Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 01-j

Root Locus Design

Contents

Reference:

1. Schaum's Outline Series – Feedback Control Systems

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1. The Design Problem

The root-locus method can be quite effective in the design of either continuous or discrete-time feedback control systems, because it graphically illustrates the variation of the system closed-loop poles as a function of the open-loop gain factor K. In its simplest form, design is accomplished by choosing a value of K which results in satisfactory closed-loop behavior. This is called *gain factor compensation* (also see Section 12.2). Specifications on allowable steady state errors usually take the form of a minimum value of K, expressed in terms of error constants, for example, K_p, K_v, and K_a (Chapter 9). If it is not possible to meet all system specifications using gain factor compensation alone, other forms of compensation can be added to the system to alter the root-locus as needed, for example, lag, lead, lag-lead networks, or PID controllers.

Going back, let us concentrate on 2nd order system only

In the study of control systems, linear constant-coefficient second-order differential equations of the form:

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = \omega_n^2 u \tag{3.22}$$

are important because higher-order systems can often be approximated by second-order systems. The constant ζ is called the **damping ratio**, and the constant ω_n is called the **undamped natural frequency** of the system. The forced response of this equation for inputs u belonging to the class of singularity functions is of particular interest. That is, the *forced response* to a unit impulse, unit step, or unit ramp is the same as the *unit impulse response*, *unit step response*, or *unit ramp response* of a system represented by this equation.

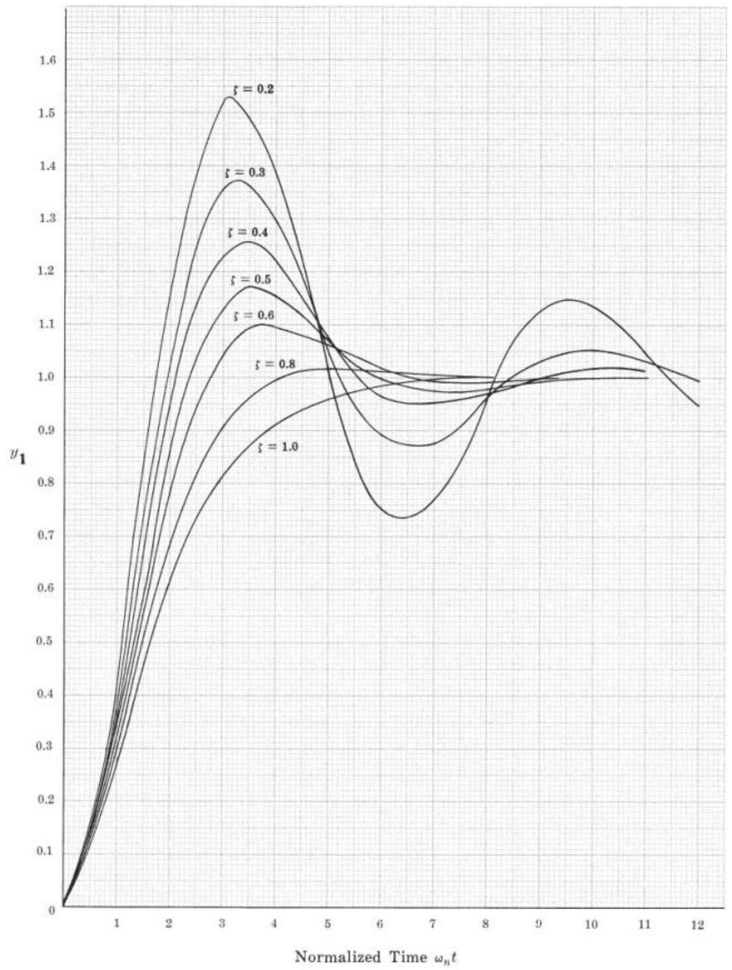


Fig. 3-4

The positive coefficient ω_n is called the **undamped natural frequency** and the coefficient ζ is the **damping ratio** of the system.

The Laplace transform of $y(t)$, when the initial conditions are zero, is

$$Y(s) = \left[\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right]U(s)$$

where $U(s) = \mathscr{L}[u(t)]$. The poles of the function $Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$ are

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Note that:

1. If $\zeta > 1$, both poles are negative and real.
2. If $\zeta = 1$, the poles are equal, negative, and real ($s = -\omega_n$).
3. If $0 < \zeta < 1$, the poles are complex conjugates with negative real parts ($s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$).
4. If $\zeta = 0$, the poles are imaginary and complex conjugate ($s = \pm j\omega_n$).
5. If $\zeta < 0$, the poles are in the right half of the s-plane (RHP).

Of particular interest in this book is Case 3, representing an **underdamped second-order system**. The poles are complex conjugates with negative real parts and are located at

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

or at

$$s = -\alpha \pm j\omega_d$$

where $1/\alpha \equiv 1/\zeta\omega_n$ is called the **time constant** of the system and $\omega_d \equiv \omega_n\sqrt{1 - \zeta^2}$ is called the **damped natural frequency** of the system. For fixed ω_n, Fig. 4-4 shows the locus of these poles as a function of ζ, $0 < \zeta < 1$. The locus is a semicircle of radius ω_n. The angle θ is related to the damping ratio by $\theta = \cos^{-1}\zeta$.

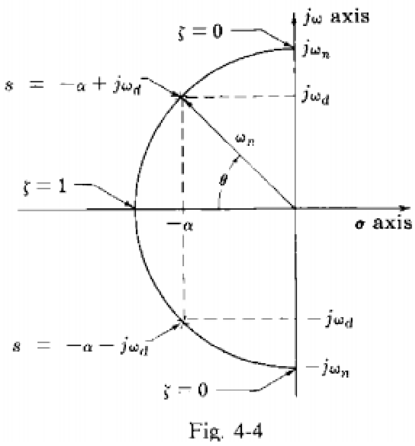


Fig. 4-4

Design Example: Gain Factor Design

EXAMPLE 14.1. Consider the design of a continuous unity feedback system with the plant $G = K/(s+1)(s+3)$ and the following specifications: (1) Overshoot less than 20%, (2) $K_p \geq 4$, (3) 10 to 90% rise time less than 1 sec.

The root-locus for this system is shown in Fig. 14-1. The system closed-loop transfer function may be written as

$$\frac{C}{R} = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

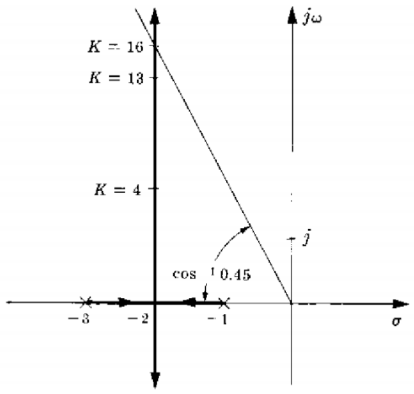


Fig. 14-1

Design target:

- ζ and ω_n can be determined from the root-locus for a given value of K.
- For this system, K_p is given by $K/3$.
- Therefore K > 12
- The rise time is a function of both ζ and ω_n.
- If K=13 is chosen, then…
- $\zeta = 0.5$, $\omega_n = 4$, and the rise time is 0.5 sec.
- Then all specifications could be met. Done!

2. The Cancelling Compensation

If the pole-zero configuration of the plant is such that the system specifications cannot be met by an adjustment of the open-loop gain factor, a more complicated cascade compensator, as shown below:

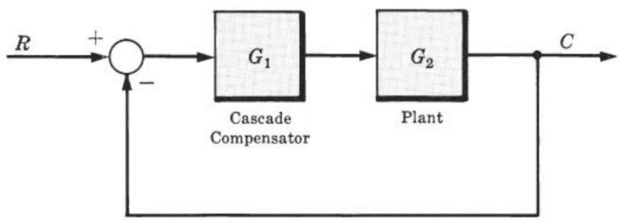


Fig. 14-2

The difficulty encountered in applying this scheme is that it is not always apparent what open-loop pole-zero configuration is desirable from the standpoint of meeting specifications on closed-loop system performance.

Some situations where cancellation compensation can be used to advantage are the following:

1. If the specifications on system rise time or bandwidth cannot be met without compensation, cancellation of low-frequency poles and replacement with high-frequency poles is helpful.
2. If the specifications on allowable steady state errors cannot be met, a low-frequency pole can be cancelled and replaced with a lower-frequency pole, yielding a larger forward-loop gain at low frequencies.
3. If poles with small damping ratios are present in the plant transfer function, they may be cancelled and replaced with poles which have larger damping ratios.

3. Phase Compensation – Lead and Lag Networks

Root–locus approach to control system design. The root-locus method is a graphical method for determining the locations of all closed-loop poles from knowledge of the locations of the open-loop poles and zeros as some parameter (usually the gain) is varied from zero to infinity. The method yields a clear indication of the effects of parameter adjustment.

In practice, the root-locus plot of a system may indicate that the desired performance cannot be achieved just by the adjustment of gain. In fact, in some cases, the system may not be stable for all values of gain. Then it is necessary to reshape the root loci to meet the performance specifications.

Effects of the addition of poles. The addition of a pole to the open-loop transfer function has the effect of pulling the root locus to the right, tending to lower the system's relative stability and to slow down the settling of the response. (Remember that the addition of integral control adds a pole at the origin, thus making the system less stable.) Figure 7–2 shows examples of root loci illustrating the effects of the addition of a pole to a single-pole system and the addition of two poles to a single-pole system.

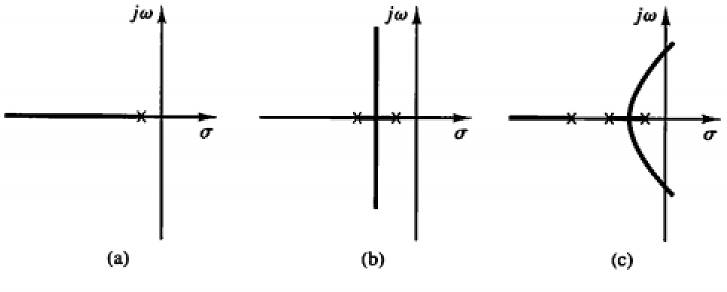


Figure 7–2
(a) Root-locus plot of a single-pole system; (b) root-locus plot of a two-pole system; (c) root-locus plot of a three-pole system.

Effects of the addition of zeros. The addition of a zero to the open-loop transfer function has the effect of pulling the root locus to the left, tending to make the system more stable and to speed up the settling of the response. (Physically, the addition of a zero in the feedforward transfer function means the addition of derivative control to the system. The effect of such control is to introduce a degree of anticipation into the system and speed up the transient response.)

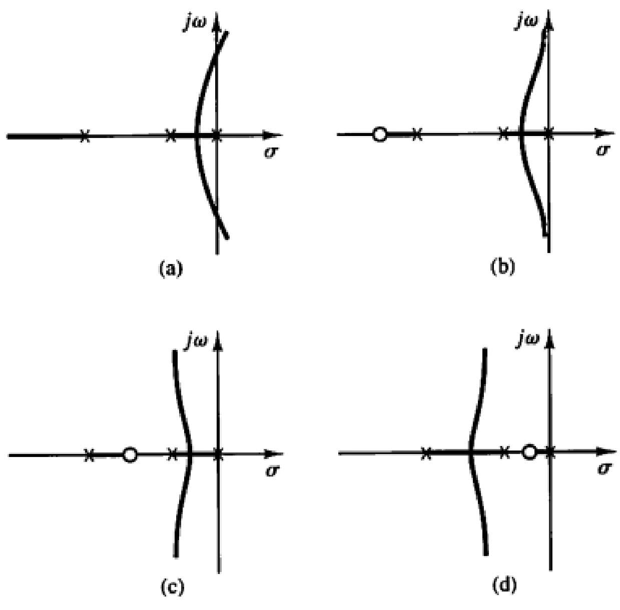


Figure 7–3
(a) Root-locus plot of a three-pole system; (b), (c), and (d) root-locus plots showing effects of addition of a zero to the three-pole system.

<u>Lead Compensation</u>

The transfer function for a continuous system lead network, presented in Equation (*6.2*), is

$$P_{\text{lead}} = \frac{s+a}{s+b}$$

where $a < b$.

The lead network provides compensation by virtue of its phase lead property in the low-to-medium-frequency range and its negligible attenuation at high frequencies. The low-to-medium-frequency range is defined as the vicinity of the resonant frequency ω_p. Several lead networks may be cascaded if a large phase lead is required.

Lead compensation generally increases the *bandwidth* of a system.

<u>Lag Compensation</u>

The transfer function for a continuous system lag network, presented in Equation (*6.3*), is

$$P_{\text{Lag}} = \frac{a}{b}\left[\frac{s+b}{s+a}\right]$$

where $a < b$.

Effects of the lag compensator:

1. The bandwidth of the system is usually decreased.
2. The dominant time constant τ of the system is usually increased, producing a more sluggish system.
3. For a given relative stability, the value of the error constant is increased.
4. For a given value of error constant, relative stability is improved.

Lag-Lead Compensation

The transfer function for a continuous system lag-lead network, presented in Equation (*6.4*), is

$$P_{LL} = \frac{(s+a_1)(s+b_2)}{(s+b_1)(s+a_2)}$$

where $a_1 b_2 / b_1 a_2 = 1$, $b_1/a_1 = b_2/a_2 > 1$, $a_i, b_i > 0$.

Lag-lead compensation has all of the advantages of both lag compensation and lead compensation, and only a minimum of their usually undesirable characteristics. Satisfaction of many system specifications is possible without the burden of excessive bandwidth and small dominant time constants.

It is not easy to generalize the method of application of Lag-Lead compensation.

Pole Zero Map

A cascade compensator can be added to a system to alter the phase characteristics of the open-loop transfer function in a manner which favorably affects system performance. These effects were illustrated in the frequency domain for lead, lag, and lag-lead networks using Polar Plots in Chapter 12, Sections 12.4 through 12.7, which summarize the general effects of these networks.

The pole-zero maps of continuous system lead and lag networks are shown in Figs. 14-3 and 14-4. Note that a lead network makes a positive, and a lag network a negative phase contribution. A lag-lead

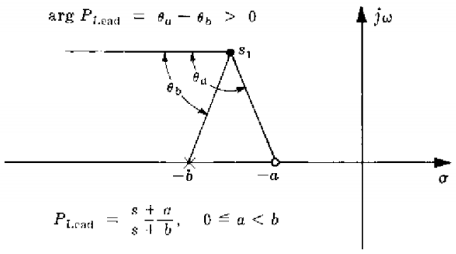


Fig. 14-3

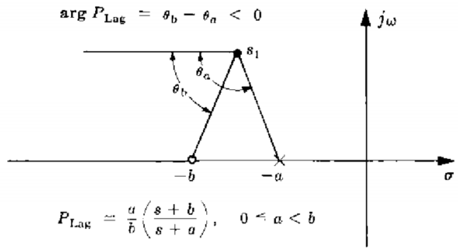


Fig. 14-4

network may be obtained by appropriately combining a lag and a lead network in series, or from the implementation described in Problem 6.14.

Since the compensated system root-locus is determined by the points in the complex plane for which the phase angle of $G = G_1 G_2$ is equal to $-180°$, the branches of the locus can be moved by proper selection of the phase angle contributed by the compensator. In general, lead compensation has the effect of moving the loci to the left.

Compensation Effect

EXAMPLE 14.2. The phase lead compensator $G_1 = (s+2)/(s+8)$ alters the root-locus of the system with the plant $G_2 = K/(s+1)^2$, as illustrated in Fig. 14-5.

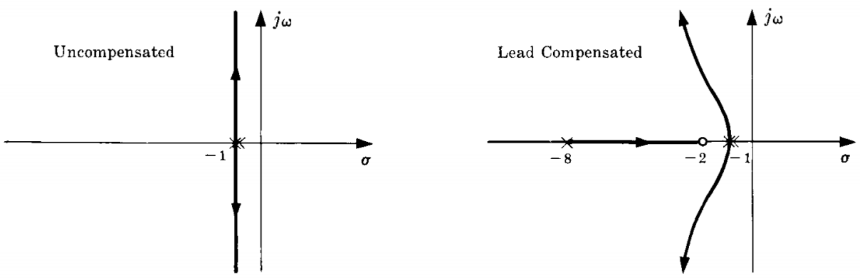


Fig. 14-5

EXAMPLE 14.3. The use of *simple lag compensation* (one pole at −1, no zero) to alter the breakaway angles of a root-locus from a pair of complex poles is illustrated in Fig. 14-6.

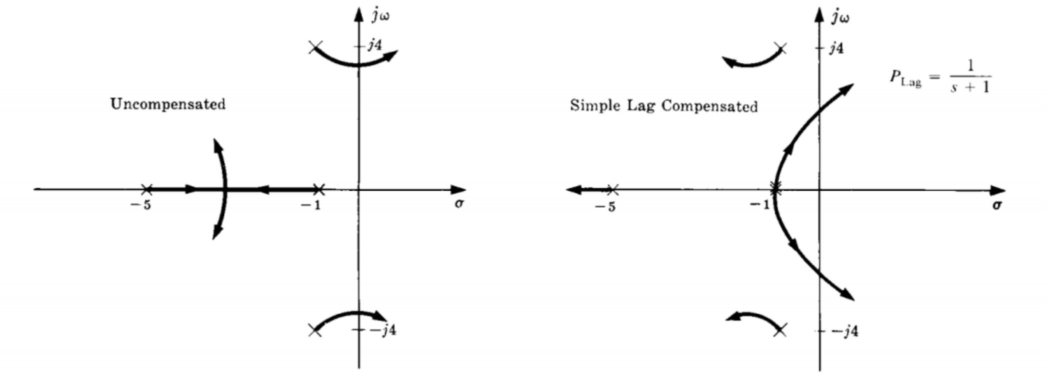
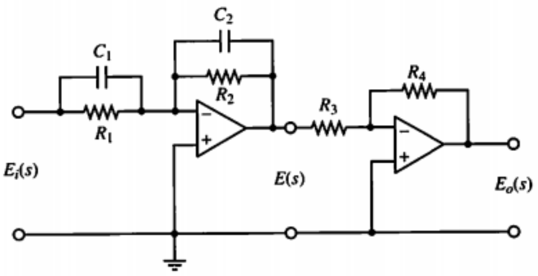


Fig. 14-6

<u>Constructing the lag and lead controller hardware</u>

Figure 7–4 shows an electronic circuit using operational amplifiers. The transfer function for this circuit was obtained in Chapter 5 as follows:

Figure 7–4
Electronic circuit that is a lead network if $R_1C_1 > R_2C_2$ and a lag network if $R_1C_1 < R_2C_2$.



$$\frac{E_o(s)}{E_i(s)} = \frac{R_2R_4}{R_1R_3}\frac{R_1C_1s+1}{R_2C_2s+1} = \frac{R_4C_1}{R_3C_2}\frac{s+\frac{1}{R_1C_1}}{s+\frac{1}{R_2C_2}}$$

$$= K_c\alpha\frac{Ts+1}{\alpha Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} \qquad (7\text{–}1)$$

where

$$T = R_1C_1, \qquad \alpha T = R_2C_2, \qquad K_c = \frac{R_4C_1}{R_3C_2}$$

Notice that

$$K_c\alpha = \frac{R_4C_1}{R_3C_2}\frac{R_2C_2}{R_1C_1} = \frac{R_2R_4}{R_1R_3}, \qquad \alpha = \frac{R_2C_2}{R_1C_1}$$

This network has a dc gain of $K_c\alpha = R_2R_4/(R_1R_3)$.

From Equation (7–1), we see that this network is a lead network if $R_1C_1 > R_2C_2$, or $\alpha < 1$. It is a lag network if $R_1C_1 < R_2C_2$. The pole–zero configurations of this network when $R_1C_1 > R_2C_2$ and $R_1C_1 < R_2C_2$ are shown in Figure 7–5(a) and (b), respectively.

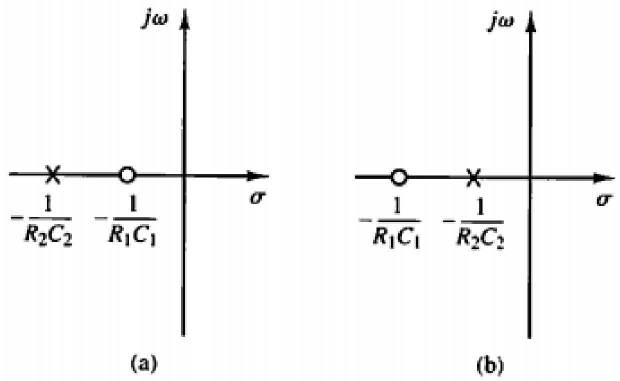


Figure 7–5
Pole–zero configurations: (a) lead network; (b) lag network.

Procedure for Root Locus Design

VERY IMPORTANT: Please study the video

1. Designing a lead compensator with root locus – Brian Douglas
2. Designing a lag compensator with root locus – Brian Douglas

Lead Compensator

1. From the performance specifications, determine the desired location for the dominant closed-loop poles.

2. By drawing the root-locus plot, ascertain whether or not the gain adjustment alone can yield the desired closed-loop poles. If not, calculate the angle deficiency ϕ. This angle must be contributed by the lead compensator if the new root locus is to pass through the desired locations for the dominant closed-loop poles.

3. Assume the lead compensator $G_c(s)$ to be

$$G_c(s) = K_c\alpha\frac{Ts+1}{\alpha Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}, \qquad (0<\alpha<1)$$

where α and T are determined from the angle deficiency. K_c is determined from the requirement of the open-loop gain.

4. If static error constants are not specified, determine the location of the pole and zero of the lead compensator so that the lead compensator will contribute the necessary angle ϕ. If no other requirements are imposed on the system, try to make the value of α as large as possible. A larger value of α generally results in a larger value of K_v, which is desirable. (If a particular static error constant is specified, it is generally simpler to use the frequency-response approach.)

5. Determine the open-loop gain of the compensated system from the magnitude condition.

Lag Compensator

Design procedures for lag compensation by the root–locus method. The procedure for designing lag compensators for the system shown in Figure 7–13 by the root-locus method may be stated as follows (we assume that the uncompensated system meets the transient-response specifications by simple gain adjustment; if this is not the case, refer to Section 7–5):

1. Draw the root-locus plot for the uncompensated system whose open-loop transfer function is $G(s)$. Based on the transient-response specifications, locate the dominant closed-loop poles on the root locus.

2. Assume the transfer function of the lag compensator to be

$$G_c(s) = \hat{K}_c\beta\frac{Ts+1}{\beta Ts+1} = \hat{K}_c\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

Then the open-loop transfer function of the compensated system becomes $G_c(s)G(s)$.

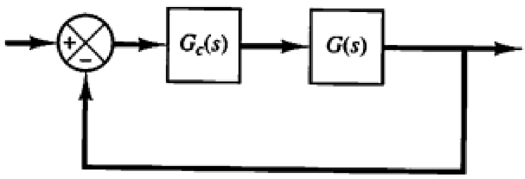


Figure 7–13
Control system.

3. Evaluate the particular static error constant specified in the problem.

4. Determine the amount of increase in the static error constant necessary to satisfy the specifications.

5. Determine the pole and zero of the lag compensator that produce the necessary increase in the particular static error constant without appreciably altering the original root loci. (Note that the ratio of the value of gain required in the specifications and the gain found in the uncompensated system is the required ratio between the distance of the zero from the origin and that of the pole from the origin.)

6. Draw a new root-locus plot for the compensated system. Locate the desired dominant closed-loop poles on the root locus. (If the angle contribution of the lag network is very small, that is, a few degrees, then the original and new root loci are almost identical. Otherwise, there will be a slight discrepancy between them. Then locate, on the new root locus, the desired dominant closed-loop poles based on the transient-response specifications.)

7. Adjust gain $\hat{K}_c$ of the compensator from the magnitude condition so that the dominant closed-loop poles lie at the desired location.

Glossary – English/Chinese Translation

English	Chinese
root locus design	根位点设计
gain factor	增益因数
open loop gain	开环增益
phase lead	相引线
phase lag	相位滞后
undamped natural frequency	无阻尼固有频率
damped natural frequency	阻尼固有频率
damping ratio	阻尼比
second order system	二阶系统
unit step response	单元阶跃响应
overshoot	过头
phase compensation	相位补偿
lag lead compensation	滞后超前补偿