Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 02-h

AC Phasors

Contents

Reference:

Introductory Circuit Analysis 14th edition, Boylesad & Olivari

Basic Circuit Analysis – Schaum's Outline Series

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2. Complex Numbers, Rectangular Form and Polar Forms

A **complex number** represents a point in a two-dimensional plane located with reference to two distinct axes. This point can also determine a radius vector drawn from the origin to the point. The horizontal axis is called the *real* axis, while the vertical axis is called the *imaginary* axis. Both are labeled in Fig. 14.35. Every number from zero to $\pm\infty$ can be represented by some point along the real axis. Prior to the development of this system of complex numbers, it was believed that any number not on the real axis did not exist—hence the term *imaginary* for the vertical axis.

Two forms are used to represent a complex number: **rectangular** and **polar**. Each can represent a point in the plane or a radius vector drawn from the origin to that point.

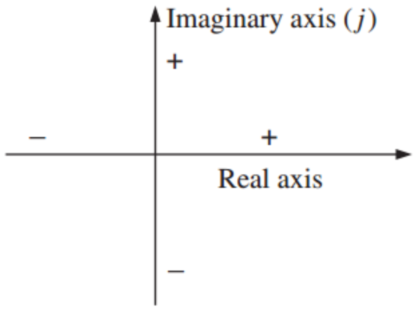


FIG. 14.35
Defining the real and imaginary axes of a complex plane.

Rectangular Form

The format for the **rectangular form** is

$$C = X + jY \tag{14.21}$$

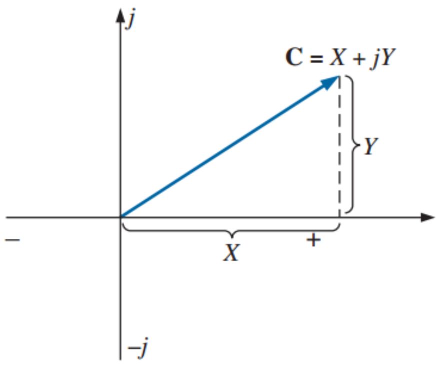


FIG. 14.36
Defining the rectangular form.

Polar Form

The format for the **polar form** is

$$\mathbf{C} = Z \angle \theta \tag{14.22}$$

with the letter Z chosen from the sequence X, Y, Z.

Z indicates magnitude only, and θ is *always measured counterclockwise (CCW) from the positive real axis,* as shown in Fig. 14.40. Angles measured in the clockwise direction from the positive real axis must have a negative sign associated with them.

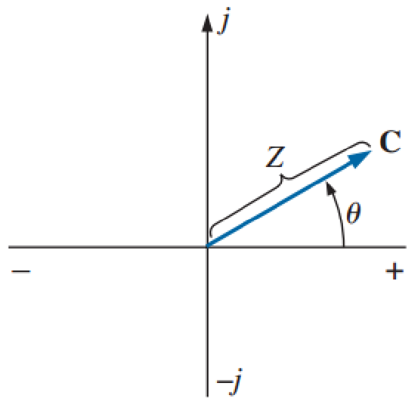


FIG. 14.40
Defining the polar form.

Conversion between Forms

Rectangular to Polar

$$Z = \sqrt{X^2 + Y^2} \tag{14.24}$$

$$\theta = \tan^{-1}\frac{Y}{X} \tag{14.25}$$

Polar to Rectangular

$$X = Z\cos\theta \tag{14.26}$$

$$Y = Z\sin\theta \tag{14.27}$$

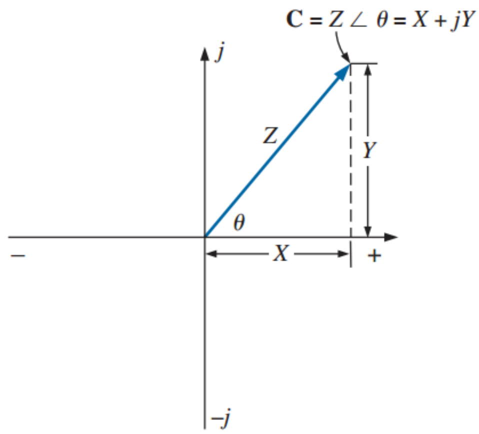


FIG. 14.45

Operation of complex numbers

By definition,

$$j = \sqrt{-1} \tag{14.28}$$

Thus,

$$j^2 = -1 \tag{14.29}$$

and

$$\frac{1}{j} = -j \tag{14.30}$$

Complex Conjugate

The **conjugate** or **complex conjugate** of a complex number can be found by simply changing the sign of the imaginary part in the rectangular form or by using the negative of the angle of the polar form. For example, the conjugate of

$$\mathbf{C} = 2 + j3$$

is

$$2 - j3$$

Reciprocal

The **reciprocal** of a complex number is 1 divided by the complex number. For example, the reciprocal of

$$\mathbf{C} = X + jY$$

is

$$\frac{1}{X + jY}$$

and that of $Z \angle\theta$ is

$$\frac{1}{Z \angle\theta}$$

Addition

$$\mathbf{C}_1 = \pm X_1 \pm jY_1 \quad \text{and} \quad \mathbf{C}_2 = \pm X_2 \pm jY_2$$

then
$$\boxed{\mathbf{C}_1 + \mathbf{C}_2 = (\pm X_1 \pm X_2) + j(\pm Y_1 \pm Y_2)} \qquad \textbf{(14.31)}$$

Subtraction

$$\mathbf{C}_1 = \pm X_1 \pm jY_1 \quad \text{and} \quad \mathbf{C}_2 = \pm X_2 \pm jY_2$$

then
$$\boxed{\mathbf{C}_1 - \mathbf{C}_2 = [\pm X_1 - (\pm X_2)] + j[\pm Y_1 - (\pm Y_2)]} \qquad \textbf{(14.32)}$$

Addition or subtraction cannot be performed in polar form unless the complex numbers have the same angle θ or unless they differ only by multiples of 180°.

Multiplication

$$\mathbf{C}_1 = X_1 + jY_1 \quad \text{and} \quad \mathbf{C}_2 = X_2 + jY_2$$

then $\mathbf{C}_1 \cdot \mathbf{C}_2$:

$$\begin{array}{r} X_1 + jY_1 \\ X_2 + jY_2 \\ \hline X_1X_2 + jY_1X_2 \\ + jX_1Y_2 + j^2Y_1Y_2 \\ \hline X_1X_2 + j(Y_1X_2 + X_1Y_2) + Y_1Y_2(-1) \end{array}$$

and
$$\boxed{\mathbf{C}_1 \cdot \mathbf{C}_2 = (X_1X_2 - Y_1Y_2) + j(Y_1X_2 + X_1Y_2)} \qquad \textbf{(14.33)}$$

In *polar* form, the magnitudes are multiplied and the angles added algebraically. For example, for

$$\mathbf{C}_1 = Z_1 \angle\theta_1 \quad \text{and} \quad \mathbf{C}_2 = Z_2 \angle\theta_2$$

we write
$$\boxed{\mathbf{C}_1 \cdot \mathbf{C}_2 = Z_1Z_2 \angle\theta_1 + \theta_2} \qquad \textbf{(14.34)}$$

Division

$$\mathbf{C}_1 = X_1 + jY_1 \quad \text{and} \quad \mathbf{C}_2 = X_2 + jY_2$$

then
$$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{(X_1 + jY_1)(X_2 - jY_2)}{(X_2 + jY_2)(X_2 - jY_2)}$$
$$= \frac{(X_1X_2 + Y_1Y_2) + j(X_2Y_1 - X_1Y_2)}{X_2^2 + Y_2^2}$$

and
$$\boxed{\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{X_1X_2 + Y_1Y_2}{X_2^2 + Y_2^2} + j\frac{X_2Y_1 - X_1Y_2}{X_2^2 + Y_2^2}} \qquad \textbf{(14.35)}$$

In *polar* form, division is accomplished by dividing the magnitude of the numerator by the magnitude of the denominator and subtracting the angle of the denominator from that of the numerator. That is, for

$$\mathbf{C}_1 = Z_1 \angle\theta_1 \quad \text{and} \quad \mathbf{C}_2 = Z_2 \angle\theta_2$$

we write
$$\boxed{\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{Z_1}{Z_2} \angle\theta_1 - \theta_2} \qquad \textbf{(14.36)}$$

EXAMPLE 14.26 Perform the following operations, leaving the answer in polar or rectangular form:

a. $$\frac{(2 + j3) + (4 + j6)}{(7 + j7) - (3 - j3)} = \frac{(2 + 4) + j(3 + 6)}{(7 - 3) + j(7 + 3)}$$
$$= \frac{(6 + j9)(4 - j10)}{(4 + j10)(4 - j10)}$$
$$= \frac{[(6)(4) + (9)(10)] + j[(4)(9) - (6)(10)]}{4^2 + 10^2}$$
$$= \frac{114 - j24}{116} = \mathbf{0.98 - j0.21}$$

b. $$\frac{(50 \angle 30°)(5 + j5)}{10 \angle -20°} = \frac{(50 \angle 30°)(7.07 \angle 45°)}{10 \angle -20°} = \frac{353.5 \angle 75°}{10 \angle -20°}$$
$$= 35.35 \angle 75° - (-20°) = \mathbf{35.35 \angle 95°}$$

c. $$\frac{(2 \angle 20°)^2(3 + j4)}{8 - j6} = \frac{(2 \angle 20°)(2 \angle 20°)(5 \angle 53.13°)}{10 \angle -36.87°}$$
$$= \frac{(4 \angle 40°)(5 \angle 53.13°)}{10 \angle -36.87°} = \frac{20 \angle 93.13°}{10 \angle -36.87°}$$
$$= 2 \angle 93.13° - (-36.87°) = \mathbf{2.0 \angle 130°}$$

d. $$3 \angle 27° - 6 \angle -40° = (2.673 + j1.362) - (4.596 - j3.857)$$
$$= (2.673 - 4.596) + j(1.362 + 3.857)$$
$$= \mathbf{-1.92 + j5.22}$$

2. Phasors

the addition (or subtraction) of two sinusoidal voltages of the same frequency and phase angle is simply the sum (or difference) of the peak values of each with the sum (or difference) having the same phase angle.

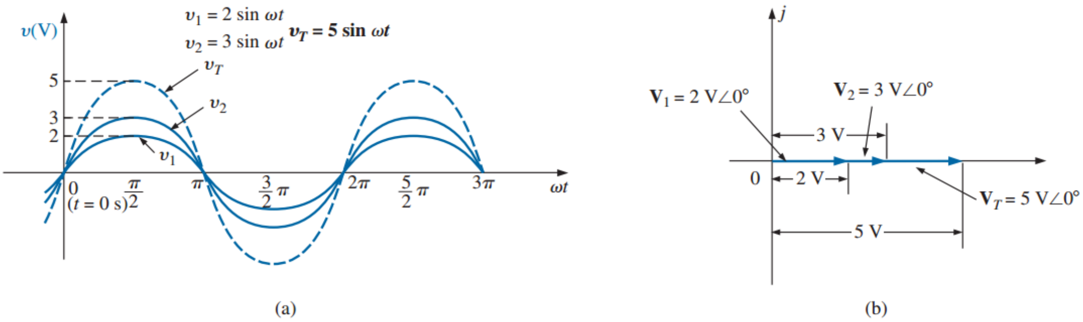


FIG. 14.66
Finding the sum of two sinusoidal waveforms with the same frequency and phase angle.

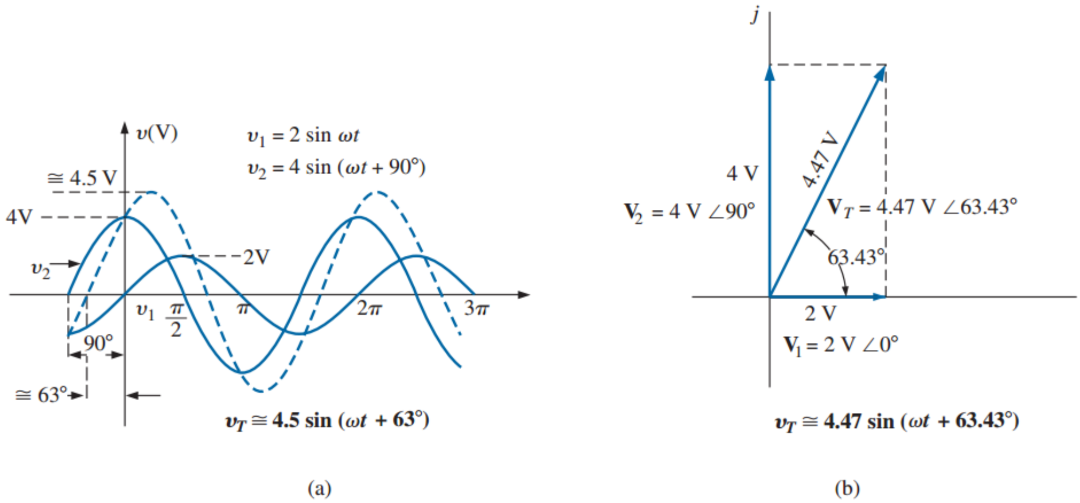


FIG. 14.67
Finding the sum of two sinusoidal waveforms that are out of phase.

Since the rms, rather than the peak, values are used almost exclusively in the analysis of ac circuits, the phasor will now be redefined for the purposes of practicality and uniformity as having a magnitude equal to the *rms value* of the sine wave it represents. The angle associated with the phasor will remain as previously described—the phase angle.

In general, for all of the analyses to follow, the phasor form of a sinusoidal voltage or current will be

$$\mathbf{V} = V\angle\theta \quad \text{and} \quad \mathbf{I} = I\angle\theta$$

where V and I are rms values and θ is the phase angle. It should be pointed out that in phasor notation, the sine wave is always the reference, and the frequency is not represented.

Phasor algebra for sinusoidal quantities is applicable only for waveforms having the same frequency.

EXAMPLE 14.27 Find the sum of the following sinusoidal functions

$$i_1 = 5\ \sin(\omega t + 30°)$$
$$i_2 = 6\ \sin(\omega t + 60°)$$

a. Using a phasor approach
b. Using a graphical approach

Solutions:

a. The two waveforms and the resultant sum appear in Fig. 14.68. It was obviously a tedious process to add the two waveforms with this approach. Take note that the position of each vector generating the waveforms shown is a snapshot of their position at $\theta = 0°(t = 0\ s)$. The sum of the two waveforms is obviously a vector addition of the two waveforms as shown to the left of Fig. 14.68.

b. In phasor form:

$$i_1 = 5\ \sin(\omega t + 30°) \Rightarrow 5\ A\angle 30°$$
$$i_2 = 6\ \sin(\omega t + 60°) \Rightarrow 6\ A\angle 60°$$
$$\mathbf{I}_T = \mathbf{I}_1 + \mathbf{I}_2$$
$$= 5\ A\angle 30° + 6\ A\angle 60°$$
$$= (4.33\ A + j2.5\ A) + (3\ A + j5.2\ A)$$
$$= 7.33\ A + j7.7\ A$$
$$= 10.63\ A\angle 46.41°$$

and $\quad i_T = \mathbf{10.63\ sin(\omega t + 46.41°)}$ as obtained graphically.

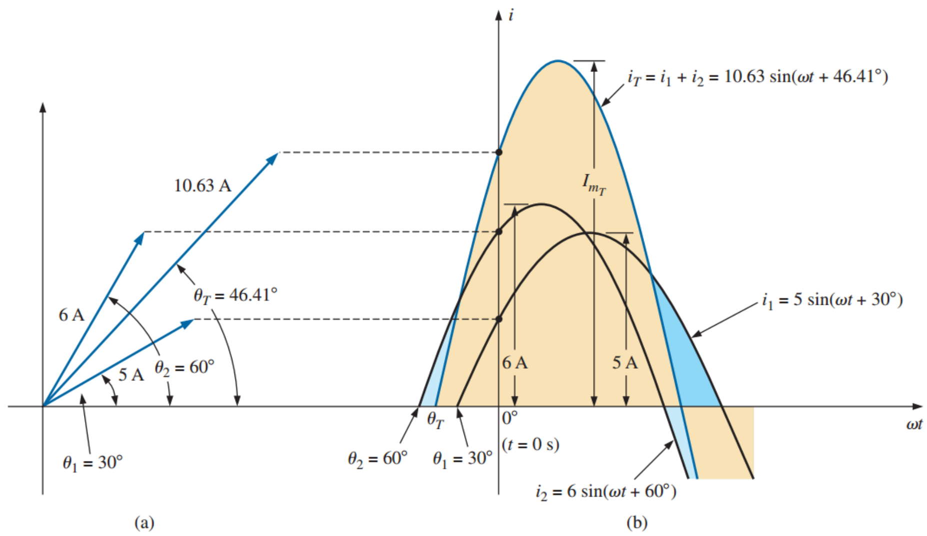


FIG. 14.68
Example 14.27.

EXAMPLE 14.30 Find the input voltage of the circuit in Fig. 14.70 if

$$\left.\begin{aligned} v_a &= 50\sin(377t + 30^\circ) \\ v_b &= 30\sin(377t + 60^\circ) \end{aligned}\right\} f = 60\text{ Hz}$$

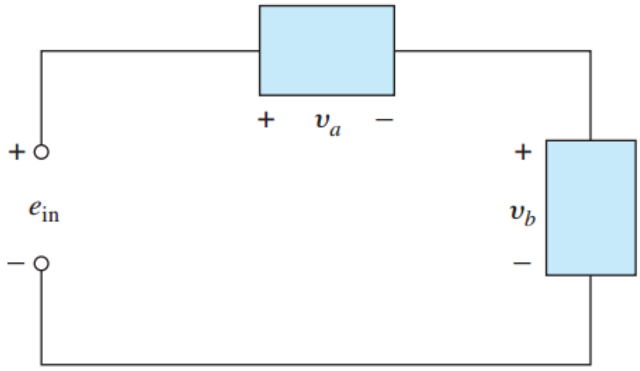


FIG. 14.70
Example 14.30.

Solution: Applying Kirchhoff's voltage law, we have

$$e_{in} = v_a + v_b$$

Converting from the time to the phasor domain yields

$$v_a = 50\sin(377t + 30^\circ) \Rightarrow \mathbf{V}_a = 35.35\text{ V}\angle 30^\circ$$

$$v_b = 30\sin(377t + 60^\circ) \Rightarrow \mathbf{V}_b = 21.21\text{ V}\angle 60^\circ$$

Converting from polar to rectangular form for addition yields

$$\mathbf{V}_a = 35.35\text{ V}\angle 30^\circ = 30.61\text{ V} + j17.68\text{ V}$$

$$\mathbf{V}_b = 21.21\text{ V}\angle 60^\circ = 10.61\text{ V} + j18.37\text{ V}$$

Then

$$\begin{aligned} \mathbf{E}_{in} = \mathbf{V}_a + \mathbf{V}_b &= (30.61\text{ V} + j17.68\text{ V}) + (10.61\text{ V} + j18.37\text{ V}) \\ &= 41.22\text{ V} + j36.05\text{ V} \end{aligned}$$

Converting from rectangular to polar form, we have

$$\mathbf{E}_{in} = 41.22\text{ V} + j36.05\text{ V} = 54.76\text{ V}\angle 41.17^\circ$$

Converting form the phasor form to the time domain, we obtain

$$\mathbf{E}_{in} = 54.76\text{ V}\angle 41.17^\circ \Rightarrow e_{in} = \sqrt{2}(54.76)\sin(377t + 41.17^\circ)$$

and $$e_{in} = \mathbf{77.43\sin(377}\boldsymbol{t}\mathbf{ + 41.17^\circ)}$$

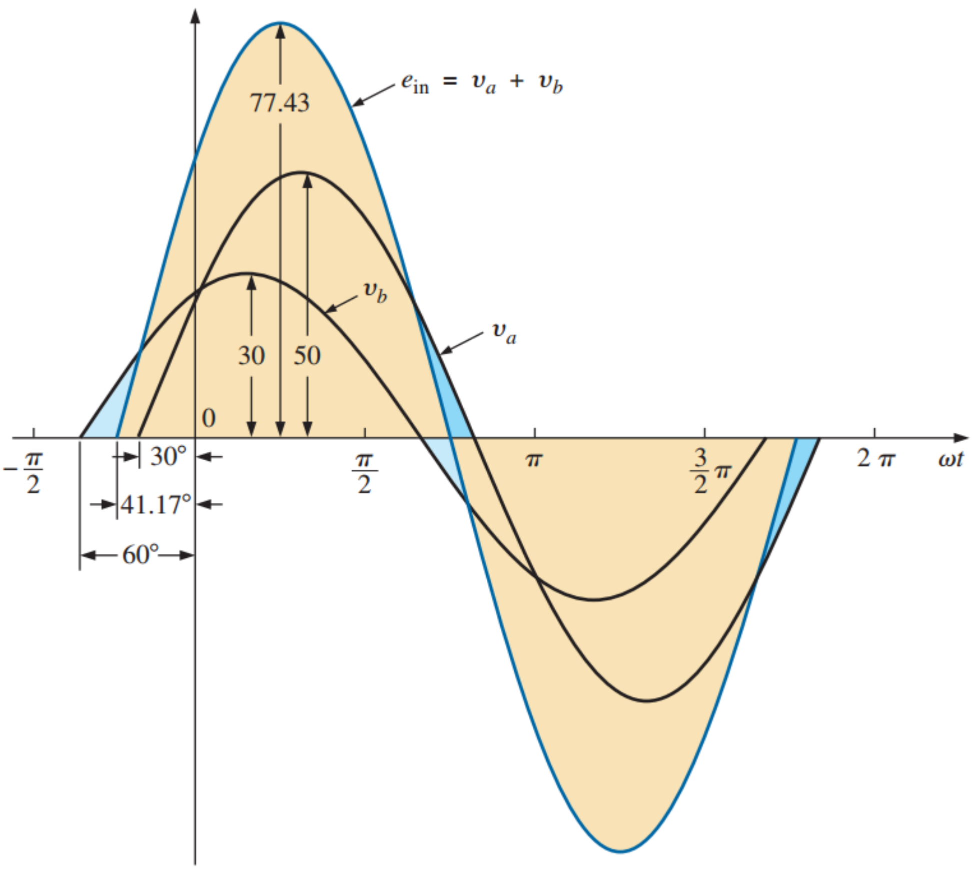


FIG. 14.71
Solution to Example 14.30.

3. Using Phasors in R, L, and C

Resistive Elements

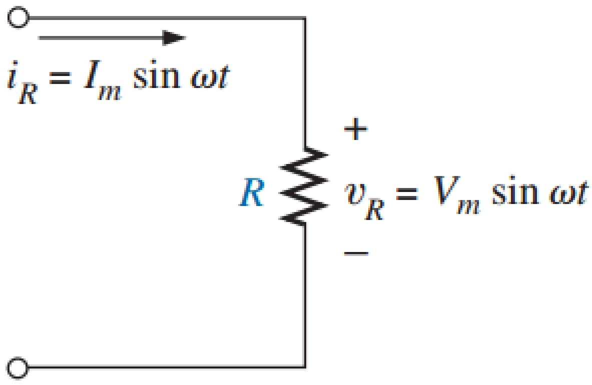


FIG. 15.2
Resistive ac circuit.

If we apply phasor algebra as follows,

$$\mathbf{I}_R = \frac{\mathbf{V}_R}{\mathbf{R}}$$

we find that the format is such that we need to assign an angle to the resistive component in order to apply phasor algebra. For the moment let us assign the angle θ_R to the resistive component so we end up with the following:

$$\mathbf{I}_R = \frac{\mathbf{V}_R}{\mathbf{R}} = \frac{V_R \angle 0°}{R \angle \theta_R} = \frac{V_R}{R} \underline{\angle 0° - \theta_R}$$

Now since we know the angle associated with the current must also be zero degrees, the angle θ_R must be zero degrees. Now if we apply phasor algebra we obtain the following:

$$\mathbf{I}_R = \frac{\mathbf{V}_R}{\mathbf{R}} = \frac{V_R \angle 0°}{R \angle 0°} = \frac{V_R}{R} \angle 0°$$

so that in the time domain

$$i_R = \sqrt{2}\left(\frac{V_R}{R}\right) \sin \omega t$$

which agrees with the development of Chapter 14.

For the future, therefore, whenever we encounter a resistor in the ac domain, we will assign an angle of zero degrees to form a complex number notation. The standard format will therefore be

$$\boxed{Z_R = R\angle 0°} \qquad \textbf{(15.1)}$$

It is important to realize, however, that Z_R is not a phasor, even though the format $R\angle 0°$ is very similar to the phasor notation for sinusoidal currents and voltages. The term phasor is reserved for quantities that vary with time, and R and its associated angle of $0°$ are fixed, nonvarying quantities.

Example

EXAMPLE 15.1 Using complex algebra,

a. Find the current i_R for the circuit in Fig. 15.3.
b. Sketch the waveforms of i_R and v_R.

Solution:

a. $$v_R = 100 \sin \omega t \Rightarrow \text{phasor form } \mathbf{V}_R = 70.71 \text{ V} \angle 0^\circ$$

$$\mathbf{I}_R = \frac{\mathbf{V}_R}{\mathbf{Z}_R} = \frac{V_R \angle \theta}{R \angle 0^\circ} = \frac{70.71 \text{ V} \angle 0^\circ}{5 \ \Omega \angle 0^\circ} = 14.14 \text{ A} \angle 0^\circ$$

$$\text{and } i_R = \sqrt{2}(14.14) \sin \omega t = \mathbf{20 \sin \omega t}$$

b. Note Fig. 15.4.

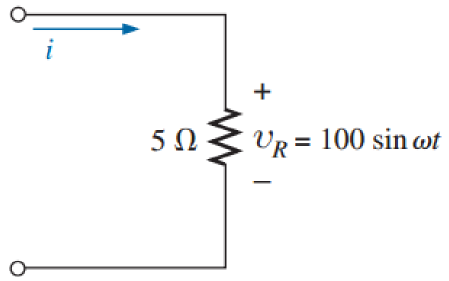


FIG. 15.3
Example 15.1.

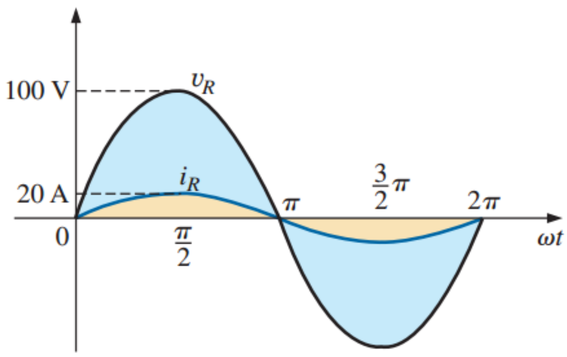
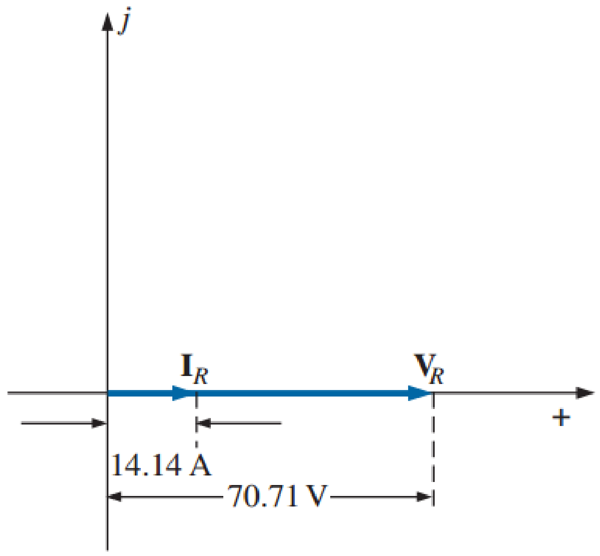


FIG. 15.4
Waveforms for Example 15.1.



EXAMPLE 15.2 Using complex algebra,

a. Find the voltage v_R for the circuit in Fig. 15.5.
b. Sketch the waveforms of v_R and i_R.

Solution:

a. $$i_R = 4\sin(\omega t + 30^\circ) \Rightarrow \text{phasor form } \mathbf{I}_R = 2.828\text{ A}\angle 30^\circ$$

$$\mathbf{V}_R = \mathbf{I}_R\mathbf{Z}_R = (I\angle\theta)(R\angle 0^\circ) = (2.828\text{ A}\angle 30^\circ)(2\,\Omega\angle 0^\circ)$$

$$= 5.656\text{ V}\angle 30^\circ$$

and $v_R = \sqrt{2}(5.656)\sin(\omega t + 30^\circ) = \mathbf{8.0\ sin\ (\boldsymbol{\omega} t + 30^\circ)}$

b. Note Fig. 15.6.

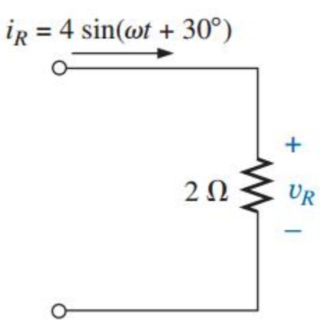


FIG. 15.5
Example 15.2.

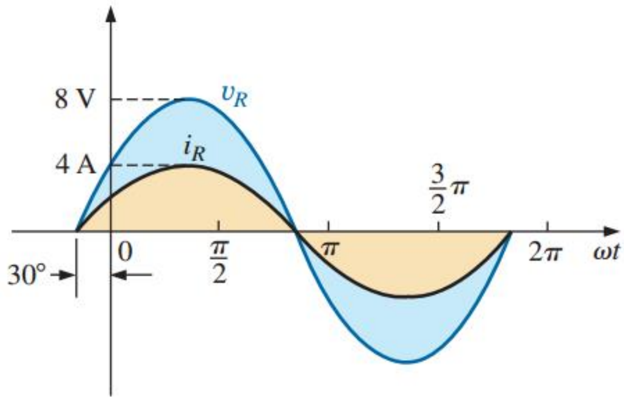
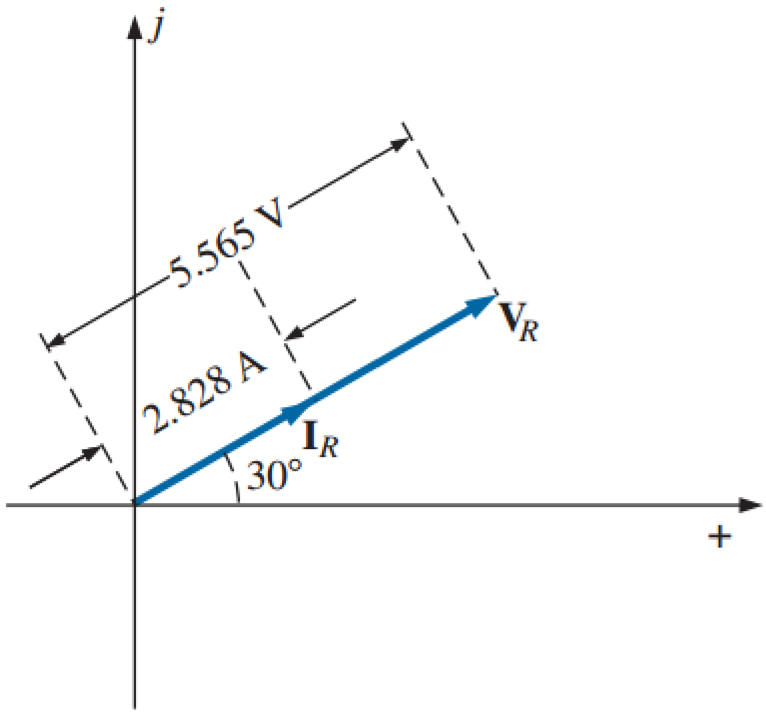


FIG. 15.6
Waveforms for Example 15.2.



Inductive Elements

We learned in Chapter 14 that for the pure inductor in Fig. 15.8, the voltage leads the current by 90° and that the reactance of the coil X_L is determined by ωL. We have

$$v_L = V_m \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = V \angle 0°$$

Applying Ohm's law and following a similar path to that applied to the resistive element, we find that

$$\mathbf{I}_L = \frac{\mathbf{V}_L}{\mathbf{X}_L} = \frac{V_L \angle 0°}{X_L \angle \theta_L} = \frac{V_L}{X_L} \underline{\angle 0° - \theta_L}$$

where V_L and I_L are the effective values in the phasor format. Since v_L leads i_L by 90°, i_L must have an angle of $-90°$ associated with it. To satisfy this condition, θ_L must equal $+90°$. Substituting $\theta_L = 90°$, we obtain

$$\mathbf{I}_L = \frac{V_L \angle 0°}{X_L \angle 90°} = \frac{V_L}{X_L} \underline{\angle 0° - 90°} = \frac{V_L}{X_L} \angle -90°$$

so that in the time domain,

$$i_L = \sqrt{2}\left(\frac{V_L}{X_L}\right) \sin(\omega t - 90°)$$

We use the fact that $\theta_L = 90°$ in the following polar format for inductive reactance to ensure the proper phase relationship between the voltage and current of an inductor:

$$\boxed{\mathbf{Z}_L = X_L \angle 90°} \qquad \textbf{(15.2)}$$

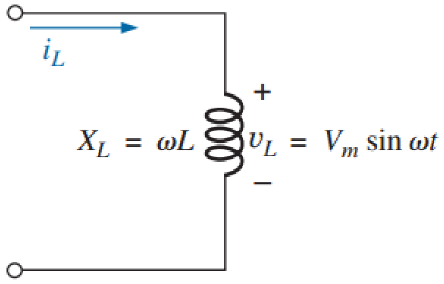


FIG. 15.8
Inductive ac circuit.

EXAMPLE 15.4 Using complex algebra,

a. Find the voltage υ_L for the circuit in Fig. 15.11.
b. Sketch the υ_L and i_L curves.

Solution:

a. $i_L = 5\sin(\omega t + 30^\circ) \Rightarrow$ phasor form $\mathbf{I}_L = 3.535\text{ A}\angle 30^\circ$

$$\mathbf{V}_L = \mathbf{I}_L\mathbf{Z}_L = (I_L\angle\theta)(X_L\angle 90^\circ) = (3.535\text{ A}\angle 30^\circ)(4\,\Omega\angle{+90^\circ})$$
$$= 14.140\text{ V}\angle 120^\circ$$

and $\upsilon_L = \sqrt{2}(14.140)\sin(\omega t + 120^\circ) = \mathbf{20\ sin(\boldsymbol{\omega t} + 120^\circ)}$

b. Note Fig. 15.12.

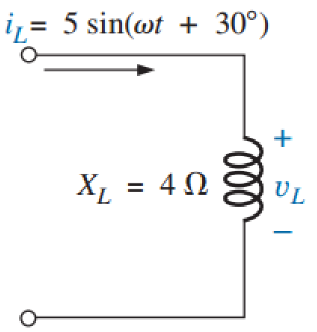


FIG. 15.11
Example 15.4.

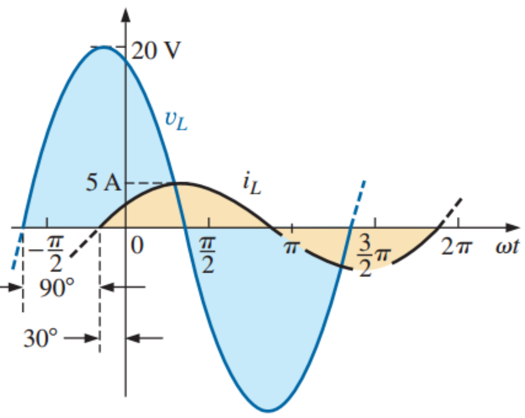
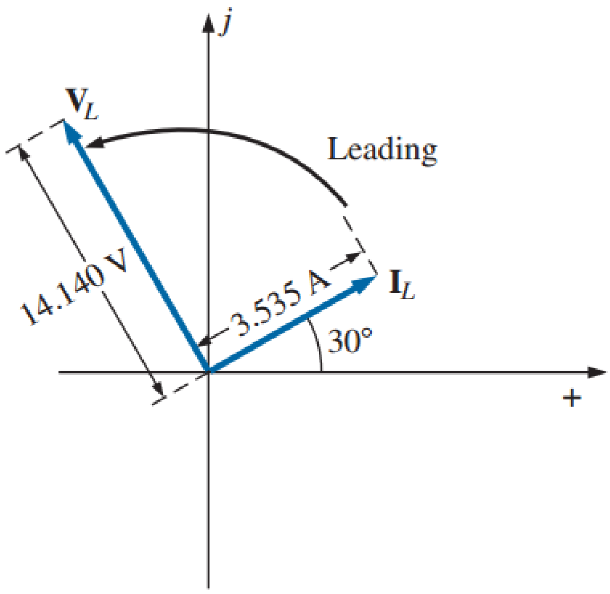


FIG. 15.12
Waveforms for Example 15.4.



Capacitive Elements

$$v_C = V_m \sin \omega t \Rightarrow \text{phasor form } \mathbf{V}_C = V_C \angle 0°$$

Applying Ohm's law and continuing as before, we find

$$\mathbf{I}_C = \frac{\mathbf{V}_C}{\mathbf{X}_C} = \frac{V_C \angle 0°}{X_C \angle \theta_C} = \frac{V_C}{X_C} \angle 0° - \theta_C$$

where V_C and I_C are effective values in the phasor notation.
Since i_C leads v_C by 90°, i_C must have an angle of +90° associated with it. To satisfy this condition, θ_C must equal −90°. Substituting $\theta_C = -90°$ yields

$$\mathbf{I}_C = \frac{\mathbf{V}_C}{\mathbf{X}_C} = \frac{V_C \angle 0°}{X_C \angle -90°} = \frac{V_C}{X_C} \angle 0° - (-90°) = \frac{V_C}{X_C} \angle 90°$$

so, in the time domain,

$$i_C = \sqrt{2}\left(\frac{V_C}{X_C}\right) \sin (\omega t + 90°)$$

We use the fact that $\theta_C = -90°$ in the following polar format for capacitive reactance to ensure the proper phase relationship between the voltage and current of a capacitor:

$$\mathbf{Z}_C = X_C \angle -90° \qquad \textbf{(15.3)}$$

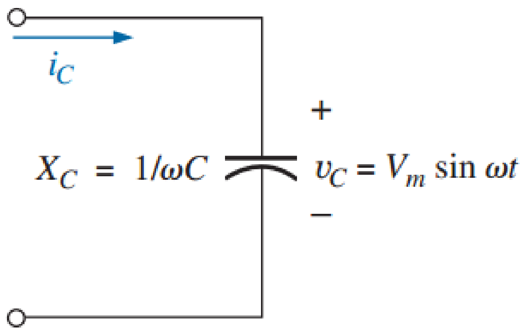


FIG. 15.14
Capacitive ac circuit.

EXAMPLE 15.6 Using complex algebra,

a. Find the voltage v_C for the circuit in Fig. 15.17.
b. Sketch the v_C and i_C curves.

Solution:

a. $i_C = 6 \sin(\omega t - 60°) \Rightarrow$ phasor notation $\mathbf{I}_C = 4.242 \text{ A} \angle -60°$

$$\mathbf{V}_C = \mathbf{I}_C \mathbf{Z}_C = (I_C \angle \theta)(X_C \angle -90°) = (4.242 \text{ A} \angle -60°)(0.5 \ \Omega \angle -90°)$$

$$= 2.121 \text{ V} \angle -150°$$

and $v_C = \sqrt{2}(2.121) \sin(\omega t - 150°) = \mathbf{3.0 \sin(\omega t - 150°)}$

b. Note Fig. 15.18.

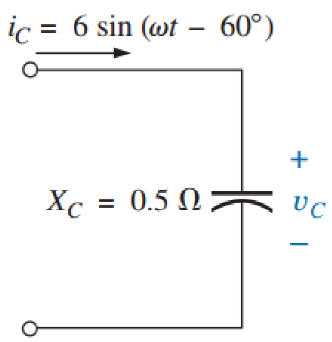


FIG. 15.17
Example 15.6.

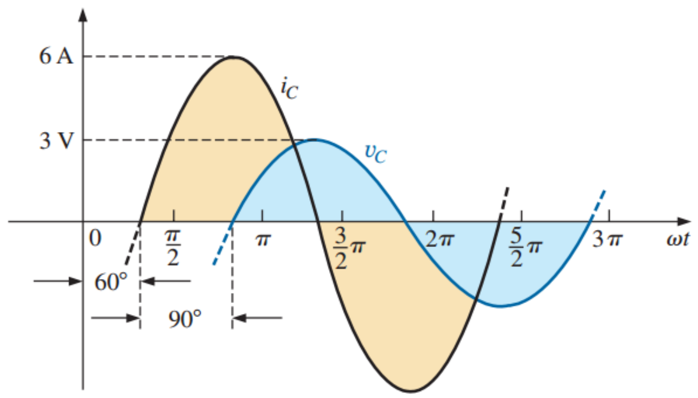
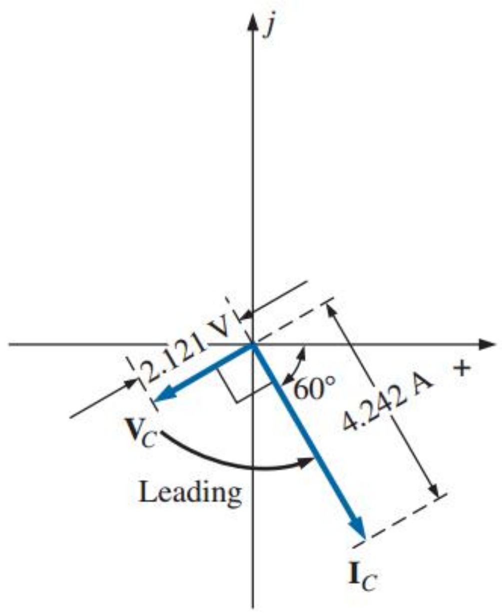


FIG. 15.18
Waveforms for Example 15.6.



Impedance Diagram

Now that an angle is associated with resistance, inductive reactance, and capacitive reactance, each can be placed on a complex plane diagram, as shown in Fig. 15.20. For any network, the resistance will *always* appear on the positive real axis, the inductive reactance on the positive imaginary axis, and the capacitive reactance on the negative imaginary axis. The result is an **impedance diagram** that can reflect the individual and total impedance levels of an ac network.

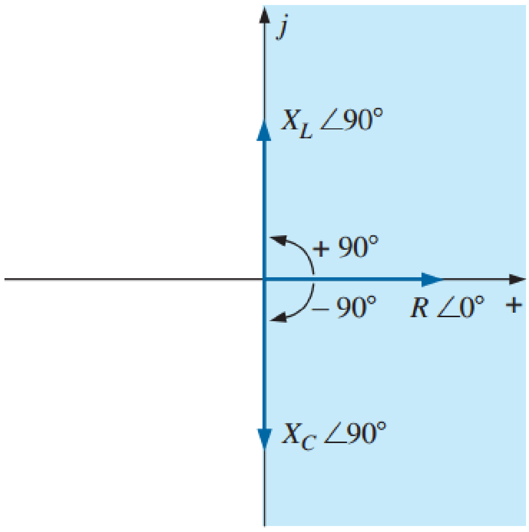


FIG. 15.20
Impedance diagram.

For any configuration (series, parallel, series-parallel, and so on), the angle associated with the total impedance is the phase angle by which the applied voltage leads the source current. For inductive networks, θ_T will be positive, whereas for capacitive networks, θ_T will be negative.

It is important to be aware that if the total impedance of a network has a positive angle associated with it, the network is inductive in nature and has a **lagging power factor** ***and the applied voltage will lead the current drawn by the network. If the total impedance of a network has a negative angle associated with it, the network is capacitive in nature and has a*** **leading power factor** ***and the applied voltage will lag the current drawn by the network.***

Examples

EXAMPLE 15.7 Sketch the impedance diagram for a 22 ohm resistor.

Solution: Note Fig. 15.21.

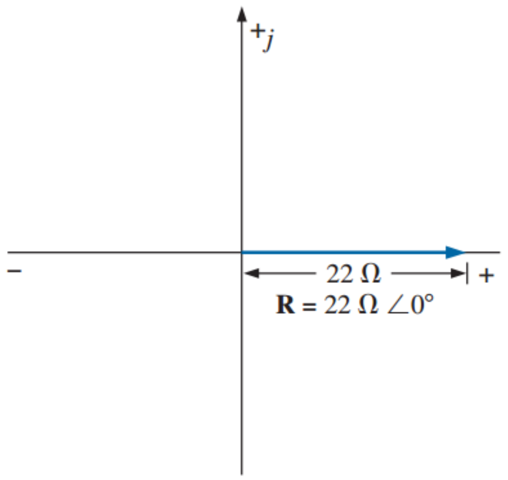


FIG. 15.21
The impedance diagram for a 22 Ω resistor.

EXAMPLE 15.8 Sketch the impedance diagram of a 1.2 kΩ inductive reactance and 2 kΩ capacitive reactance.

Solution: Note Fig. 15.22.

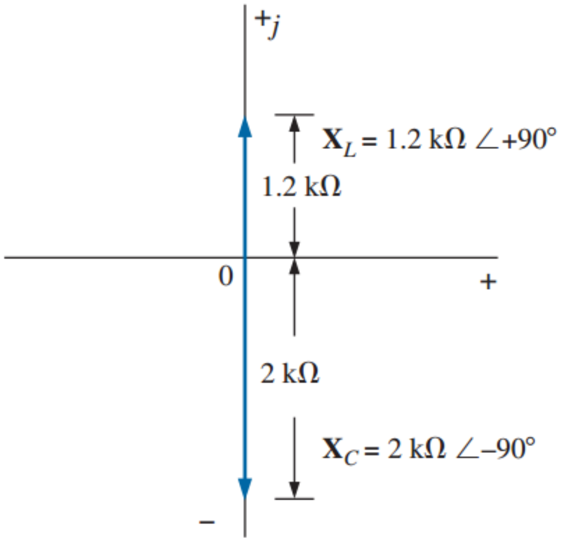


FIG. 15.22
The impedance diagram for a 1.2 kΩ inductive reactance and a 2 kΩ capacitive reactance.

4. Glossary – English/Chinese Translation

English	Chinese
Impedance diagram	阻抗图
Phasor diagram	相量图
Series configuration	系列配置
Voltage divider rule	分压器规则
Phasor	相
Phasor diagram	相量图
Polar form	极性形式
Power factor	功率因数
Reactance	电抗
Reciprocal	倒数
Rectangular form	矩形形式
AC Impedance	交流阻抗
AC Reactance	交流电抗