# Dr. Norbert Cheung's Lecture Series

## Level 1 Topic no: 01-f

# Routh Hurwitz and Stability

**Contents**



**Reference:**

1. "Control Engineering 2nd edition", W Bolton
2. Schaum's Outline Series – Feedback Control Systems

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## 1. Poles and Zeros

The closed-loop transfer function $G(s)$ of a system can, in general, be represented by

$$G(s) = \frac{K(s^m + a_{m-1}s^{m-1} + a_{m-2}s^{m-2} + \ldots a_1 s + a_0)}{(s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots b_1 s + b_0)} \quad [1]$$

and, if the roots of the denominator and numerator are established, as

$$G(s) = \frac{K(s + z_1)(s + z_2) \ldots (s + z_m)}{(s + p_1)(s + p_2) \ldots (s + p_n)} \quad [2]$$

where the roots of the numerator are $z_1, z_2, \ldots z_m$ and are called *zeros* and the roots of the denominator are $p_1, p_2, \ldots p_n$ and are called *poles*, $K$ is a constant multiplier or gain of the system.

**Example 3**

What are the transfer functions of the systems having the following poles and zeros?
(*a*) Poles −1, −2; no zero.
(*b*) Poles +1, −2; zero 0.
(*c*) Poles (−2 ± j1), zero + 1.

*Answer*

(*a*) The denominator will be $(s + 1)(s + 2)$ and the numerator 1 (in the absence of any information about the gain $K$). Hence

$$G(s) = \frac{1}{(s + 1)(s + 2)}$$

(*b*) The denominator will be $(s - 1)(s + 2)$ and the numerator $(s - 0)$. Hence

$$G(s) = \frac{s}{(s - 1)(s + 2)}$$

(*c*) The denominator will be

$$\begin{aligned}&[s - (-2 + j1)][s - (-2 - j1)] \\ &\quad = [s^2 - (-2 + j1)s - (-2 - j1)s + (-2 + j1)(-2 - j1)] \\ &\quad = [s^2 + 4s + 5]\end{aligned}$$

The numerator is $(s - 1)$, and so

$$G(s) = \frac{s - 1}{s^2 + 4s + 5}$$

The two-dimensional plot is known as the *s-plane*. Poles or zeros in the left-hand side of the plot are all negative, poles or zeros in the right-hand side positive. Poles or zeros are either real or occur in pairs as (σ ± jω).

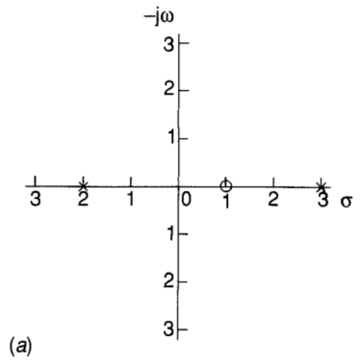
**Example 4**

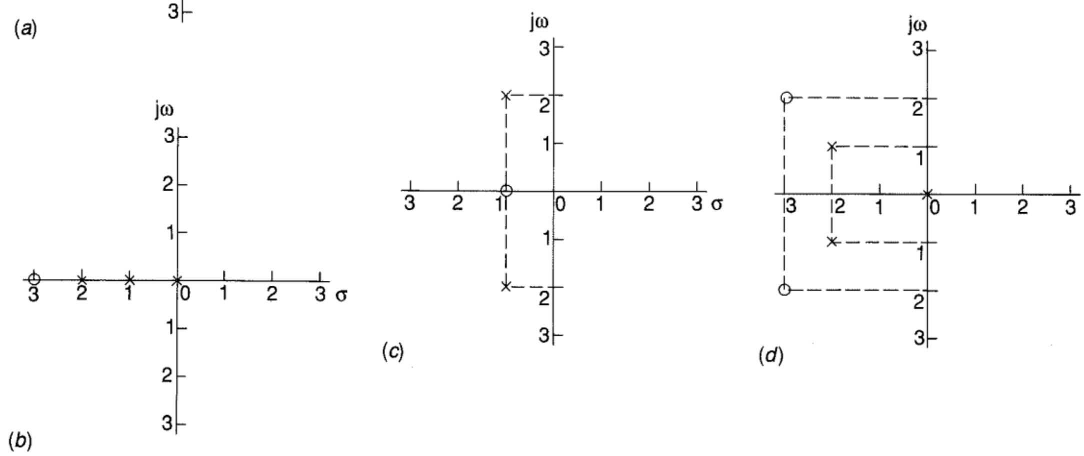
Sketch the pole-zero plots for systems having the following poles and zeros.

(*a*) Poles −2, + 3; zero +1.
(*b*) Poles 0, −1, −2; zero −3.
(*c*) Poles −1 ± j2; zero −1.
(*d*) Poles −2 ± j1, 0; zero −3 ± j2.

*Answer*

See Fig. 8.3.





In general, when an impulse is applied to a system the output is in the form of the summation of a number of exponential terms. If just one of the exponential terms is of an exponential growth, i.e. the exponential of a positive function of $t$ such as $e^{2t}$, then the output continually grows with time and the system is unstable. This situation will arise if any one of the poles has a real part which is positive and so the denominator of the transfer function includes a term $(s - a)$. When there are pole pairs involving ±jω then the output is always an oscillation. Such an oscillation is stable if the real part of the pole pair is negative and unstable if positive.

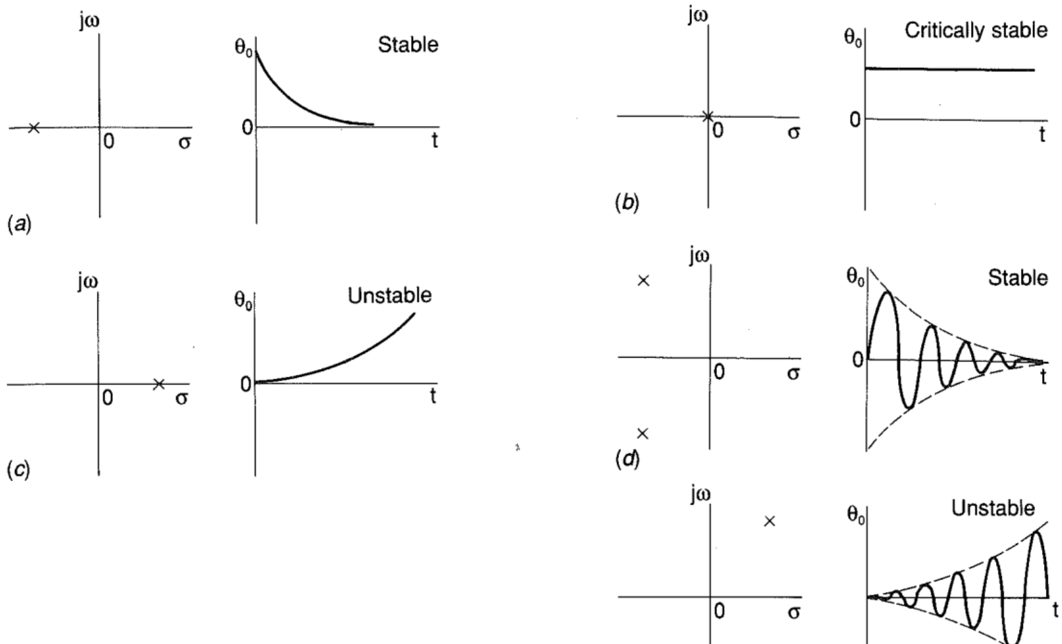



**Fig. 8.5** Outputs for different pole positions with an impulse input

## 2. The Routh Stability Criterion

The Routh criterion is a method for determining continuous system stability, for systems with an $n$th-order characteristic equation of the form:

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

The criterion is applied using a **Routh table** defined as follows:

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ \cdot & b_1 & b_2 & b_3 & \cdots \\ \cdot & c_1 & c_2 & c_3 & \cdots \\ \cdot & \cdots & \cdots & \cdots & \cdots \end{array}$$

where $a_n, a_{n-1}, \ldots, a_0$ are the coefficients of the characteristic equation and

$$b_1 \equiv \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \qquad b_2 \equiv \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \qquad \text{etc.}$$

$$c_1 \equiv \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1} \qquad c_2 \equiv \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1} \qquad \text{etc.}$$

The table is continued horizontally and vertically until only zeros are obtained. Any row can be multiplied by a positive constant before the next row is computed without disturbing the properties of the table.

**The Routh Criterion:** *All the roots of the characteristic equation have negative real parts if and only if the elements of the first column of the Routh table have the same sign. Otherwise, the number of roots with positive real parts is equal to the number of changes of sign.*

The determination of the stability of a system given its transfer function involves the determination of the roots of the denominator of the function and a consideration of whether any of them are positive. However, the roots may well not be easily obtained if the denominator is in the form

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots a_1 s + a_0 \qquad [4]$$

and $n$ is more than 3 or 4. The *Routh–Hurwitz criterion*, however, represents a method which can be used in such situations.

The first test that is applied is to inspect the coefficients, i.e. the values of the terms in the above expression. If they are all positive and none are zero then the system can be stable. If any coefficient is negative then the system is unstable. If any coefficient is zero then, at the best, it is critically stable.

Thus, for example, the denominator $(s^3 + 2s^2 + 3s + 1)$ can be stable since all coefficients are present and all are positive. However, $(s^3 - 2s^2 + 3s + 1)$ is unstable since there is a negative coefficient. With $(s^3 + 2s^2 + 3s)$ there is a term missing and so at the best the system is critically stable.

For systems which have denominators which could be stable then a second test is carried out. The coefficients of equation [4] are written in a particular order called the *Routh array*. This is

| | | | | |
|---|---|---|---|---|
| $s^n$ | $a_n$ | $a_{n-2}$ | $a_{n-4}$ | $\ldots$ |
| $s^{n-1}$ | $a_{n-1}$ | $a_{n-3}$ | $a_{n-5}$ | $\ldots$ |

Further rows in the array are determined by calculation from elements in the two rows immediately above. Successive rows are calculated until only zeros appear. The array should then contain $(n + 1)$ rows, a row corresponding to each of the terms $s^n$ to $s^0$.

| | | | | |
|---|---|---|---|---|
| $s^n$ | $a_n$ | $a_{n-2}$ | $a_{n-4}$ | $\cdots$ |
| $s^{n-1}$ | $a_{n-1}$ | $a_{n-3}$ | $a_{n-5}$ | $\cdots$ |
| $s^{n-2}$ | $b_1$ | $b_2$ | $b_3$ | $\cdots$ |
| $s^{n-3}$ | $c_1$ | $c_2$ | $c_3$ | $\cdots$ |
| . | . | . | . | . . . |
| . | . | . | . | . . |
| . | . | . | . | . |
| $s^2$ | $x_1$ | $x_2$ | $x_3$ | |
| $s^1$ | $y_1$ | $y_2$ | | |
| $s^0$ | $z_1$ | | | |

Elements in the third row are obtained from elements in the previous two rows by

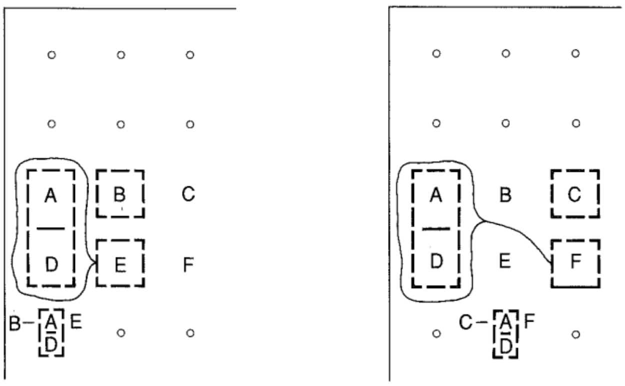
$$b_1 = a_{n-2} - \left(\frac{a_n}{a_{n-1}}\right) a_{n-3} \qquad [5]$$

$$b_2 = a_{n-4} - \left(\frac{a_n}{a_{n-1}}\right) a_{n-5} \qquad [6]$$

Elements in the fourth row are obtained from elements in the previous two rows by

$$c_1 = a_{n-3} - \left(\frac{a_{n-1}}{b_1}\right) b_2 \qquad [7]$$

$$c_2 = a_{n-5} - \left(\frac{a_{n-1}}{b_1}\right) b_3 \qquad [8]$$




Determining the elements in the Routh Array

When the array has been completed it is inspected. If all the elements in the first column of the array are positive the roots all have negative real parts, and so lie in the left-hand side of the pole-zero plot. The system is thus stable if all the first-column elements are positive. If there are any negative elements in the first column the number of sign changes in the first column is equal to the number of roots with positive real parts.

**Example 6**

The following are the denominators of the transfer functions of a number of systems. By inspection, which of them could be stable, which unstable and which critically stable?
(*a*) $s^4 + 3s^3 + 2s + 3$.
(*b*) $s^3 + 2s^2 + 3s + 1$.
(*c*) $s^5 - 4s^4 + 3s^3 + 2s^2 + 5s + 2$.
(*d*) $s^5 + s^4 + 5s^3 + 2s^2 + 3s + 2$.
(*e*) $s^5 + 2s^3 + 3s^2 + 4s + 5$.

*Answer*

(*b*) and (*d*) can be stable since all coefficients are positive and none are zero. (*c*) is unstable since there is a negative term. (*a*) and (*e*) are at best critically stable.

**Example 7**

Use the Routh array to determine whether the system having the following transfer function is stable.

$$G(s) = \frac{2s + 1}{s^4 + 2s^3 + 3s^2 + 4s + 1}$$

*Answer*

The denominator is $(s^4 + 2s^3 + 3s^2 + 4s + 1)$ and inspection reveals that all coefficients are positive and none are missing. It could therefore be stable. In order to be sure that it is stable the Routh array has then to be used. The first two rows of the array are

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 3 | 1 |
| $s^3$ | 2 | 4 | |

Elements in the third row of the array are calculated using equations [5] and [6].

$$b_1 = a_{n-2} - \left(\frac{a_n}{a_{n-1}}\right) a_{n-3}$$

$$b_1 = 3 - \left(\frac{1}{2}\right) 4 = 1$$

and

$$b_2 = a_{n-4} - \left(\frac{a_n}{a_{n-1}}\right) a_{n-5}$$

$$b_2 = 1 - \left(\frac{1}{2}\right) 0 = 1$$

Thus the array becomes

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 3 | 1 |
| $s^3$ | 2 | 4 | |
| $s^2$ | 1 | 1 | |

Elements in the fourth row of the array are calculated using equations [7] and [8].

$$c_1 = a_{n-3} - \left(\frac{a_{n-1}}{b_1}\right) b_2$$

$$c_1 = 4 - \left(\frac{2}{1}\right) 1 = 2$$

and

$$c_2 = a_{n-5} - \left(\frac{a_{n-1}}{b_1}\right) b_3$$

$$c_2 = 0 - \left(\frac{2}{1}\right) 0 = 0$$

Thus the array becomes

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 3 | 1 |
| $s^3$ | 2 | 4 | |
| $s^2$ | 1 | 1 | |
| $s^1$ | 2 | | |

The fourth row element can be calculated using

$$d_1 = b_2 - \left(\frac{b_1}{c_1}\right) c_2$$

$$d_1 = 1 - \left(\frac{1}{2}\right) 0 = 1$$

Thus the array becomes

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 3 | 1 |
| $s^3$ | 2 | 4 | |
| $s^2$ | 1 | 1 | |
| $s^1$ | 2 | | |
| $s^0$ | 1 | | |

The first column has all positive elements and so the system is stable.

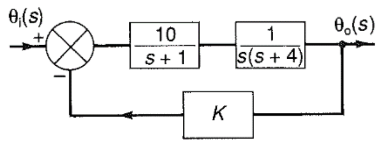



**Fig. 8.8** Example 10

**Example 10**

For the system shown in Fig. 8.8, what range of $K$ will result in stability?

*Answer*

The overall system transfer function is given by

$$\frac{G(s)}{1 + G(s)H(s)}$$

and so, since the forward path transfer function is $10/[s(s+1)(s+4)]$, is

$$\frac{10/[s(s+1)(s+4)]}{1 + 10K/[s(s+1)(s+4)]} = \frac{10}{s^3 + 5s + 4s + 10K}$$

Hence the Routh array for the denominator is

| | | |
|---|---|---|
| $s^3$ | 1 | 4 |
| $s^2$ | 5 | $10K$ |
| $s^1$ | $4 - 2K$ | |
| $s^0$ | $10K$ | |

For the first column to only have positive values we must have

$$4 - 2K > 0$$

and

$$10K > 0$$

This means that $K$ must lie between 0 and 2.

## 2. The Hurwitz Stability Criterion

The Hurwitz criterion is another method for determining whether all the roots of the characteristic equation of a continuous system have negative real parts. This criterion is applied using determinants formed from the coefficients of the characteristic equation. It is assumed that the first coefficient, $a_n$, is positive. The determinants $\Delta_i$, $i = 1, 2, \ldots, n-1$, are formed as the principal minor determinants of the determinant

$$\Delta_n = \begin{vmatrix} a_{n-1} & a_{n-3} & \cdots & \begin{bmatrix} a_0 & \text{if } n \text{ odd} \\ a_1 & \text{if } n\text{even} \end{bmatrix} & 0 & \cdots & 0 \\ a_n & a_{n-2} & \cdots & \begin{bmatrix} a_1 & \text{if } n \text{ odd} \\ a_0 & \text{if } n \text{ even} \end{bmatrix} & 0 & \cdots & 0 \\ 0 & a_{n-1} & a_{n-3} & \cdots & \cdots & \cdots & 0 \\ 0 & a_n & a_{n-2} & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & a_0 \end{vmatrix}$$

The determinants are thus formed as follows:

$$\Delta_1 = a_{n-1}$$

$$\Delta_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} = a_{n-1}a_{n-2} - a_n a_{n-3}$$

$$\Delta_3 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix} = a_{n-1}a_{n-2}a_{n-3} + a_n a_{n-1} a_{n-5} - a_n a_{n-3}^2 - a_{n-4} a_{n-1}^2$$

and so on up to $\Delta_{n-1}$.

**Hurwitz Criterion:** *All the roots of the characteristic equation have negative real parts if and only if* $\Delta_i > 0$, $i = 1, 2, \ldots, n$.

**EXAMPLE 5.5.** For $n = 3$,

$$\Delta_3 = \begin{vmatrix} a_2 & a_0 & 0 \\ a_3 & a_1 & 0 \\ 0 & a_2 & a_0 \end{vmatrix} = a_2 a_1 a_0 - a_0^2 a_3, \qquad \Delta_2 = \begin{vmatrix} a_2 & a_0 \\ a_3 & a_1 \end{vmatrix} = a_2 a_1 - a_0 a_3, \qquad \Delta_1 = a_2$$

Thus all the roots of the characteristic equation have negative real parts if

$$a_2 > 0 \qquad a_2 a_1 - a_0 a_3 > 0 \qquad a_2 a_1 a_0 - a_0^2 a_3 > 0$$

**5.13.** Determine if the characteristic equation below represents a stable or an unstable system.

$$s^3 + 8s^2 + 14s + 24 = 0$$

The Hurwitz determinants for this system are

$$\Delta_3 = \begin{vmatrix} 8 & 24 & 0 \\ 1 & 14 & 0 \\ 0 & 8 & 24 \end{vmatrix} = 2112 \qquad \Delta_2 = \begin{vmatrix} 8 & 24 \\ 1 & 14 \end{vmatrix} = 88 \qquad \Delta_1 = 8$$

Since each determinant is positive, the system is stable. Note that the general formulation of Example 5.5 could have been used to check the stability in this case by substituting the appropriate values for the coefficients $a_0$, $a_1$, $a_2$, and $a_3$.

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## Glossary – English/Chinese Translation

| English | Chinese |
|---|---|
| Routh Stability Criterion | Routh 稳定性准则 |
| Hurwitz Stability Criterion | Hurwitz 稳定性准则 |
| Poles and Zeros | 极点和零点 |
| denominator and numerator | 分母和分子 |
| s plane | S 平面 |
| Matrix Determinant | 矩阵行列式 |

**Your Notes:**