02-k

Frequency Response of Resonant Circuits

Contents

Reference:

Introductory Circuit Analysis 14th edition, Boylesad & Olivari
Basic Circuit Analysis – Schaum's Outline Series

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1. Introduction

This chapter introduces the very important *resonant* (or *tuned*) *circuit*, which is fundamental to the operation of a wide variety of electrical and electronic systems in use today. The resonant circuit is a combination of *R*, *L*, and *C* elements having a frequency response characteristic similar to the one appearing in Fig. 21.1. Note in the figure that the response is a maximum for the frequency f_r, decreasing to the right and left of this frequency. In other words, for a particular range of frequencies, the response will be near or equal to the maximum.

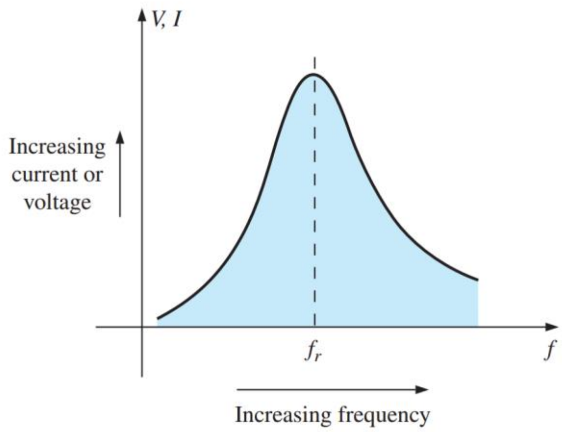

FIG. 21.1
Resonance curve.

There are two types of resonant circuits: *series* and *parallel*. As the name implies, a series resonant circuit is a combination of series elements that includes a resistor, inductor, and capacitor. As shown in Fig. 21.2(a), a voltage source of fixed magnitude over the given frequency range is applied to the circuit. As the applied frequency increases, there will be a range of frequencies where the current through the circuit will peak as shown in the same figure. In other words,

a series resonant circuit is one where the resonant curve of interest is the current through the circuit due to an applied voltage source.

A parallel resonant circuit has the same component list but in a parallel combination of elements and the applied source is a current source of fixed magnitude as shown in Fig. 21.2(b). In other words,

for parallel resonance the resonant curve of interest is the voltage across the output terminals of the network due to an applied current source.

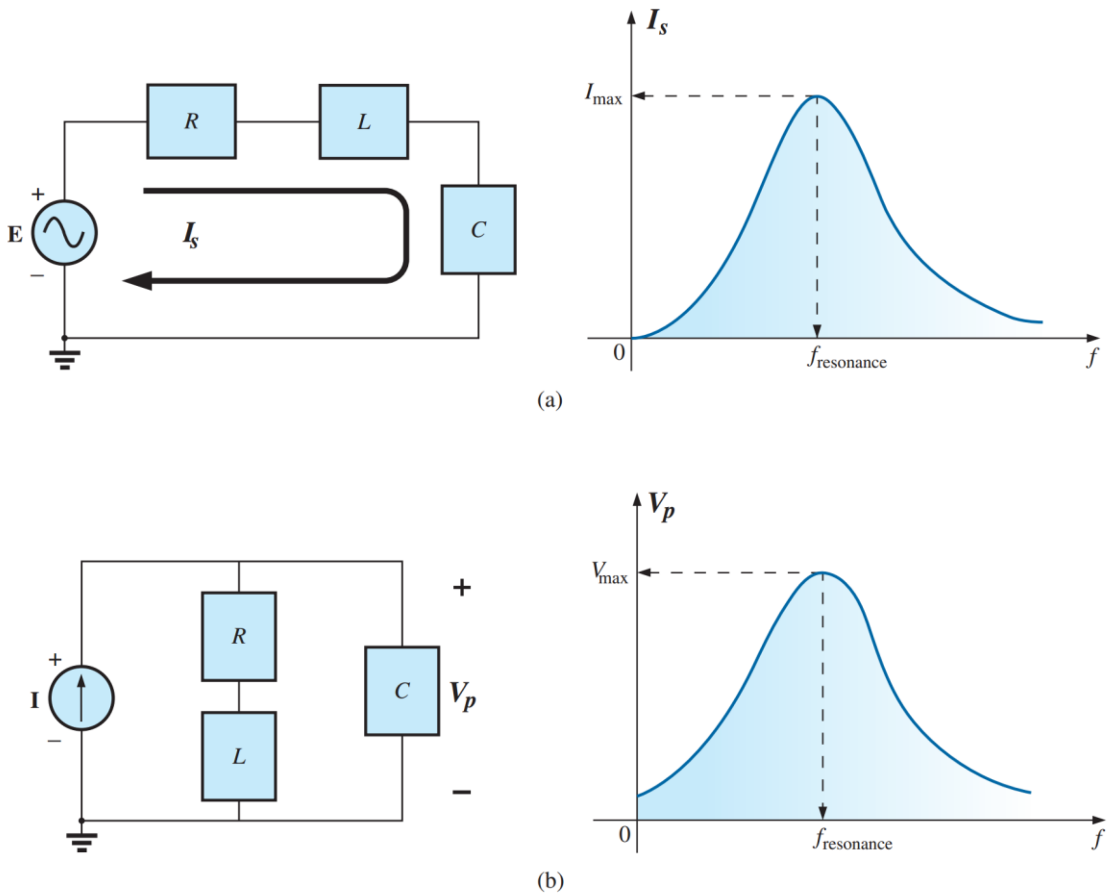

FIG. 21.2
Resonance: (a) series; (b) parallel.

2. Series Resonant Circuit

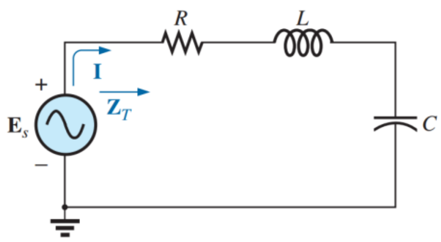

FIG. 21.3
Series resonant circuit.

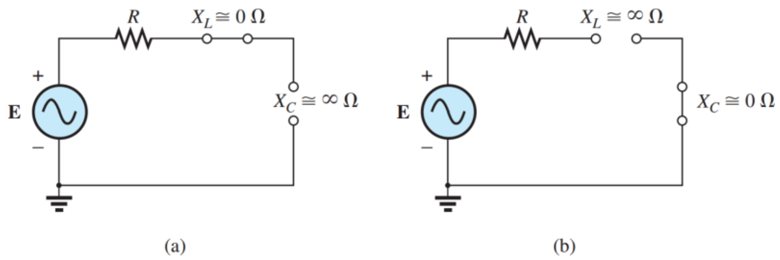

FIG. 21.4
(a) Very low frequencies. (b) Very high frequencies.

For the mid-range of frequencies, the total impedance is defined by

$$Z_T = R + jX_L - jX_C = R + j(X_L - X_C) \quad \textbf{(21.1)}$$

For the series, resonant circuit resonance is defined by the condition that

$$X_L = X_C \quad \textbf{(21.2)}$$

Inserting this equivalence into Eq. (21.1) will result in a total impedance of simply the resistance

$$Z_{T_s} = R \quad \textbf{(21.3)}$$

where the subscript s denotes resonant value.

Resonant Frequency The resonant frequency can be determined in terms of the inductance and capacitance by examining the defining equation for resonance Eq. (21.2):

$$X_L = X_C$$

Substituting yields

$$\omega L = \frac{1}{\omega C} \quad \text{so that} \quad \omega^2 = \frac{1}{LC}$$

and

$$\omega_s = \frac{1}{\sqrt{LC}} \quad \textbf{(21.4a)}$$

or

$$f_s = \frac{1}{2\pi\sqrt{LC}} \qquad \begin{aligned} f &= \text{hertz (Hz)} \\ L &= \text{henries (H)} \\ C &= \text{farads (F)} \end{aligned} \quad \textbf{(21.4b)}$$

Peak Resonant Current The current through the circuit at resonance is

$$I_{\max} = \frac{E\angle 0^\circ}{R\angle 0^\circ} = \frac{E}{R}\angle 0^\circ$$

which is the maximum current for the circuit in Fig. 21.3 for an applied voltage **E** since $\mathbf{Z}_T$ is a minimum value. Consider also that *the input voltage and current are in phase at resonance.*

Resonant Voltage Levels Since the current is the same through the capacitor and inductor, the voltage across each is equal in magnitude but 180° out of phase at resonance:

$$\left.\begin{aligned}\mathbf{V}_L &= (I\angle 0°)(X_L\angle 90°) = IX_L\angle 90° \\ \mathbf{V}_C &= (I\angle 0°)(X_C\angle -90°) = IX_C\angle -90°\end{aligned}\right\} 180° \text{ out of phase}$$

and, since $X_L = X_C$, the magnitude of V_L equals V_C at resonance; that is,

$$V_{L_s} = V_{C_s} \qquad \textbf{(21.5)}$$

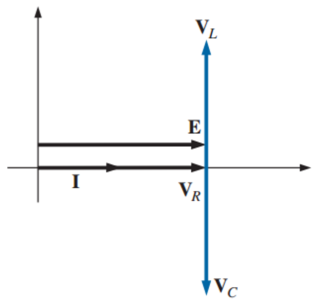

FIG. 21.5
Phasor diagram for the series resonant circuit at resonance.

Phasor Diagram at Resonance Fig. 21.5, a phasor diagram of the voltages and current, clearly indicates that the voltage across the resistor at resonance is the input voltage, and **E**, **I**, and $\mathbf{V}_R$ are in phase at resonance.

Power Diagram at Resonance The average power to the resistor at resonance is equal to I^2R, and the reactive power to the capacitor and inductor are I^2X_C and I^2X_L, respectively.

The power triangle at resonance (Fig. 21.6) shows that the total apparent power is equal to the average power dissipated by the resistor since $Q_L = Q_C$. The power factor of the circuit at resonance is

$$F_p = \cos\theta = \frac{P}{S}$$

and

$$F_{p_s} = 1 \qquad \textbf{(21.6)}$$

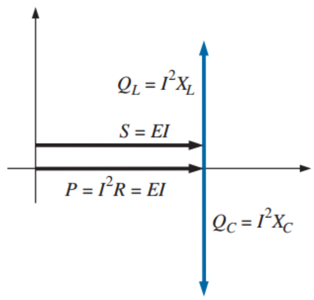

FIG. 21.6
Power triangle for the series resonant circuit at resonance.

Plotting the power curves of each element on the same set of axes (Fig. 21.7), we note that, **even though the total reactive power at any instant is equal to zero (note that $t = t'$), energy is still being absorbed and released by the inductor and capacitor at resonance**.

A closer examination reveals that the energy absorbed by the inductor from time 0 to t_1 is the same as the energy released by the capacitor from 0 to t_1. The reverse occurs from t_1 to t_2, and so on. Therefore, the total apparent power continues to be equal to the average power, even though

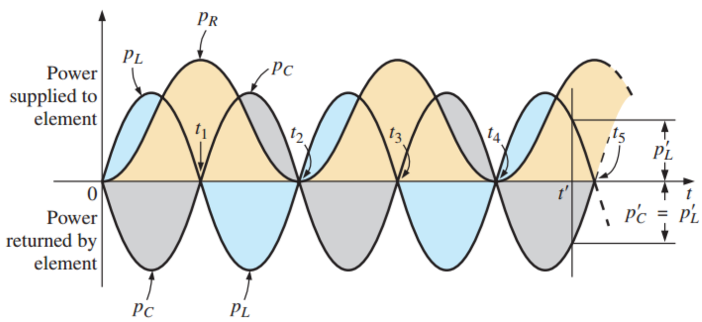

FIG. 21.7
Power curves at resonance for the series resonant circuit.

the inductor and capacitor are absorbing and releasing energy. This condition occurs only at resonance. The slightest change in frequency introduces a reactive component into the power triangle, which increases the apparent power of the system above the average power dissipation, and resonance no longer exists.

21.3 THE QUALITY FACTOR (Q)

The **quality factor** Q of a series resonant circuit is defined as the ratio of the reactive power of either the inductor or the capacitor to the average power of the resistor at resonance; that is,

$$Q_s = \frac{\text{reactive power}}{\text{average power}} \qquad \textbf{(21.7)}$$

The quality factor is also an indication of how much energy is placed in storage (continual transfer from one reactive element to the other) compared to that dissipated. The lower the level of dissipation for the same reactive power, the larger is the Q_s factor and the more concentrated and intense is the region of resonance.

Both low and high Q series resonant response curves appear in Fig. 21.8.

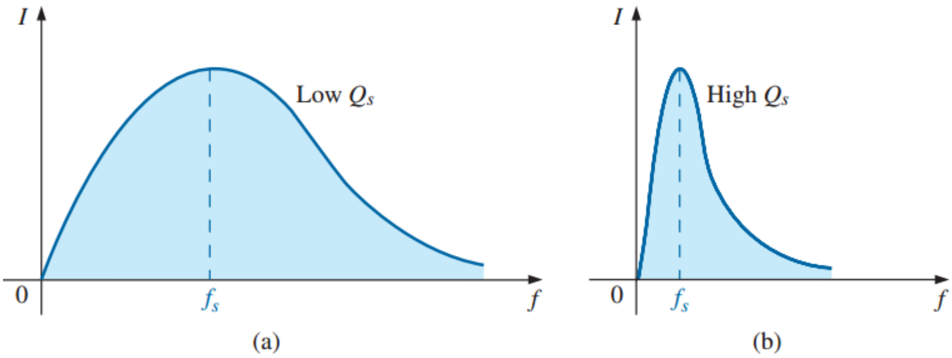

FIG. 21.8
Effect of Q_s as shape of resonant curve.

In general, the higher the quality factor, the higher the voltage across the capacitor or inductor at resonance. In fact, it can be significantly high and perhaps of concern.

Substituting for an inductive reactance in Eq. (21.7) at resonance gives us

$$Q_s = \frac{I^2 X_L}{I^2 R}$$

and

$$Q_s = \frac{X_L}{R} = \frac{\omega_s L}{R} \qquad \textbf{(21.8)}$$

If the resistance R is just the resistance of the coil (R_l) then

$$Q_s = Q_{\text{coil}} = Q_l = \frac{X_L}{R_l} \quad R = R_l \qquad \textbf{(21.9)}$$

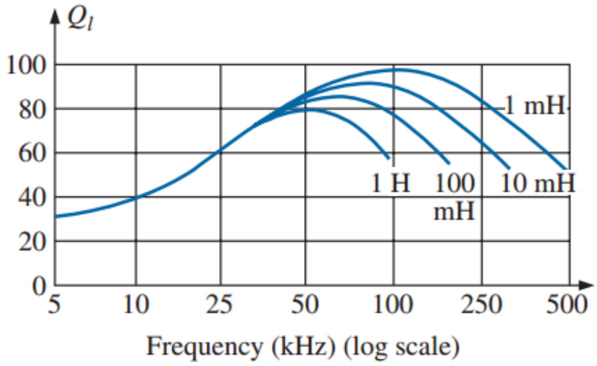

FIG. 21.9

Q_l versus frequency for a series of inductors of similar construction.

If we substitute

$$\omega_s = 2\pi f_s$$

and then

$$f_s = \frac{1}{2\pi\sqrt{LC}}$$

into Eq. (21.8), we have

$$Q_s = \frac{\omega_s L}{R} = \frac{2\pi f_s L}{R} = \frac{2\pi}{R}\left(\frac{1}{2\pi\sqrt{LC}}\right) L$$

$$= \frac{L}{R}\left(\frac{1}{\sqrt{LC}}\right) = \left(\frac{\sqrt{L}}{\sqrt{L}}\right)\frac{L}{R\sqrt{LC}}$$

and

$$\boxed{Q_s = \frac{1}{R}\sqrt{\frac{L}{C}}} \qquad \textbf{(21.10)}$$

providing Q_s in terms of the circuit parameters.

For series resonant circuits used in communication systems, Q_s is usually greater than 1. By applying the voltage divider rule to the circuit in Fig. 21.3, we obtain

$$V_L = \frac{X_L E}{Z_T} = \frac{X_L E}{R} \text{ (at resonance)}$$

and

$$\boxed{V_{L_s} = Q_s E} \qquad \textbf{(21.11)}$$

or

$$V_C = \frac{X_C E}{Z_T} = \frac{X_C E}{R}$$

and

$$\boxed{V_{C_s} = Q_s E} \qquad \textbf{(21.12)}$$

In the circuit in Fig. 21.10, for example, which is in the state of resonance,

$$Q_s = \frac{X_L}{R} = \frac{480\ \Omega}{6\ \Omega} = 80$$

and
$$V_L = V_C = Q_s E = (80)(10\ \text{V}) = \mathbf{800\ V}$$

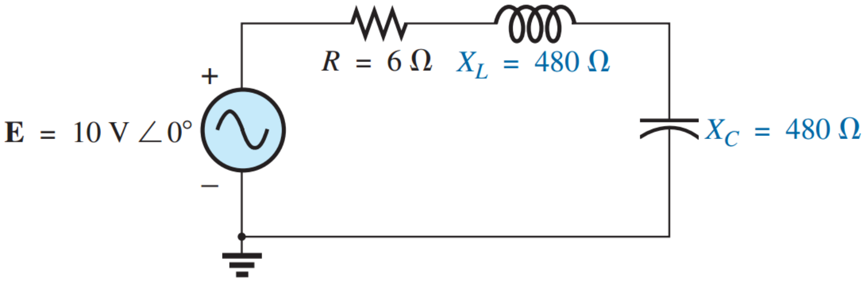

FIG. 21.10
High-Q series resonant circuit.

21.4 Z_T VERSUS FREQUENCY

The total impedance of the series *R-L-C* circuit in Fig. 21.3 at any frequency is determined by

$$\mathbf{Z}_T = R + jX_L - jX_C \qquad \text{or} \qquad \mathbf{Z}_T = R + j(X_L - X_C)$$

The magnitude of the impedance $\mathbf{Z}_T$ versus frequency is determined by

$$Z_T = \sqrt{R^2 + (X_L - X_C)^2}$$

The total-impedance-versus-frequency curve for the series resonant circuit in Fig. 21.3 can be found by applying the impedance-versus-frequency curve for each element of the equation just derived, written in the following form:

$$Z_T(f) = \sqrt{[R(f)]^2 + [X_L(f) - X_C(f)]^2} \qquad \textbf{(21.13)}$$

where $Z_T(f)$ "means" the total impedance as a *function* of frequency.

For the frequency range of interest, we assume that the resistance R does not change with frequency, resulting in the plot in Fig. 21.11. The curve for the inductance, as determined by the reactance equation, is a straight line intersecting the origin with a slope sensitive to the inductance of the coil. The mathematical expression for any straight line in a two-dimensional plane is given by

$$y = mx + b$$

Thus, for the coil,

$$\begin{array}{ccccccc} X_L = 2\pi fL + 0 = & (2\pi L) & (f) & + & 0 \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ y = & m & \cdot\ x & + & b \end{array}$$

(where $2\pi L$ is the slope), producing the results shown in Fig. 21.12.

For the capacitor,

$$X_C = \frac{1}{2\pi fC} \qquad \text{or} \qquad X_C f = \frac{1}{2\pi C}$$

which becomes $yx = k$, the equation for a hyperbola, where

$$y(\text{variable}) = X_C$$

$$x(\text{variable}) = f$$

$$k(\text{constant}) = \frac{1}{2\pi C}$$

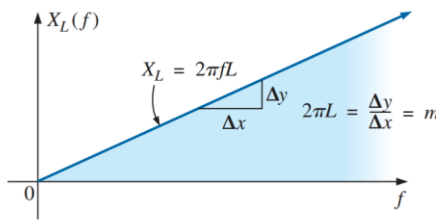

FIG. 21.12
Inductive reactance versus frequency.

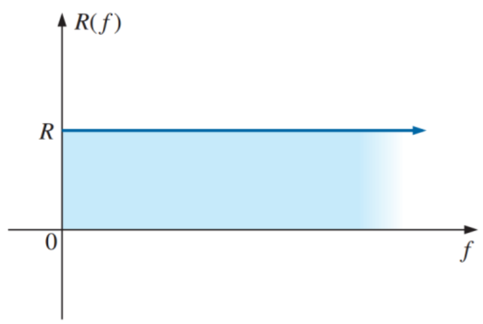

FIG. 21.11
Resistance versus frequency.

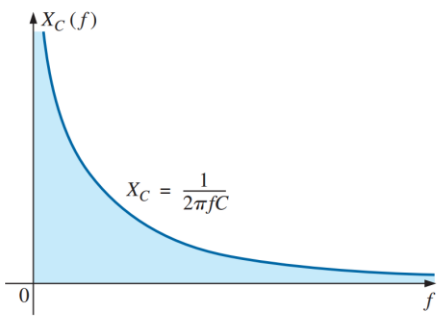

FIG. 21.13
Capacitive reactance versus frequency.

$$Z_T(f) = \sqrt{[R(f)]^2 + [X_L(f) - X_C(f)]^2}$$
$$= \sqrt{[R(f)]^2 + [X(f)]^2}$$

to the curves in Fig. 21.14, where $X(f) = X_L(f) - X_C(f)$, we obtain the curve for $Z_T(f)$ as shown in Fig. 21.15. The minimum impedance occurs at the resonant frequency and is equal to the resistance *R*. Note that the curve is not symmetrical about the resonant frequency (especially at higher values of Z_T).

The phase angle associated with the total impedance is

$$\theta = \tan^{-1}\frac{(X_L - X_C)}{R} \quad \textbf{(21.14)}$$

For the $\tan^{-1}x$ function (resulting when $X_L > X_C$), the larger *x* is, the larger is the angle θ (closer to 90°). However, for regions where $X_C > X_L$, one must also be aware that

$$\tan^{-1}(-x) = -\tan^{-1}x \quad \textbf{(21.15)}$$

At low frequencies, $X_C > X_L$, and θ approaches −90° (capacitive), as shown in Fig. 21.16, whereas at high frequencies, $X_L > X_C$, and θ approaches 90°. In general, therefore, for a series resonant circuit:

$f < f_s$: network capacitive; **I** leads **E**
$f > f_s$: network inductive; **E** leads **I**
$f = f_s$: network resistive; **E** and **I** are in phase

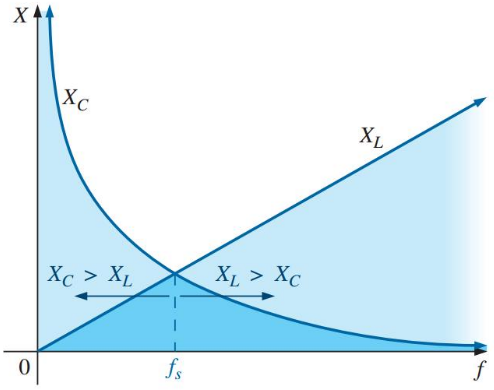

FIG. 21.14
Placing the frequency response of the inductive and capacitive reactance of a series R-L-C circuit on the same set of axes.

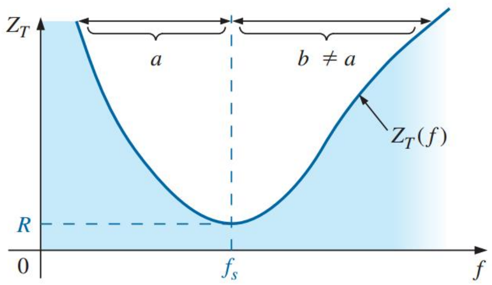

FIG. 21.15
Z_T versus frequency for the series resonant circuit.

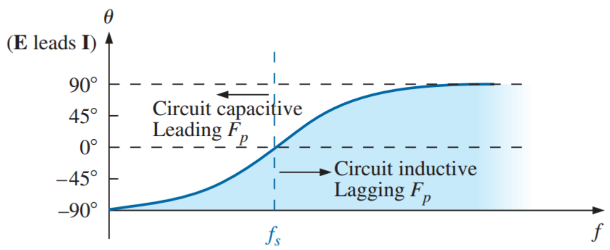

FIG. 21.16
Phase plot for the series resonant circuit.

21.5 SELECTIVITY

If we now plot the magnitude of the current $I = E/Z_T$ versus frequency for a *fixed* applied voltage *E*, we obtain the curve shown in Fig. 21.17, which rises from zero to a maximum value of E/R (where Z_T is a minimum) and then drops toward zero (as Z_T increases) at a slower rate than it rose to its peak value. The curve is actually the inverse of the impedance-versus-frequency curve. Since the Z_T curve is not absolutely symmetrical about the resonant frequency, the curve of the current versus frequency has the same property.

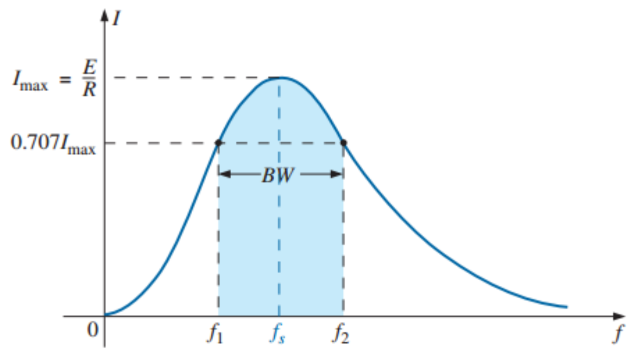

FIG. 21.17
I versus frequency for the series resonant circuit.

There is a definite range of frequencies at which the current is near its maximum value and the impedance is at a minimum. Those frequencies corresponding to 0.707 of the maximum current are called the **band frequencies, cutoff frequencies, half-power frequencies,** or **corner frequencies**. They are indicated by f_1 and f_2 in Fig. 21.17. The range of frequencies between the two is referred to as the **bandwidth** (abbreviated *BW*) of the resonant circuit.

Half-power frequencies are those frequencies at which the power delivered is one-half that delivered at the resonant frequency; that is,

$$P_{HPF} = \frac{1}{2} P_{max} \qquad \textbf{(21.16)}$$

The above condition is derived using the fact that

$$P_{max} = I^2_{max} R$$

and

$$P_{HPF} = I^2 R = (0.707 I_{max})^2 R = (0.5)(I^2_{max} R) = \frac{1}{2} P_{max}$$

Since the resonant circuit is adjusted to "select" a band of frequencies, the curve in Fig. 21.17 is called the **selectivity curve**. The term is derived from the fact that one must be *selective* in choosing the frequency to ensure that it is in the bandwidth.

The smaller the bandwidth, the higher is the selectivity.

The shape of the curve, as shown in Fig. 21.18, depends on each element of the series *R-L-C* circuit. If the resistance is made smaller with a fixed inductance and capacitance, the bandwidth decreases and the selectivity increases. Similarly, if the ratio L/C increases with fixed resistance, the bandwidth again decreases with an increase in selectivity.

In terms of Q_s, if R is larger for the same X_L, then Q_s is less, as determined by the equation $Q_s = \omega_s L/R$.

A small Q_s, therefore, is associated with a resonant curve having a large bandwidth and a low level of selectivity, while a large Q_s indicates the opposite.

For circuits where $Q_s \geq 10$ (indicating a tight curve around the resonant frequency), a widely accepted approximation is that the resonant frequency bisects the bandwidth and that the resonant curve is symmetrical about the resonant frequency.

These conditions are shown in Fig. 21.19, indicating that the cutoff frequencies are then equidistant from the resonant frequency.

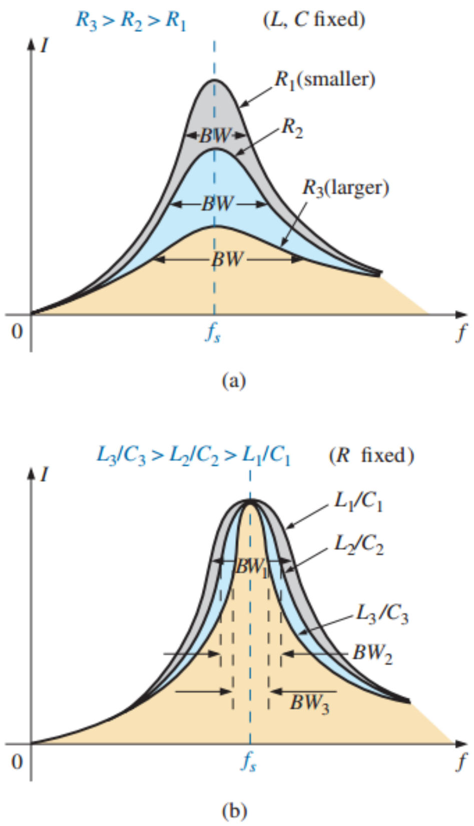

FIG. 21.18
Effect of R, L, and C on the selectivity curve for the series resonant circuit.

For any Q_s, the preceding is not true. The cutoff frequencies f_1 and f_2 can be found for the general case (any Q_s) by first using the fact that a drop in current to 0.707 of its resonant value corresponds to an increase in impedance equal to $1/0.7071 = \sqrt{2}$ times the resonant value, which is R.

Substituting $\sqrt{2}R$ into the equation for the magnitude of Z_T, we find that

$$Z_T = \sqrt{R^2 + (X_L - X_C)^2}$$

becomes

$$\sqrt{2}R = \sqrt{R^2 + (X_L - X_C)^2}$$

or, squaring both sides, that

$$2R^2 = R^2 + (X_L - X_C)^2$$

and

$$R^2 = (X_L - X_C)^2$$

Taking the square root of both sides gives us

$$R = X_L - X_C \quad \text{or} \quad R - X_L + X_C = 0$$

Let us first consider the case where $X_L > X_C$, which relates to f_2 or ω_2. Substituting $\omega_2 L$ for X_L and $1/\omega_2 C$ for X_C and bringing both quantities to the left of the equal sign, we have

$$R - \omega_2 L + \frac{1}{\omega_2 C} = 0 \quad \text{or} \quad R\omega_2 - \omega_2^2 L + \frac{1}{C} = 0$$

which can be written as

$$\omega_2^2 - \frac{R}{L}\omega_2 - \frac{1}{LC} = 0$$

Solving the quadratic, we have

$$\omega_2 = \frac{-(-R/L) \pm \sqrt{[-(R/L)]^2 - [-(4/LC)]}}{2}$$

and

$$\omega_2 = +\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} + \frac{4}{LC}}$$

with

$$f_2 = \frac{1}{2\pi}\left[\frac{R}{2L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}\right] \text{ (Hz)} \qquad \textbf{(21.17)}$$

The negative sign in front of the second factor was dropped because $(1/2)\sqrt{(R/L)^2 + 4/LC}$ is always greater than $R/(2L)$. If it were not dropped, there would be a negative solution for the radian frequency ω_2.

If we repeat the same procedure for $X_C > X_L$, which relates to ω_1 or f_1 such that $Z_T = \sqrt{R^2 + (X_C - X_L)^2}$, the solution f_1 becomes

$$f_1 = \frac{1}{2\pi}\left[-\frac{R}{2L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}\right] \text{ (Hz)} \qquad \textbf{(21.18)}$$

The bandwidth (BW) is

$$BW = f_2 - f_1 = \text{Eq. (21.17)} - \text{Eq. (21.18)}$$

and

$$BW = f_2 - f_1 = \frac{R}{2\pi L} \qquad \textbf{(21.19)}$$

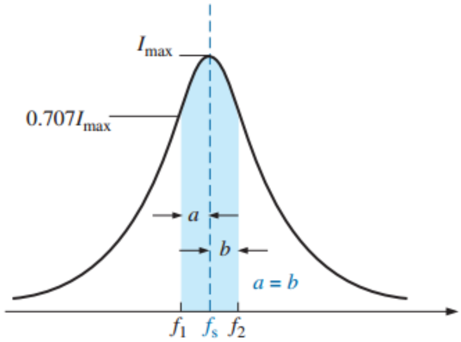

FIG. 21.19
Approximate series resonance curve for $Q_s \geq 10$.

Substituting $R/L = \omega_s/Q_s$ from $Q_s = \omega_s L/R$ and $1/2\pi = f_s/\omega_s$ from $\omega_s = 2\pi f_s$ gives us

$$BW = \frac{R}{2\pi L} = \left(\frac{1}{2\pi}\right)\left(\frac{R}{L}\right) = \left(\frac{f_s}{\omega_s}\right)\left(\frac{\omega_s}{Q_s}\right)$$

or

$$BW = \frac{f_s}{Q_s} \qquad \textbf{(21.20)}$$

which is a very convenient form since it relates the bandwidth to the Q_s of the circuit. As mentioned earlier, Eq. (21.20) verifies that the larger the Q_s, the smaller is the bandwidth, and vice versa.

Written in a slightly different form, Eq. (21.20) becomes

$$\frac{f_2 - f_1}{f_s} = \frac{1}{Q_s} \qquad \textbf{(21.21)}$$

The ratio $(f_2 - f_1)/f_s$ is sometimes called the *fractional bandwidth*, providing an indication of the width of the bandwidth compared to the resonant frequency.

It can also be shown through mathematical manipulations of the pertinent equations that the resonant frequency is related to the geometric mean of the band frequencies; that is,

$$f_s = \sqrt{f_1 f_2} \qquad \textbf{(21.22)}$$

21.7 PRACTICAL CONSIDERATIONS

In the real world, the circuit of Fig. 21.22 should appear as shown in Fig. 21.22. The resistance R used in all the equations in this chapter up to this point must include the source resistance R_s, the resistance of the inductor R_l, and any resistance R_d introduced by design to control the shape of the resonant curve. For the future, therefore,

$$R = R_s + R_d + R_l \qquad \textbf{(21.23)}$$

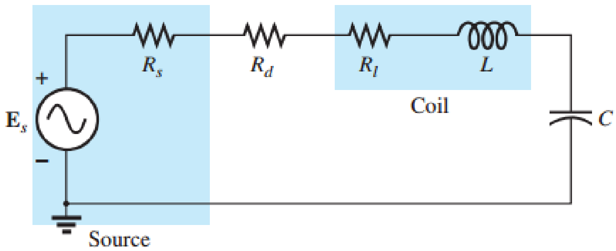

FIG. 21.22
Series resonant circuit.

5. Glossary – English/Chinese Translation

English	Chinese
frequency response	频率响应
resonant circuit	谐振电路
resonance	共鸣
series resonance circuits	串联谐振电路
tuned circuit	调谐电路
quality factor	品质因数
selectivity	选择性
bandwidth	带宽
symmetrical	对称