Tutorial on Mathematical Modeling

A–2–4. Figure 2–37(a) shows a schematic diagram of an automobile suspension system. As the car moves along the road, the vertical displacements at the tires act as the motion excitation to the automobile suspension system. The motion of this system consists of a translational motion of the center of mass and a rotational motion about the center of mass. Mathematical modeling of the complete system is quite complicated.

A very simplified version of the suspension system is shown in Fig. 2–27(b). Assuming that the motion x_i at point P is the input to the system and the vertical motion x_o of the body is the output,

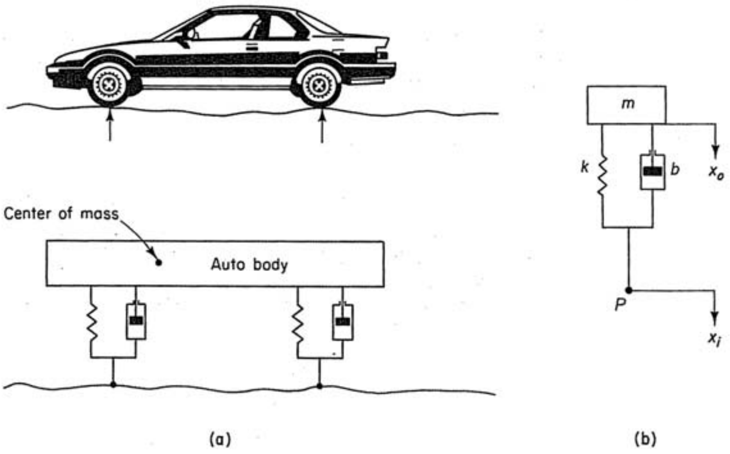
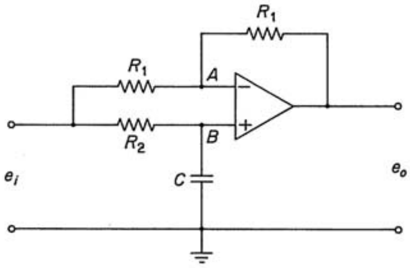


Figure 2–37
(a) Automobile suspension system; (b) simplified suspension system.

obtain the transfer function $X_o(s)/X_i(s)$. (Consider the motion of the body only in the vertical direction.) Displacement x_o is measured from the equilibrium position in the absence of input x_i.

Q2 Obtain the transfer function of the circuit



Q3

A–2–11. Obtain the closed-loop transfer function for the positional servo system shown in Figure 2–45. Assume that the input and output of the system are the input shaft position and the output shaft position, respectively. Assume the following numerical values for system constants:

r = angular displacement of the reference input shaft, radians
c = angular displacement of the output shaft, radians
θ = angular displacement of the motor shaft, radians
K_1 = gain of the potentiometric error detector = $24/\pi$ volts/rad
K_p = amplifier gain = 10 volts/volt
e_a = applied armature voltage, volts
e_b = back emf, volts
R_a = armature-winding resistance = 0.2 ohms
L_a = armature-winding inductance = negligible
i_a = armature-winding current, amperes
K_b = back emf constant = 5.5×10^{-2} volts-sec/rad
K = motor torque constant = 6×10^{-5} N-m/ampere
J_m = moment of inertia of the motor = 1×10^{-5} kg-m^2
b_m = viscous-friction coefficient of the motor = negligible
J_L = moment of inertia of the load = 4.4×10^{-3} kg-m^2
b_L = viscous-friction coefficient of the load = 4×10^{-2} N-m/rad/sec
n = gear ratio N_1/N_2 = 1/10

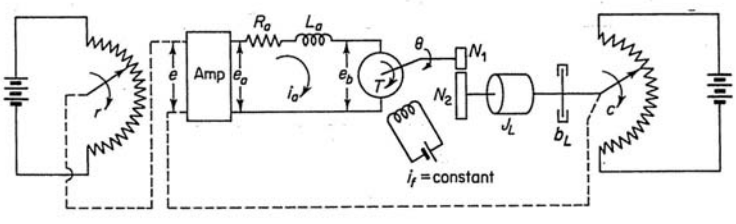
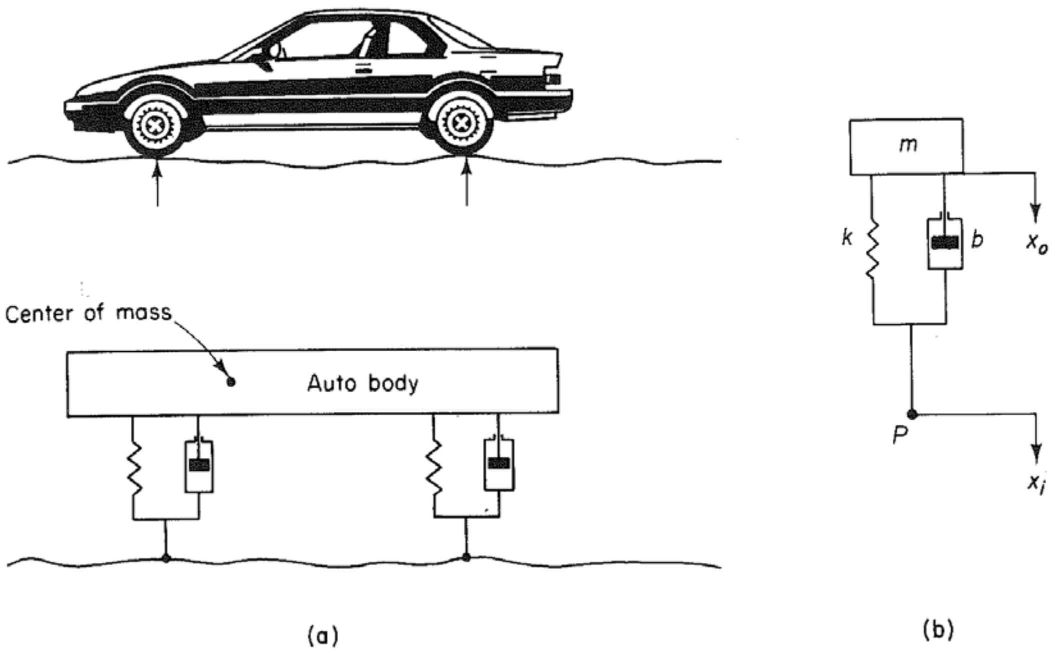


Figure 2–45
Positional servo system.

Answer Q1



Solution. The equation of motion for the system shown in Fig. 2–37(b) is

$$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0$$

or

$$m\ddot{x}_o + b\dot{x}_o + kx_o = b\dot{x}_i + kx_i$$

Taking the Laplace transform of this last equation, assuming zero initial conditions, we obtain

$$(ms^2 + bs + k)X_o(s) = (bs + k)X_i(s)$$

Hence the transfer function $X_o(s)/X_i(s)$ is given by

$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Answer Q2

Solution. The voltage at point A is

$$e_A = \frac{1}{2}(e_i - e_o) + e_o$$

The Laplace transformed version of this last equation is

$$E_A(s) = \frac{1}{2}\left[E_i(s) + E_o(s)\right]$$

The voltage at point B is

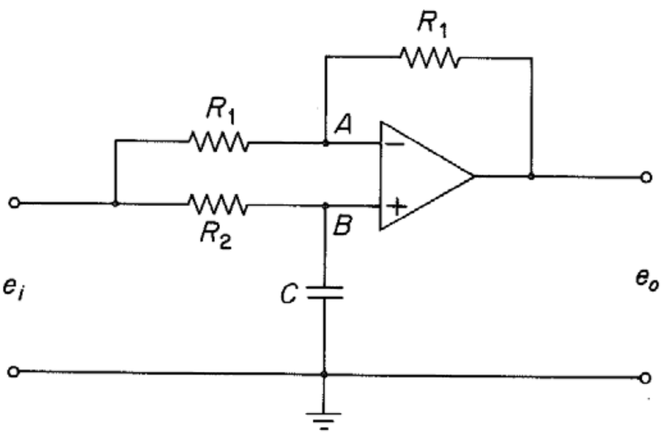
$$E_B(s) = \frac{\frac{1}{Cs}}{R_2 + \frac{1}{Cs}} E_i(s) = \frac{1}{R_2Cs + 1} E_i(s)$$

Since $[E_B(s) - E_A(s)]K = E_o(s)$ and $K \gg 1$, we must have $E_A(s) = E_B(s)$. Thus

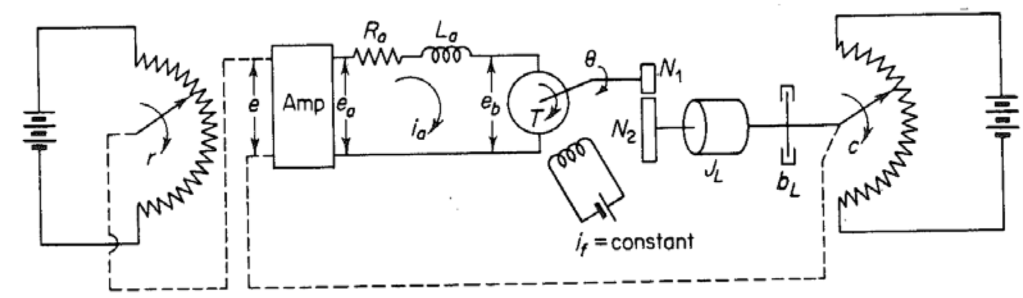
$$\frac{1}{2}[E_i(s) + E_o(s)] = \frac{1}{R_2Cs + 1} E_i(s)$$

Hence

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2Cs - 1}{R_2Cs + 1} = -\frac{s - \frac{1}{R_2C}}{s + \frac{1}{R_2C}}$$



Answer Q3



Solution. The equations describing the system dynamics are as follows:
For the potentiometric error detector;

$$E(s) = K_1[R(s) - C(s)] = 7.64[R(s) - C(s)] \quad (2\text{–}94)$$

For the amplifier,

$$E_a(s) = K_p E(s) = 10E(s) \quad (2\text{–}95)$$

For the armature-controlled dc motor, the equivalent moment of inertia J and equivalent viscous-friction coefficient b referred to the motor shaft are, respectively,

$$J = J_m + n^2 J_L$$
$$= 1 \times 10^{-5} + 4.4 \times 10^{-5} = 5.4 \times 10^{-5}$$
$$b = b_m + n^2 b_L$$
$$= 4 \times 10^{-4}$$

Referring to Equation (2–44), we obtain

$$\frac{\Theta(s)}{E_a(s)} = \frac{K}{s(T_m s + 1)}$$

where

$$K_m = \frac{K}{R_a b + KK_b} = \frac{6 \times 10^{-5}}{(0.2)(4 \times 10^{-4}) + (6 \times 10^{-5})(5.5 \times 10^{-2})} = 0.72$$

$$T_m = \frac{R_a J}{R_a b + KK_b} = \frac{(0.2)(5.4 \times 10^{-5})}{(0.2)(4 \times 10^{-4}) + (6 \times 10^{-5})(5.5 \times 10^{-2})} = 0.13$$

Thus

$$\frac{\Theta(s)}{E_a(s)} = \frac{10C(s)}{E_a(s)} = \frac{0.72}{s(0.13s + 1)} \quad (2\text{–}96)$$

Using Equations (2–94), (2–95), and (2–96), we can draw the block diagram of the system as shown in Figure 2–46(a). Simplifying this block diagram, we obtain Figure 2–46(b). The closed-loop transfer function of this system is

$$\frac{C(s)}{R(s)} = \frac{5.5}{0.13s^2 + s + 5.5} = \frac{42.3}{s^2 + 7.69s + 42.3}$$

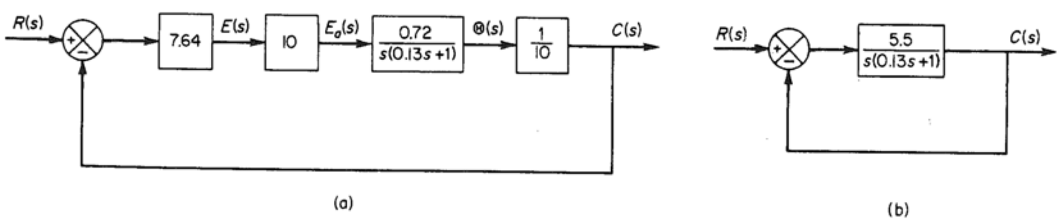


Figure 2–46 (a) Block diagram of the system shown in Fig. 2–45; (b) simplified block diagram.