Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 01-e

System Modeling

Contents

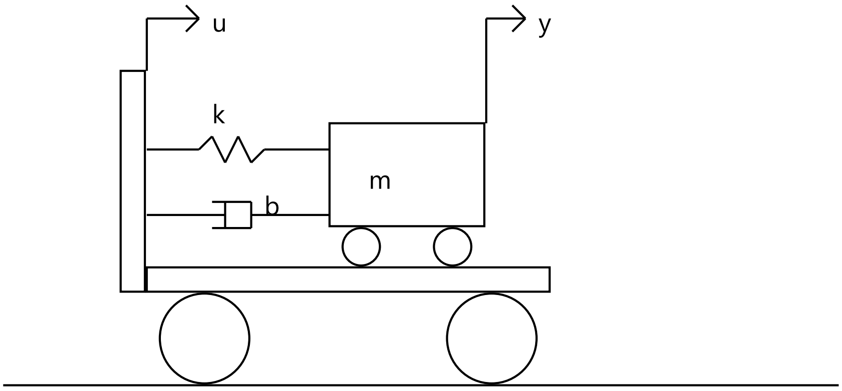
Reference:

1. Chapter 2, "Modern Control Engineering," K. Ogata, Prentice Hall
2. Schaum's Outline Series – Feedback Control Systems

Email: norbertcheung@szu.edu.cn
Web Site: http://norbert.idv.hk
Last Updated: 2024-04

1. Mechanical Systems Modeling

Mechanical Translational System



Use Newton's second law to form the equation:

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u)$$

Rearrange the equation gives:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

Then take the Laplace transform of the mechanical equation:

$$L\left[m\frac{d^2y}{dt^2}\right] = m\left[s^2Y(s) - sy(0) - \dot{y}(0)\right]$$

$$L\left[b\frac{dy}{dt}\right] = b\left[sY(s) - y(0)\right]$$

$$L[ky] = kY(s)$$

$$L\left[b\frac{du}{dt}\right] = b\left[sU(s) - u(0)\right]$$

$$L[ku] = kU(s)$$

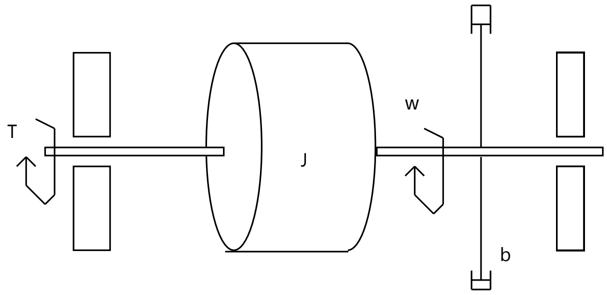
Set all the initial conditions to zero, the whole equation can be rewritten as:

$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

The transfer function is obtained by taking the ratio of *Y(s)* and *U(s)*

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{bs+k}{ms^2+bs+k}}$$

Mechanical Rotational Systems



J: Moment of inertia
ω: Angular velocity
b: Viscous friction coefficient
T: Torque

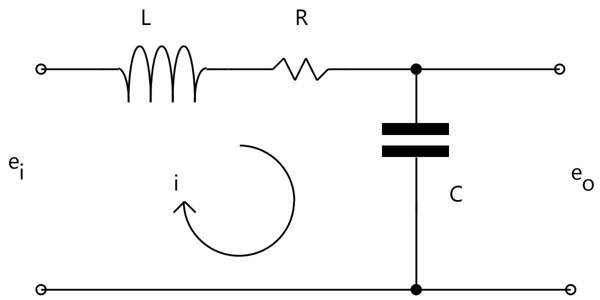
Equation of motion: $J\dot{\omega} + b\omega = T$

Laplace transform of equation: $\boxed{\dfrac{\Omega(s)}{T(s)} = \dfrac{1}{Js+b}}$

where $\Omega(s) = L[\omega(t)]$ and $T(s) = L[T(t)]$

2. Electrical Systems Modeling

LCR Circuit



Applying Kirchoff's voltage law:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i dt = e_i \qquad \text{and} \qquad \frac{1}{C}\int i dt = e_o$$

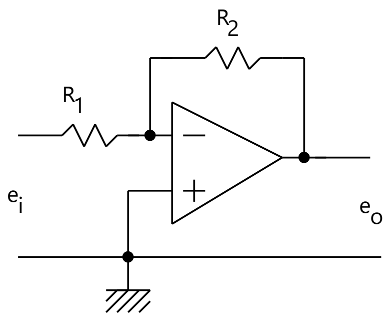
Apply the Laplace Transform:

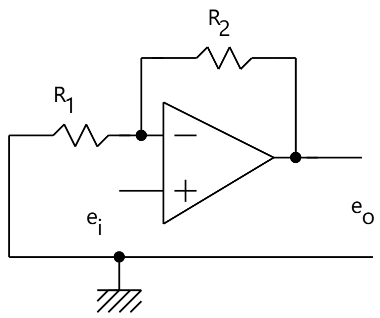
$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s) \qquad \text{and} \qquad \frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

The transfer function is:

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

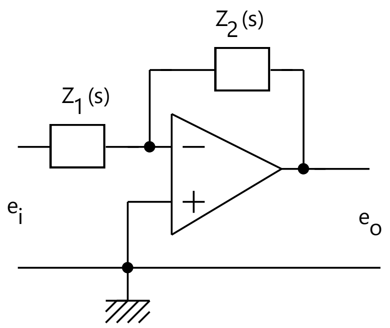
Operational Amplifier Circuits

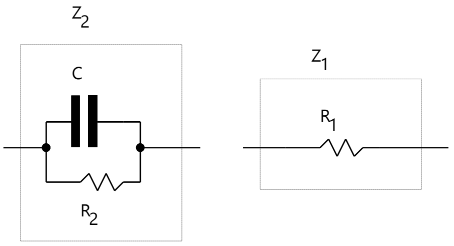




$$e_o = -\frac{R_2}{R_1} e_i \qquad\qquad e_o = (1 + \frac{R_2}{R_1}) e_i$$

Impedance Approach for Complex Transfer Functions in s domain





Since the circuit is similar to an inverter circuit, the transfer function is:

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

The two impedances are:

$$Z_1(s) = R_1 \qquad \text{and} \qquad Z_2(s) = \frac{1}{Cs + \frac{1}{R_2}} = \frac{R_2}{R_2 Cs + 1}$$

Hence the Transfer function becomes:

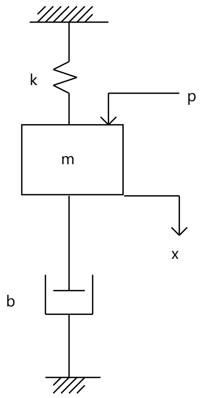
$$\frac{E_o(s)}{E_i(s)} = -\frac{\frac{R_2}{R_2 Cs + 1}}{R_1} = -\frac{R_2}{R_1} \frac{1}{R_2 Cs + 1}$$

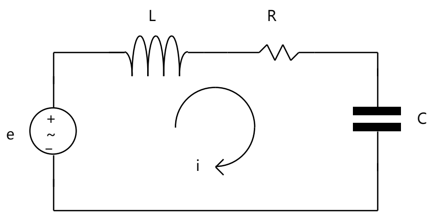
3. Analogous Systems

Systems that can be represented by the same mathematical model but that are different physically are called *analogous systems*. Analogous systems are very useful in practice.

- The solution of equation of one physical system can also be applied to any other field with the same equation.
- Since one type of system may be easier to handle experimentally, it is therefore very easy to find out the behavior of an expensive system by investigating the behavior its cheaper counter part.
- In many cases, we use simple electric circuits to simulate the behavior of complex and expensive mechanical, chemical, or thermal, or other building systems.

Mechanical electrical analogies (force-voltage relationship)





System equations:

Mechanical: $$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=p$$

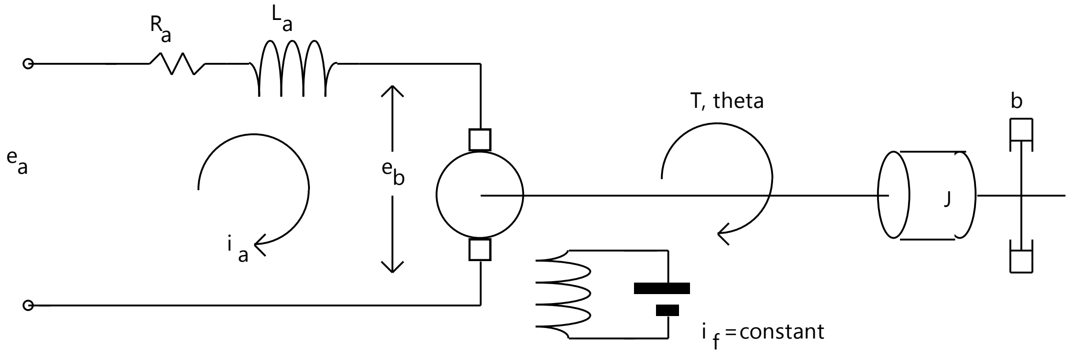
Electrical: $$L\frac{di}{dt}+Ri+\frac{1}{C}\int idt=e$$

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=e$$ (in terms of electric charge, $i=dq/dt$)

Translational Mechanical Systems	Rotational Mechanical Systems	Electrical Systems
Force (p)	Torque (T)	Voltage (e)
Mass (m)	Moment of inertia (J)	Inductance (L)
Viscous friction coeff. (b)	Viscous friction coeff. (b)	Resistance (R)
Spring constant (k)	Spring constant (k)	1/capacitance (1/C)
Displacement (x)	Angular displacement (θ)	Charge (q)
Velocity (dx/dt)	Angular velocity (ω)	Current (i)

4. Electro-mechanical Systems

Modeling of DC servomotors:



R_a = armature resistance
L_a = armature inductance
i_a = armature current
i_f = field current
e_a = applied voltage
e_b = back emf
θ = angular displacement (theta)
T = torque
J = equivalent moment of inertia
b = equivalent viscous friction

Torque developed by the motor: $T = K_f i_f i_a$

Group all constant terms into K (motor torque constant):

$T = Ki_a$ or in s-domain $T(s) = KI_a(s)$

A back emf will be generated when the motor is rotating:

$e_b = K_b \dfrac{d\theta}{dt}$ or in s-domain $K_b s\Theta(s) = E_b(s)$

The voltage equation of the motor can be written as:

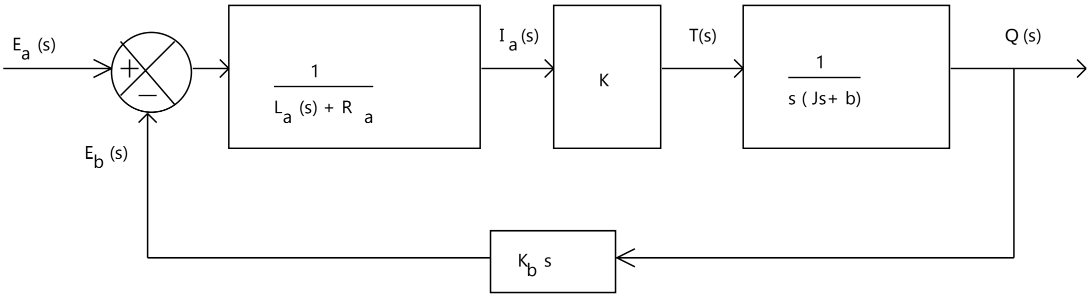
$L_a \dfrac{di_a}{dt} + R_a i_a + e_b = e_a$ or in s-domain $(L_a s + R_a)I_a(s) + E_b(s) = E_a(s)$

The mechanical equation can be described as:

$J \dfrac{d^2\theta}{dt^2} + b\dfrac{d\theta}{dt} = T$ or in s-domain $(Js^2 + bs)\Theta(s) = T(s)$

For simplicity, assume that all the initial conditions are zero.

Consider the *Θ(s)* as output, and $E_a(s)$ as input, it is possible to construct a block diagram from the above equations:



The DC motor in itself is a feedback system. The output velocity is directly related to the input voltage. The block diagram can be simplified into the following equation:

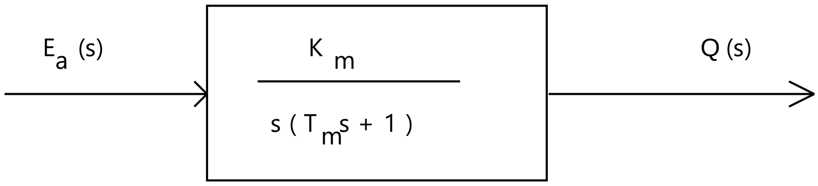
$$\frac{\Theta(s)}{E_a(s)} = \frac{K}{s\left[L_a J s^2 + \left(L_a b + R_a J\right)s + R_a b + K K_b\right]}$$

The inductor L_a is usually very small and can be neglected. The whole equation can be simplified as:

$$\frac{\Theta(s)}{E_a(s)} = \frac{K_m}{s\left(T_m s + 1\right)}$$

where $K_m = \dfrac{K}{R_a b + K K_b}$ (motor gain constant)

and $T_m = \dfrac{R_a J}{R_a b + K K_b}$ (motor time constant)



--- END ---

Glossary – English/Chinese Translation

English	Chinese
modeling	建 模
electrical system	电气系统
mechanical system	机械系统
electromechanical system	机电系统
Newton's Second Law	牛顿第二定律
Free Body Diagram	自由体图
Kirchhoff’s Voltage Law	基尔霍夫电压定律
operational amplifier	运算放大器
analogous System	类比系统
armature and field current	电枢和励磁电流
servomotor	伺服
viscous friction	粘性摩擦