Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 01-g

Root Locus Analysis

Contents

Reference:

1. Schaum's Outline Series – Feedback Control Systems

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1. Introduction to the Root Locus

Consider the canonical feedback control system given in Fig. 13-1. The closed-loop transfer function is

$$\frac{C}{R} = \frac{G}{1 + GH}$$

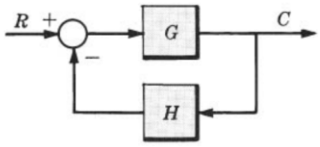


Fig. 13-1

Let the open-loop transfer function GH be represented by

$$GH \equiv \frac{KN}{D}$$

where N and D are finite polynomials in the complex variable s or z and K is the open-loop gain factor. The closed-loop transfer function then becomes

$$\frac{C}{R} = \frac{G}{1 + KN/D} = \frac{GD}{D + KN}$$

The closed-loop poles are roots of the characteristic equation

$$D + KN = 0 \qquad (13.1)$$

In general the location of these roots in the s-plane or z-plane changes as the open-loop gain factor K is varied. A locus of these roots plotted in the s-plane or z-plane as a function of K is called a **root-locus**.

for displaying the location of the poles of the closed-loop transfer function

$$\frac{G}{1 + GH}$$

as a function of the gain factor K (see Sections 6.2 and 6.6) of the open-loop transfer function GH. This method, called *root-locus analysis*, requires that only the location of the poles and zeros of GH be known, and does not require factorization of the characteristic polynomial.

EXAMPLE 13.1. Consider the continuous system open-loop transfer function

$$GH = \frac{KN(s)}{D(s)} = \frac{K(s+1)}{s^2 + 2s} = \frac{K(s+1)}{s(s+2)}$$

For $H = 1$, the closed-loop transfer function is

$$\frac{C}{R} = \frac{K(s+1)}{s^2 + 2s + K(s+1)}$$

The closed-loop poles of this system are easily determined by factoring the denominator polynomial:

$$p_1 = -\tfrac{1}{2}(2 + K) + \sqrt{1 + \tfrac{1}{4}K^2}$$

$$p_2 = -\tfrac{1}{2}(2 + K) - \sqrt{1 + \tfrac{1}{4}K^2}$$

The locus of these roots plotted as a function of K (for $K > 0$) is shown in the s-plane in Fig. 13-2. As observed in the figure, this root-locus has two *branches*: one for a closed-loop pole which moves from the open-loop pole at the origin to the open-loop zero at -1, and from the open-loop pole at -2 to the open-loop zero at $-\infty$.

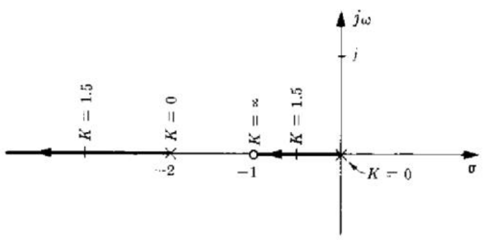


Fig. 13-2

2. Rules of the Root Locus Plot

Number of Loci

The number of loci, that is, the number of branches of the root-locus, is equal to the number of poles of the open-loop transfer function GH (for $n \geq m$).

EXAMPLE 13.2. The open-loop transfer function of the discrete-time system $GH(z) = K(z+\frac{1}{2})/z^2(z+\frac{1}{4})$ has three poles. Hence there are three loci in the root-locus plot.

Real Axis Loci

Those sections of the root-locus on the real axis in the complex plane are determined by counting the total number of finite poles and zeros of GH to the right of the points in question. The following rule depends on whether the open-loop gain factor K is positive or negative.

Rule for $K > 0$

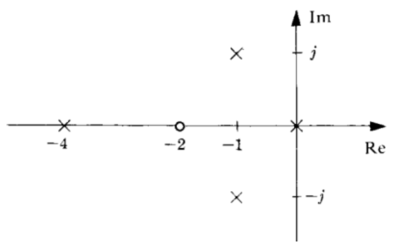
Points of the root-locus on the real axis lie to the left of an *odd* number of finite poles and zeros.

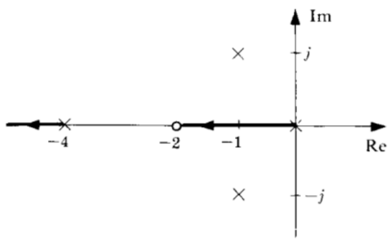
Rule for $K < 0$

Points of the root-locus on the real axis lie to the left of an *even* number of finite poles and zeros.

If no points on the real axis lie to the left of an odd number of finite poles and zeros, then no portion of the root-locus for $K > 0$ lies on the real axis. A similar statement is true for $K < 0$.

EXAMPLE 13.3. Consider the pole-zero map of an open-loop transfer function GH shown in Fig. 13-3. Since all the points on the real axis between 0 and -1 and between -1 and -2 lie to the left of an odd number of finite poles and zeros, these points are on the root-locus for $K > 0$. The portion of the real axis between $-\infty$ and -4 lies to the left of an odd number of finite poles and zeros; hence these points are also on the root-locus for $K > 0$. All portions of the root-locus for $K > 0$ on the real axis are illustrated in Fig. 13-4. All remaining portions of the real axis, that is, between -2 and -4 and between 0 and ∞, lie on the root-locus for $K < 0$.





Asymptotes

For large distances from the origin in the complex plane, the branches of a root-locus approach a set of straight-line asymptotes. These asymptotes emanate from a point in the complex plane on the real axis called the **center of asymptotes** σ_c given by

$$\sigma_c = -\frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} \tag{13.6}$$

where $-p_i$ are the poles, $-z_i$ are the zeros, n is the number of poles, and m the number of zeros of GH.

The angles between the asymptotes and the real axis are given by

$$\beta = \begin{cases} \dfrac{(2l+1)180}{n-m} \text{ degrees} & \text{for } K > 0 \\ \dfrac{(2l)180}{n-m} \text{ degrees} & \text{for } K < 0 \end{cases} \tag{13.7}$$

for $l = 0, 1, 2, \ldots, n-m-1$. This results in a number of asymptotes equal to $n-m$.

EXAMPLE 13.4. The center of asymptotes for $GH = K(s+2)/s^2(s+4)$ is located at

$$\sigma_c = -\frac{4-2}{2} = -1$$

Since $n - m = 3 - 1 = 2$, there are two asymptotes. Their angles with the real axis are 90° and 270°, for $K > 0$, as shown in Fig. 13-5.

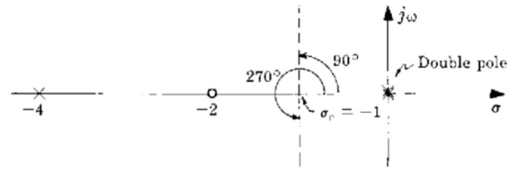


Fig. 13-5

Breakaway Points

A **breakaway point** σ_b is a point on the real axis where two or more branches of the root-locus depart from or arrive at the real axis. Two branches leaving the real axis are illustrated in the root-locus plot in Fig. 13-6. Two branches coming onto the real axis are illustrated in Fig. 13-7.

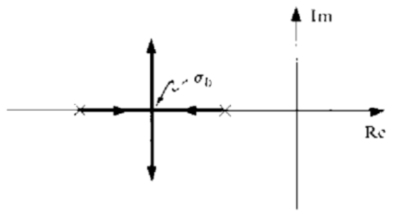


Fig. 13-6

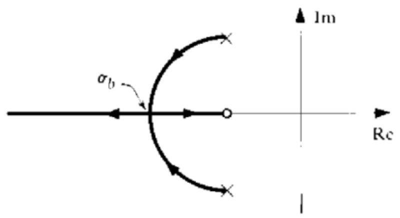


Fig. 13-7

The location of the breakaway point can be determined by solving the following equation for σ_b:

$$\sum_{i=1}^{n} \frac{1}{(\sigma_b + p_i)} = \sum_{i=1}^{m} \frac{1}{(\sigma_b + z_i)} \tag{13.8}$$

where $-p_i$ and $-z_i$ are the poles and zeros of GH, respectively. The solution of this equation requires factorization of an $(n + m - 1)$-order polynomial in σ_b. Consequently, the breakaway point can only be easily determined analytically for relatively simple GH. However, an approximate location can often be determined intuitively; then an iterative process can be used to solve the equation more exactly (see Problem 13.20). Computer programs for factorization of polynomials could also be applied.

EXAMPLE 13.5. To determine the breakaway points for $GH = K/s(s+1)(s+2)$, the following equation must be solved for σ_b:

$$\frac{1}{\sigma_b} + \frac{1}{\sigma_b + 1} + \frac{1}{\sigma_b + 2} = 0$$

$$(\sigma_b + 1)(\sigma_b + 2) + \sigma_b(\sigma_b + 2) + \sigma_b(\sigma_b + 1) = 0$$

which reduces to $3\sigma_b^2 + 6\sigma_b + 2 = 0$ whose roots are $\sigma_b = -0.423, -1.577$.

Applying the real axis rule of Section 13.5 for $K > 0$ indicates that there are branches of the root-locus between 0 and −1 and between −∞ and −2. Therefore the root at −0.423 is a breakaway point, as shown in Fig. 13-8. The value $\sigma_b = -1.577$ represents a breakaway on the root-locus for negative values of K since the portion of the real axis between −1 and −2 is on the root-locus for $K < 0$.

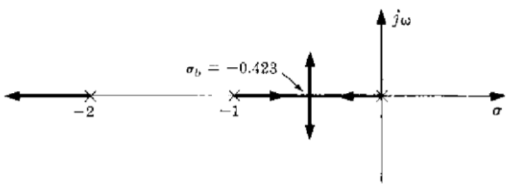


Fig. 13-8

Departure Angle

The **departure angle** of the root-locus from a *complex pole* is given by

$$\theta_D = 180° + \arg GH' \qquad (13.9)$$

where $\arg GH'$ is the phase angle of GH computed at the complex pole, but ignoring the contribution of that particular pole.

EXAMPLE 13.6. Consider the continuous system open-loop transfer function

$$GH = \frac{K(s+2)}{(s+1+j)(s+1-j)} \qquad K > 0$$

The departure angle of the root-locus from the complex pole at $s = -1 + j$ is determined as follows. The angle of GH for $s = -1 + j$, ignoring the contribution of the pole at $s = -1 + j$, is $-45°$. Therefore the departure angle is

$$\theta_D = 180° - 45° = 135°$$

and is illustrated in Fig. 13-9.

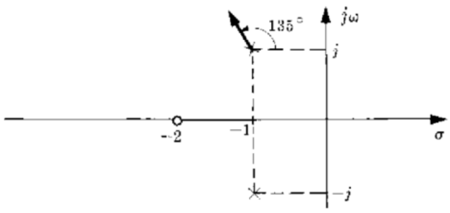


Fig. 13-9

The **angle of arrival** of the root-locus at a *complex zero* is given by

$$\theta_A = 180° - \arg GH'' \qquad (13.10)$$

where $\arg GH''$ is the phase angle of GH at the complex zero, ignoring the effect of that zero.

3. Construction of a Root Locus

EXAMPLE 13.8. The root-locus for the closed-loop continuous system with open-loop transfer function

$$GH = \frac{K}{s(s+2)(s+4)} \qquad K > 0$$

is constructed as follows. Applying the real axis rule of Section 13.5, the portions of the real axis between 0 and -2 and between -4 and $-\infty$ lie on the root-locus for $K > 0$. The center of asymptotes is determined from Equation (*13.6*) to be $\sigma_c = -(2+4)/3 = -2$, and there are three asymptotes located at angles of $\beta = 60°$, $180°$, and $300°$.

Since two branches of the root-locus for $K > 0$ come together on the real axis between 0 and -2, a breakaway point exists on that portion of the real axis. Hence the root-locus for $K > 0$ may be sketched by estimating the location of the breakaway point and continuing the branches of the root-locus to the asymptotes, as shown in Fig. 13-11. To improve the accuracy of this plot, the exact location of the breakaway point is determined from Equation (*13.8*):

$$\frac{1}{\sigma_b} + \frac{1}{\sigma_b + 2} + \frac{1}{\sigma_b + 4} = 0$$

which simplifies to $3\sigma_b^2 + 12\sigma_b + 8 = 0$. The appropriate solution of this equation is $\sigma_b = -0.845$.

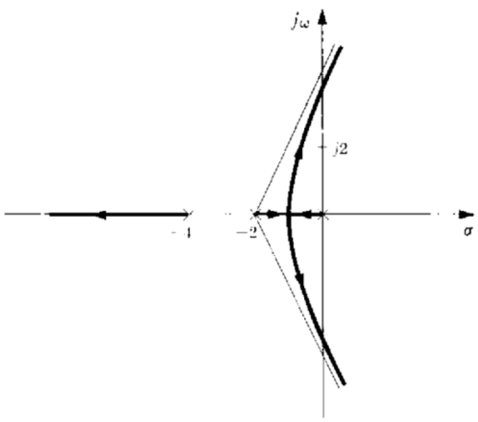


Fig. 13-11

The angle criterion is applied to points in the vicinity of the approximate root-locus to improve the accuracy of the location of the branches in the complex part of the s-plane; the magnitude criterion is used to determine the values of K along the root-locus. The resulting root-locus plot for $K > 0$ is shown in Fig. 13-12.

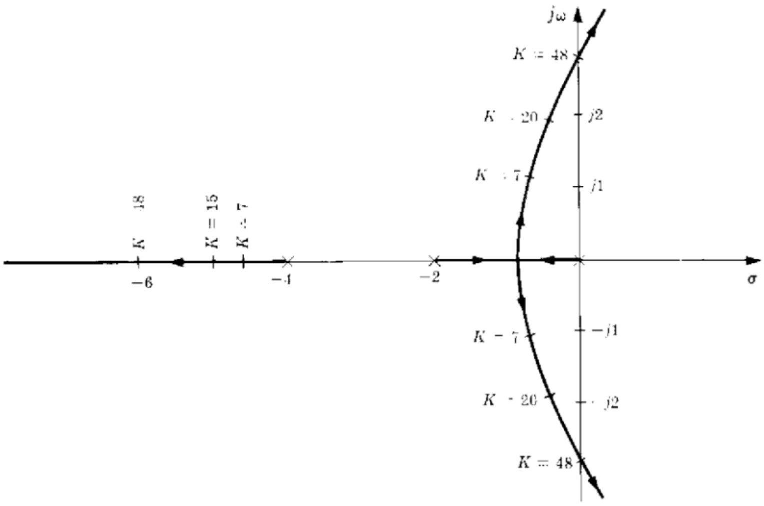


Fig. 13-12

The root-locus for $K<0$ is constructed in a similar manner. In this case, however, the portions of the real axis between 0 and ∞ and between -2 and -4 lie on the root-locus; the breakaway point is located at -3.155; and the asymptotes have angles of 0°, 120°, and 240°. The root-locus for $K<0$ is shown in Fig. 13-13.

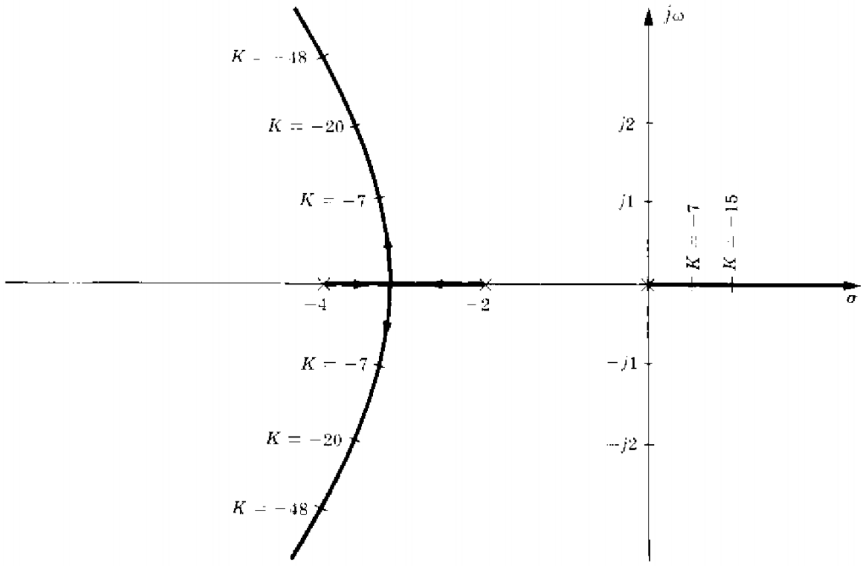


Fig. 13-13

4. Root Locus Plot Analysis

Closed Loop Transfer Function

Consider the closed-loop transfer function C/R for the canonical *unity* (*negative*) *feedback* system

$$\frac{C}{R}=\frac{G}{1+G} \tag{13.11}$$

Open-loop transfer functions which are rational algebraic expressions can be written (for continuous systems) as

$$G=\frac{KN}{D}=\frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{13.12}$$

G has the same form for discrete-time systems, with z replacing s in Equation (*13.12*). In Equation (*13.12*), $-z_i$ are the zeros, $-p_i$ are the poles of G, $m\le n$, and N and D are polynomials whose roots are $-z_i$ and $-p_i$, respectively. Then

$$\frac{C}{R}=\frac{KN}{D+KN} \tag{13.13}$$

and it is clear that C/R and G have the same zeros but not the same poles (unless $K=0$). Hence

$$\frac{C}{R}=\frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+\alpha_1)(s+\alpha_2)\cdots(s+\alpha_n)}$$

where $-\alpha_i$ denote the n closed-loop poles. The location of these poles is by definition determined directly from the root-locus plot for a specified value of open-loop gain K.

EXAMPLE 13.9. Consider the continuous system whose open-loop transfer function is

$$G=\frac{K(s+2)}{(s+1)^2} \qquad K>0$$

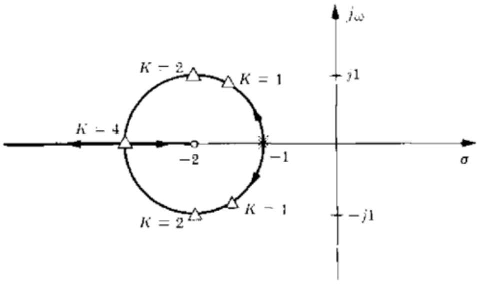


Fig. 13-14

Several values of gain factor K are shown at points on the loci denoted by *small triangles*. These points are the *closed-loop poles* corresponding to the specified values of K. For $K = 2$, the closed-loop poles are $-\alpha_1 = -2 + j$ and $-\alpha_2 = -2 - j$. Therefore

$$\frac{C}{R} = \frac{2(s+2)}{(s+2+j)(s+2-j)}$$

When the system is not unity feedback, then

$$\frac{C}{R} = \frac{G}{1+GH} \tag{13.14}$$

and

$$GH = \frac{KN}{D} \tag{13.15}$$

EXAMPLE 13.10. Consider the continuous system described by

$$G = \frac{K(s+2)}{s+1} \qquad H = \frac{1}{s+1} \qquad GH = \frac{K(s+2)}{(s+1)^2} \qquad K > 0$$

and

$$\frac{C}{R} = \frac{K(s+1)(s+2)}{(s+1)^2 + K(s+2)} = \frac{K(s+1)(s+2)}{(s+\alpha_1)(s+\alpha_2)}$$

The root-locus plot for this example is the same as that for Example 13.9. Hence for $K = 2$, $\alpha_1 = 2 + j$ and $\alpha_2 = 2 - j$. Thus

$$\frac{C}{R} = \frac{2(s+1)(s+2)}{(s+2+j)(s+2-j)}$$

Gain Margin

The **gain margin** is the factor by which the gain factor K can be multiplied before the closed-loop system becomes unstable. It can be determined from the root-locus using the following formula:

$$\text{gain margin} = \frac{\text{value of } K \text{ at the stability boundary}}{\text{design value of } K} \tag{13.16}$$

where the stability boundary is the $j\omega$-axis in the s-plane, or the unit circle in the z-plane. If the root-locus does not cross the stability boundary, the gain margin is infinite.

EXAMPLE 13.12. Consider the continuous system in Fig. 13-16. The design value for the gain factor is 8, producing the closed-loop poles (denoted by small triangles) shown in the root-locus of Fig. 13-17. The gain factor at the $j\omega$-axis crossing is 64; hence the gain margin for this system is $64/8 = 8$.

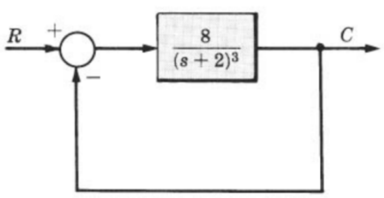


Fig. 13-16

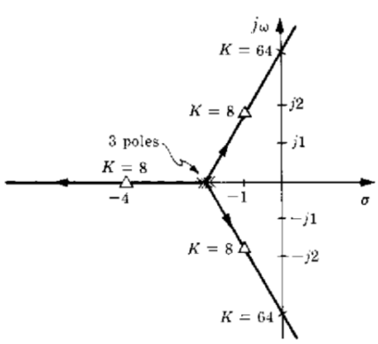


Fig. 13-17

Finding the Damping Ratio

13.12 DAMPING RATIO FROM THE ROOT-LOCUS FOR CONTINUOUS SYSTEMS

The gain factor K required to give a specified damping ratio ζ (or vice versa) for the second-order continuous system

$$GH = \frac{K}{(s+p_1)(s+p_2)} \qquad K, p_1, p_2 > 0$$

is easily determined from the root-locus. Simply draw a line from the origin at an angle of plus or minus θ with the negative real axis, where

$$\theta = \cos^{-1}\zeta \tag{13.18}$$

(See Section 4.13.) The gain factor at the point of intersection with the root-locus is the required value of K. This procedure can be applied to any pair of complex conjugate poles, for systems of second or higher order. For higher-order systems, the damping ratio determined by this procedure for a *specific pair* of complex poles does not necessarily determine the damping (predominant time constant) of the system.

EXAMPLE 13.16. Consider the third-order system of Example 13.15. The damping ratio ζ of the *complex poles* for $K = 24$ is easily determined by drawing a line from the origin to the point on the root-locus where $K = 24$, as shown in Fig. 13-20. The angle θ is measured as 60°; hence

$$\zeta = \cos\theta = 0.5$$

This value of ζ is a good approximation for the damping of the third-order system with $K = 24$ because the complex poles dominate the response.

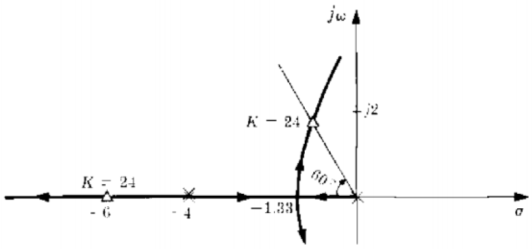


Fig. 13-20

Glossary – English/Chinese Translation

English	Chinese
root locus	根位点
numerator and denominator	分子和分母
open loop transfer function	开环传递函数
characteristic equation	特征方程
polynomial	多项式
loci	基因
real and imaginary axis	实轴和虚轴
asymptote	渐近线
break away point	突破点
poles and zeros	极点和零点
gain margin	获得利润
damping ratio	阻尼比