Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 02-c

DC Circuit Analysis

Contents

Reference:
Basic Circuit Analysis – Schaum's Outline Series

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1. Cramer's Rule and Calculator Solution

Cramer's Rule

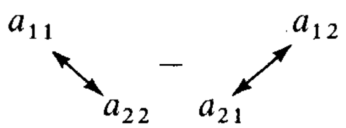
A knowledge of determinants is necessary for using Cramer's rule, which is a popular method for solving the simultaneous equations that occur in the analysis of a circuit.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

A determinant with two rows and columns is a second-order determinant. One with three rows and columns is a third-order determinant, and so on. Determinants have values. The value of the second-order determinant.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

is $a_{11}a_{22} - a_{21}a_{12}$, which is the product of the numbers on the principal diagonal minus the product of the numbers on the other diagonal:



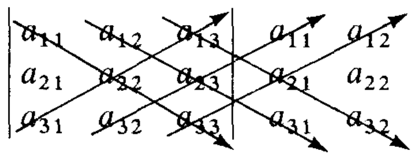
For example, the value of:

$$\begin{vmatrix} 8 & -2 \\ 6 & -4 \end{vmatrix}$$

is $8(-4)-6(-2) = -32 +12= -20.$

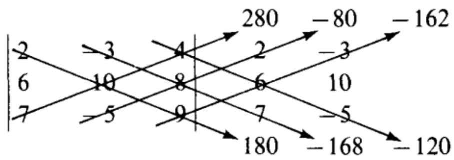
A convenient method for evaluating a third-order determinant is to repeat the first two columns to the right of the third column and then take the sum of the products of the numbers on the diagonal indicated by downward arrows, as follows, and subtract from this the sum of the products of the numbers on the diagonals indicated by upward arrows. The result is

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$



For example, the value of:

$$\begin{vmatrix} 2 & -3 & 4 \\ 6 & 10 & 8 \\ 7 & -5 & 9 \end{vmatrix}$$



is $180 - 168 - 120 - (280 - 80 - 162) = -146$.

Before Cramer's rule can be applied to solve for the unknowns in a set of equations, the equations must be arranged with the unknowns on one side, say the left, of the equal signs and the knowns on the right-hand side. The unknowns should have the same order in each equation. For example, 7, may be the first unknown in each equation, 72 the second, and so on. Then, by Cramer's rule, each unknown is the ratio of two determinants. The denominator determinants are the same, being formed from the coefficients of the unknowns. Each numerator determinant differs from the denominator determinant in only one column. For the first unknown, the numerator determinant has a *first* column that is the right-hand side of the equations. For the second unknown, the numerator determinant has a *second* column that is the right-hand side of the equations, and so on.

$$\begin{aligned} 10I_1 - 2I_2 - 4I_3 &= 32 \\ -2I_1 + 12I_2 - 9I_3 &= -43 \\ -4I_1 - 9I_2 + 15I_3 &= 13 \end{aligned}$$

$$I_1 = \frac{\begin{vmatrix} 32 & -2 & -4 \\ -43 & 12 & -9 \\ 13 & -9 & 15 \end{vmatrix}}{\begin{vmatrix} 10 & -2 & -4 \\ -2 & 12 & -9 \\ -4 & -9 & 15 \end{vmatrix}} \quad I_2 = \frac{\begin{vmatrix} 10 & 32 & -4 \\ -2 & -43 & -9 \\ -4 & 13 & 15 \end{vmatrix}}{\begin{vmatrix} 10 & -2 & -4 \\ -2 & 12 & -9 \\ -4 & -9 & 15 \end{vmatrix}} \quad I_3 = \frac{\begin{vmatrix} 10 & -2 & 32 \\ -2 & 12 & -43 \\ -4 & -9 & 13 \end{vmatrix}}{\begin{vmatrix} 10 & -2 & -4 \\ -2 & 12 & -9 \\ -4 & -9 & 15 \end{vmatrix}}$$

Calculator Solution

To solve the simultaneous equations of interest here is to use an advanced scientific calculator. No programming is required, the equations are easy to enter, and solutions can be obtained just by pressing a single key. Typically, the equations must be first placed in matrix form.

$$\begin{aligned} 10I_1 - 2I_2 - 4I_3 &= 32 \\ -2I_1 + 12I_2 - 9I_3 &= -43 \\ -4I_1 - 9I_2 + 15I_3 &= 13 \end{aligned}$$

The correspondent matrix:

$$\begin{bmatrix} 10 & -2 & -4 \\ -2 & 12 & -9 \\ -4 & -9 & 15 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 32 \\ -43 \\ 13 \end{bmatrix}$$

The exact method of entering the elements depends on the calculator used but should be simple to do. Typically, the solutions are returned in a vector, and they appear in the same order as the corresponding quantity symbols in the vector of unknowns.

2. Source Transformation

Figure 4-la shows the transformation from a voltage source to an equivalent current source, and Fig. 4-1b the transformation from a current source to an equivalent voltage source.

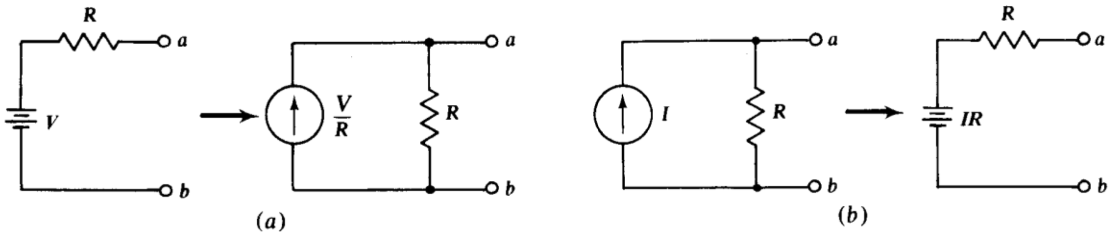


Fig. 4-1

3. Mesh and Loop Analysis

In mesh analysis, KVL is applied with mesh currents, which are currents assigned to meshes, and preferably referenced to flow clockwise, as shown below.

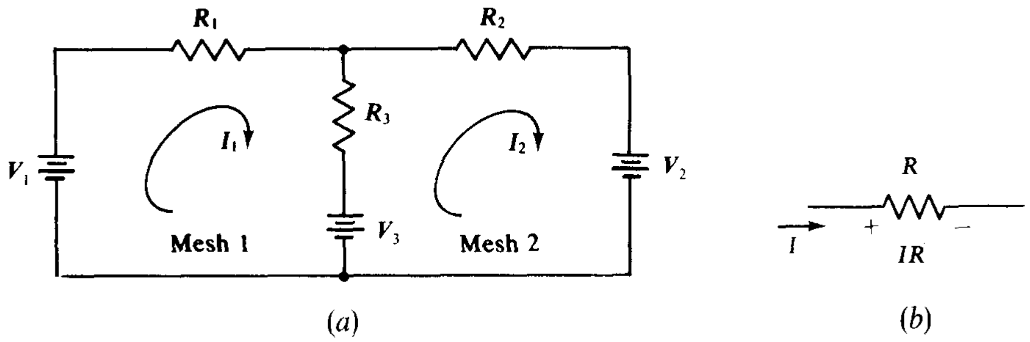


Fig. 4-2

Mesh 1 equation:

$$I_1R_1 + (I_1 - I_2)R_3 = V_1 - V_3 \qquad \text{or} \qquad (R_1 + R_3)I_1 - R_3I_2 = V_1 - V_3$$

$R_1 + R_3$, the coefficient of I_1; is the sum of the resistances of the resistors in mesh 1. This sum is called the self-resistance of mesh 1. R_3 is called the mutual resistance.

Mesh 2 equation:

$$-R_3I_1 + (R_2 + R_3)I_2 = V_3 - V_2$$

In a mesh equation, the voltage for a voltage source has a positive sign if the voltage source aids the flow of the principal mesh current—that is, if this current flow out of the positive terminal—because this aiding is equivalent to a voltage rise. Otherwise, a source voltage has a negative sign.

Loop Analysis

Loop analysis is similar to mesh analysis, the principal difference being that the current paths selected are loops that are not necessarily meshes.

Also, there is no convention on the direction of loop currents; they can be clockwise or counter-clockwise. As a result, mutual terms can be positive when KVL is applied to the loops.

4. Nodel Analysis

For nodal analysis, preferably all voltage sources are transformed to current sources and all resistances are converted to conductance.

Conventionally, voltages on all other nodes are referenced positive with respect to the ground node. As a consequence, showing node voltage polarity signs is not necessary.

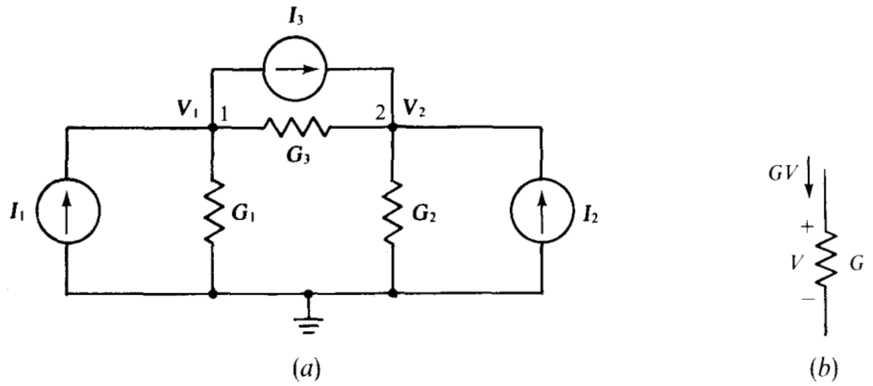


Fig. 4-3

This current is equal to the conductance times the voltage at the node at which the current enters the resistor minus the voltage at the node at which the current leaves the resistor.

For node 1:

$$G_1V_1 + G_3(V_1 - V_2) = I_1 - I_3 \qquad \text{or} \qquad (G_1 + G_3)V_1 - G_3V_2 = I_1 - I_3$$

$G_1 + G_3$ is called the <u>self-conductance</u> of node 1. G_3 is called the mutual conductance of nodes 1 and 2.

For node 2:

$$-G_3V_1 + (G_2 + G_3)V_2 = I_2 + I_3$$

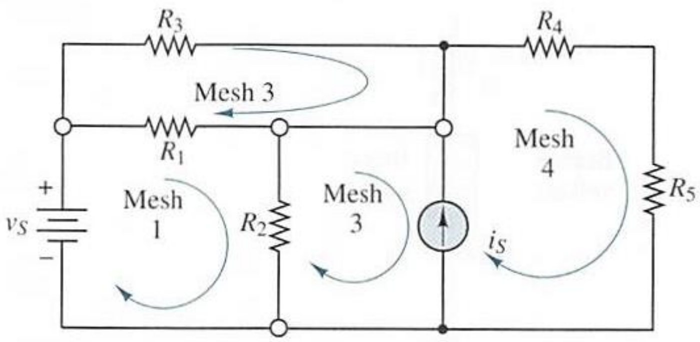
<u>5. Examples</u>

<u>Example 1</u>

How many branches, nodes, loops and meshes are there is in the circuit in Figure 4-2a, in page 5?

(Answer: 6; 5; 3; 2)

<u>Example 2</u>



How many loops can you identify in this four-mesh circuit? (Answer: 14)

Figure 2.8 Definition of a mesh

Example 3 *(from Giorgio Rizzoni, "Principles and Applications of Electrical Engineering, chapter 2)*

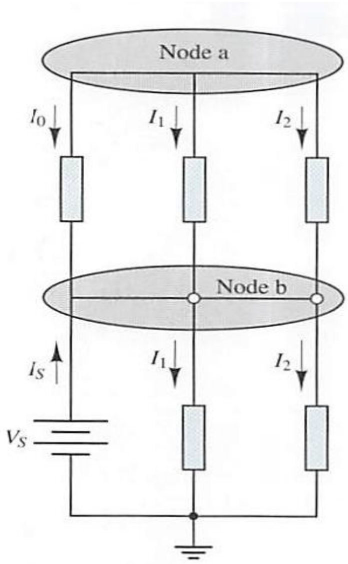


Figure 2.13 Demonstration of KCL

Known Quantities:

$$I_S = 5\text{ A} \qquad I_1 = 2\text{ A} \qquad I_2 = -3\text{ A} \qquad I_3 = 1.5\text{ A}$$

Find: I_0 and I_4.

Analysis: Two nodes are clearly shown in Figure 2.13 as node a and node b; the third node in the circuit is the reference (ground) node. In this example we apply KCL at each of the three nodes.

At node a:

$$I_0 + I_1 + I_2 = 0$$

$$I_0 + 2 - 3 = 0$$

$$\therefore \qquad I_0 = 1\text{ A}$$

Note that the three currents are all defined as flowing away from the node, but one of the currents has a negative value (i.e., it is actually flowing toward the node).

At node b:

$$I_S - I_3 - I_4 = 0$$

$$5 - 1.5 - I_4 = 0$$

$$\therefore \qquad I_4 = 3.5\text{ A}$$

Note that the current from the battery is defined in a direction opposite to that of the other two currents (i.e., toward the node instead of away from the node). Thus, in applying KCL, we have used opposite signs for the first and the latter two currents.

At the reference node: If we use the same convention (positive value for currents entering the node and negative value for currents exiting the node), we obtain the following equations:

$$-I_S + I_3 + I_4 = 0$$

$$-5 + 1.5 + I_4 = 0$$

$$\therefore \qquad I_4 = 3.5\text{ A}$$

<u>Example 4</u>*(from Giorgio Rizzoni, "Principles and Applications of Electrical Engineering, chapter 3)*

EXAMPLE 3.3 Nodal Analysis

Problem

Write the nodal equations and solve for the node voltages in the circuit of Figure 3.7.

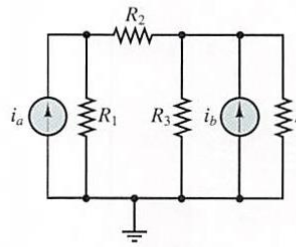


Figure 3.7

Solution

Known Quantities: Source currents, resistor values.

Find: All node voltages and branch currents.

Schematics, Diagrams, Circuits, and Given Data: $i_a = 1$ mA; $i_b = 2$ mA; $R_1 = 1$ kΩ; $R_2 = 500$ Ω; $R_3 = 2.2$ kΩ; $R_4 = 4.7$ kΩ.

Analysis: We follow the steps of the Focus on Methodology box.

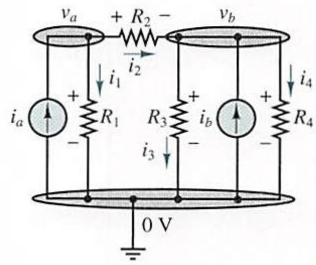


Figure 3.8

1. The reference (ground) node is chosen to be the node at the bottom of the circuit.
2. See Figure 3.8. Two nodes remain after the selection of the reference node. Let us label these a and b and define voltages v_a and v_b. Both nodes are associated with independent variables.
3. We apply KCL at each of nodes a and b:

$$i_a - \frac{v_a}{R_1} - \frac{v_a - v_b}{R_2} = 0 \qquad \text{node } a$$

$$\frac{v_a - v_b}{R_2} + i_b - \frac{v_b}{R_3} - \frac{v_b}{R_4} = 0 \qquad \text{node } b$$

and rewrite the equations to obtain a linear system:

$$\left(\frac{1}{R_1} + \frac{1}{R_2}\right) v_a + \left(-\frac{1}{R_2}\right) v_b = i_a$$

$$\left(-\frac{1}{R_2}\right) v_a + \left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) v_b = i_b$$

4. Substituting the numerical values in these equations, we get

$$3 \times 10^{-3} v_a - 2 \times 10^{-3} v_b = 1 \times 10^{-3}$$
$$-2 \times 10^{-3} v_a + 2.67 \times 10^{-3} v_b = 2 \times 10^{-3}$$

or

$$3v_a - 2v_b = 1$$
$$-2v_a + 2.67v_b = 2$$

The solution $v_a = 1.667$ V, $v_b = 2$ V may then be obtained by solving the system of equations.

Example 5 on Nodal Analysis (Ex 3.5, Rizonni)

Use the node voltage analysis to determine the voltage v in the circuit of Figure 3.9. Assume that $R_1 = 2\ \Omega$, $R_2 = 1\ \Omega$, $R_3 = 4\ \Omega$, $R_4 = 3\ \Omega$, $I_1 = 2$ A, and $I_2 = 3$ A.

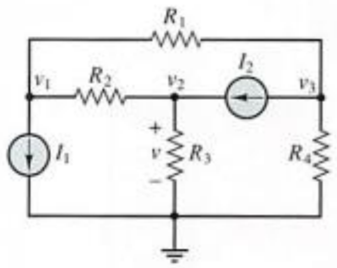


Figure 3.9 Circuit for Example 3.5

Solution

Known Quantities: Values of the resistors and the current sources.

Find: Voltage across R_3.

Analysis: Once again, we follow the steps outlined in the Focus on Methodology box.

1. The reference node is denoted in Figure 3.9.
2. Next, we define the three node voltages v_1, v_2, v_3, as shown in Figure 3.9.
3. Apply KCL at each of the $n - 1$ nodes, expressing each current in terms of the adjacent node voltages.

$$\frac{v_3 - v_1}{R_1} + \frac{v_2 - v_1}{R_2} - I_1 = 0 \qquad \text{node 1}$$

$$\frac{v_1 - v_2}{R_2} + \frac{v_2}{R_3} + I_2 = 0 \qquad \text{node 2}$$

$$\frac{v_1 - v_3}{R_1} + \frac{v_3}{R_4} - I_2 = 0 \qquad \text{node 3}$$

4. Solve the linear system of $n - 1 - m$ unknowns. Finally, we write the system of equations resulting from the application of KCL at the three nodes associated with independent variables:

$$(-1 - 2)v_1 + 2v_2 + 1v_3 = 4 \qquad \text{node 1}$$

$$4v_1 + (-1 - 4)v_2 + 0v_3 = -12 \qquad \text{node 2}$$

$$3v_1 + 0v_2 + (-2 - 3)v_3 = 18 \qquad \text{node 3}$$

The resulting system of three equations in three unknowns can now be solved. Starting with the node 2 and node 3 equations, we write

$$v_2 = \frac{4v_1 + 12}{5}$$

$$v_3 = \frac{3v_1 - 18}{5}$$

Substituting each of variables v_2 and v_3 into the node 1 equation and solving for v_1 provides

$$-3v_1 + 2 \cdot \frac{4v_1 + 12}{5} + 1 \cdot \frac{3v_1 - 18}{5} = 4 \qquad \Rightarrow \qquad v_1 = -3.5 \text{ V}$$

After substituting v_1 into the node 2 and node 3 equations, we obtain

$$v_2 = -0.4 \text{ V} \qquad \text{and} \qquad v_3 = -5.7 \text{ V}$$

Therefore, we find

$$v = v_2 = -0.4 \text{ V}$$

Example on 6 Mesh Analysis (Ex 3.7, Rizonni)

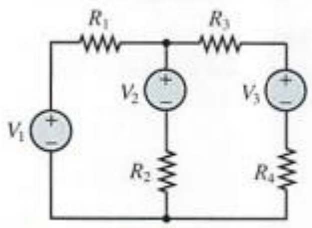


Figure 3.17

Find the mesh currents in the circuit of Figure 3.17.

VIRTUAL LAB

Solution

Known Quantities: Source voltages; resistor values.

Find: Mesh currents.

Schematics, Diagrams, Circuits, and Given Data: $V_1 = 10$ V; $V_2 = 9$ V; $V_3 = 1$ V; $R_1 = 5\ \Omega$; $R_2 = 10\ \Omega$; $R_3 = 5\ \Omega$; $R_4 = 5\ \Omega$.

Analysis: We follow the steps outlined in the Focus on Methodology box.

1. Assume clockwise mesh currents i_1 and i_2.
2. The circuit of Figure 3.17 will yield two equations in the two unknowns i_1 and i_2.
3. It is instructive to consider each mesh separately in writing the mesh equations; to this end, Figure 3.18 depicts the appropriate voltage assignments around the two meshes, based on the assumed directions of the mesh currents. From Figure 3.18, we write the mesh equations:

$$V_1 - R_1 i_1 - V_2 - R_2(i_1 - i_2) = 0$$
$$-R_2(i_2 - i_1) + V_2 - R_3 i_2 - V_3 - R_4 i_2 = 0$$

Rearranging the linear system of the equation, we obtain

$$15i_1 - 10i_2 = 1$$
$$-10i_1 + 20i_2 = 8$$

4. The equations above can be solved to obtain i_1 and i_2:

$$i_1 = 0.5 \text{ A} \qquad \text{and} \qquad i_2 = 0.65 \text{ A}$$

Comments: Note how the voltage v_2 across resistor R_2 has different polarity in Figure 3.18, depending on whether we are working in mesh 1 or mesh 2.

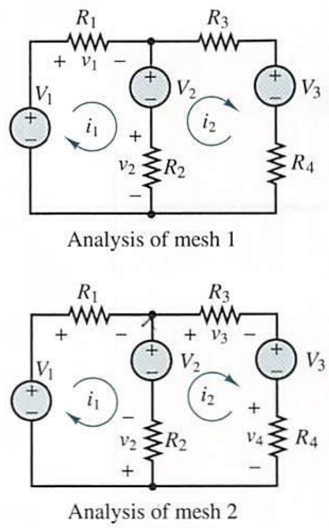


Figure 3.18

6. Glossary – English/Chinese Translation

English	Chinese
Cramer's Rule	克莱默法则
Matrix and Vector	矩阵和向量
Mesh Analysis	网格分析
Loop Analysis	环路分析
Nodal Analysis	节点分析
Determinants	因素
Simultaneous Equation	联立方程
Second Order Determinants	二阶行列式
Numerator and Denominator	分子和分母
Rows and Columns	行和列
Source Transformation	源转换
Self-Resistance	自抗性
Mutual Resistance	相互抵制
Self-Conductance	自导
Mutual Conductance	互电导