State space control.

Q1

2.1. **Find the matrix state equations in the first canonical form for the linear time-invariant differential equation**

$$\ddot{y} + 5\dot{y} + 6y = \dot{u} + u \qquad (2.41)$$

with initial conditions $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$. **Also find the initial conditions on the state variables.**

Q2

2.2. **Find the matrix state equations in the second canonical form for the equation (2.41) of Problem 2.1, and the initial conditions on the state variables.**

Q3

2.4. **Given the state equations**

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the differential equation relating the input to the output.

Q4

2.5. **Given the feedback system of Fig. 2-17 find a state space representation of this closed loop system.**

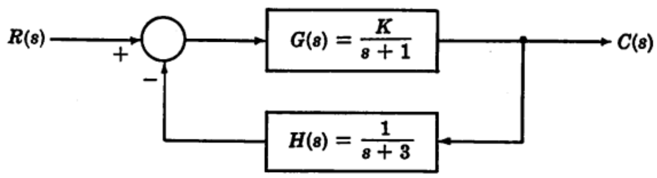

Fig. 2-17

SOLUTION

Q1

Using $p = d/dt$, equation (*2.41*) can be written as $p^2y + 5py + 6y = pu + u$. Dividing by p^2 and rearranging,

$$y = \frac{1}{p}(u - 5y) + \frac{1}{p^2}(u - 6y)$$

The flow diagram of Fig. 2-14 can be drawn starting from the output at the right.

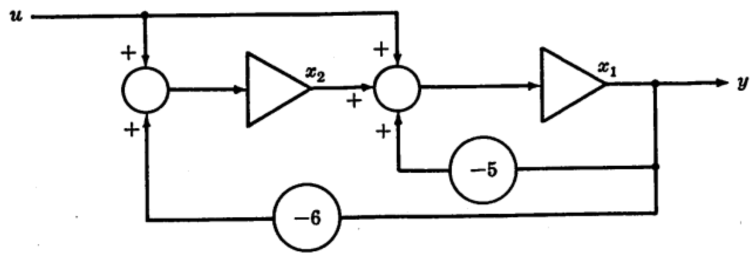

Fig. 2-14

Next, the outputs of the integrators are labeled the state variables x_1 and x_2 as shown. Now an equation can be formed using the summer on the left:

$$\dot{x}_2 = -6y + u$$

Similarly, an equation can be formed using the summer on the right:

$$\dot{x}_1 = x_2 - 5y + u$$

Also, the output equation is $y = x_1$. Substitution of this back into the previous equations gives

$$\dot{x}_1 = -5x_1 + x_2 + u$$
$$\dot{x}_2 = -6x_1 + u \qquad (2.42)$$

The state equations can then be written in matrix notation as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -5 & 1 \\ -6 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

with the output equation

$$y = (1 \ \ 0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The initial conditions on the state variables must be related to y_0 and $\dot{y}_0$, the given output initial conditions. The output equation is $x_1(t) = y(t)$, so that $x_1(0) = y(0) = y_0$. Also, substituting $y(t) = x_1(t)$ into (*2.42*) and setting $t = 0$ gives

$$\dot{y}(0) = -5y(0) + x_2(0) + u(0)$$

Use of the given initial conditions determines

$$x_2(0) = \dot{y}_0 + 5y_0 - u(0)$$

These relationships for the initial conditions can also be obtained by referring to the flow diagram at time $t = 0$.

Q2

The flow diagram (Fig. 2-14) of the previous problem is turned "backwards" to get the flow diagram of Fig. 2-15.

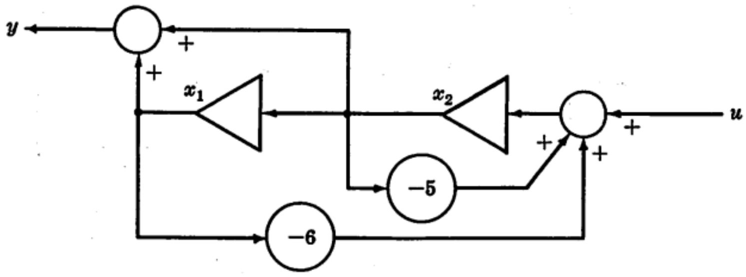

Fig. 2-15

The outputs of the integrators are labeled x_1 and x_2 as shown. These state variables are different from those in Problem 2.1, but are also denoted x_1 and x_2 to keep the state vector $\mathbf{x}(t)$ notation, as is conventional. Then looking at the summers gives the equations

$$y = x_1 + x_2 \tag{2.43}$$

$$\dot{x}_2 = -6x_1 - 5x_2 + u \tag{2.44}$$

Furthermore, the input to the left integrator is

$$\dot{x}_1 = x_2 \tag{2.45}$$

This gives the state equations

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

and

$$y = (1 \quad 1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The initial conditions are found using (2.43),

$$y_0 = x_1(0) + x_2(0) \tag{2.46}$$

and its derivative

$$\dot{y}_0 = \dot{x}_1(0) + \dot{x}_2(0)$$

Use of (2.44) and (2.45) then gives

$$\dot{y}_0 = x_2(0) - 6x_1(0) - 5x_2(0) + u(0) \tag{2.47}$$

Equations (2.46) and (2.47) can be solved for the initial conditions

$$x_1(0) = -2y_0 - \tfrac{1}{2}\dot{y}_0 + \tfrac{1}{2}u(0)$$

$$x_2(0) = 3y_0 + \tfrac{1}{2}\dot{y}_0 - \tfrac{1}{2}u(0)$$

Q3

In operator notation, the state equations are

$$px_1 = x_2$$

$$px_2 = -6x_1 - 5x_2 + u$$

$$y = x_1 + x_2$$

Eliminating x_1 and x_2 then gives

$$p^2y + 5py + 6y = pu + u$$

This is equation (2.41) of Example 2.1 and the given state equations were derived from this equation in Problem 2.2. Therefore this is a way to check the results.

Q4

The transfer function diagram is almost in flow diagram form already. Using the Jordan canonical form for the plant $G(s)$ and the feedback $H(s)$ separately gives the flow diagram of Fig. 2-18.

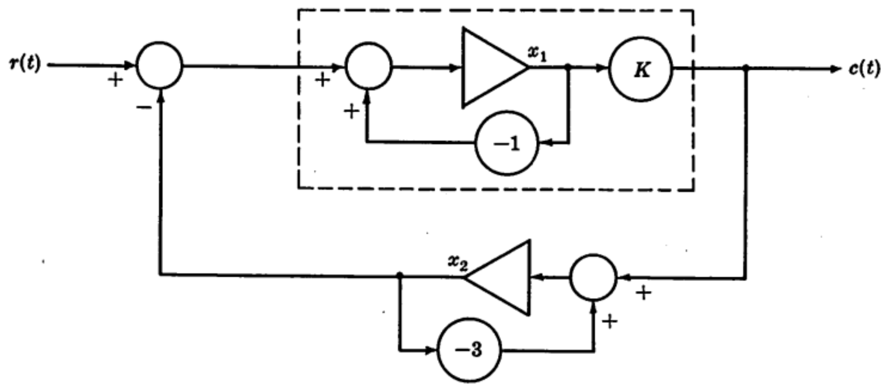

Fig. 2-18

Note $G(s)$ in Jordan form is enclosed by the dashed lines. Similarly the part for $H(s)$ was drawn, and then the transfer function diagram is used to connect the parts. From the flow diagram,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ K & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} r(t), \qquad c(t) = (K \;\; 0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$