Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 01-d

Signal Flow Graphs

Contents

Reference:

1. Chapter 3, Benjamin C. Kuo, "Automatic Control Systems," 6th Edition, Prentice Hall International Editions.
2. Schaum's Outline Series – Feedback Control Systems

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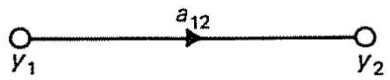
Last Updated: 2024-03

1. Introduction

A signal flow graph can be seen as a simplified version of block diagram, and it applies to linear systems only. The signal flow graph consists of nodes and branches. A signal can only transmit in the direction of the arrow.



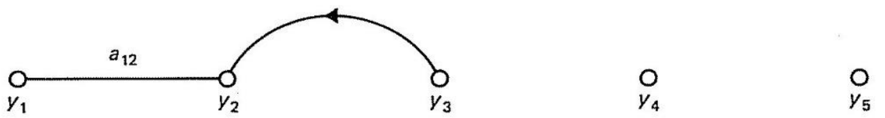
$$y_2 = a_{12} y_1$$

Example

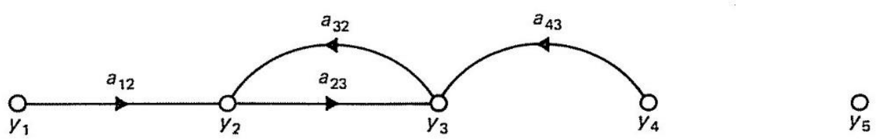
Consider the following set of equations:

$$y_2 = a_{12} y_1 + a_{32} y_3$$
$$y_3 = a_{23} y_2 + a_{43} y_4$$
$$y_4 = a_{24} y_2 + a_{34} y_3 + a_{44} y_4$$
$$y_5 = a_{25} y_2 + a_{45} y_4$$

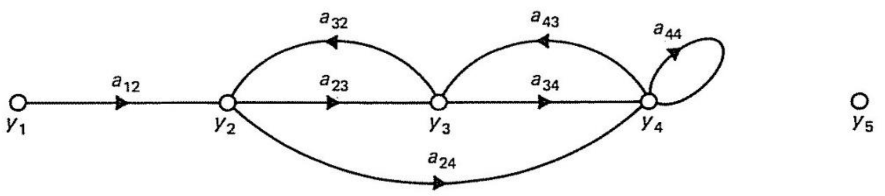
We can construct the signal flow graph as follows:



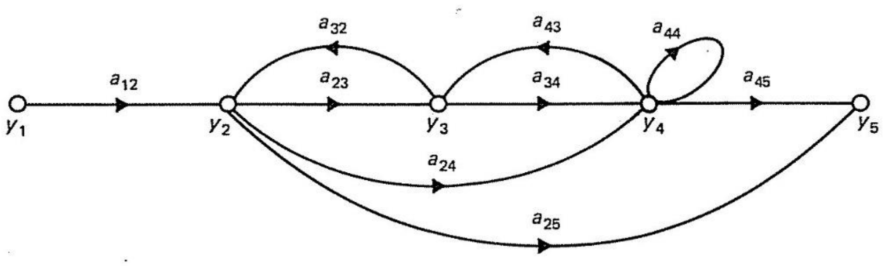
(a) $y_2 = a_{12} y_1 + a_{32} y_3$



(b) $y_2 = a_{12} y_1 + a_{32} y_3$, $y_3 = a_{23} y_2 + a_{43} y_4$



(c) $y_2 = a_{12} y_1 + a_{32} y_3$, $y_3 = a_{23} y_2 + a_{43} y_4$, $y_4 = a_{24} y_2 + a_{34} y_3 + a_{44} y_4$



(d) Complete signal-flow graph

2. Basic Properties of Signal Flow Graphs (SFG)

1. SFG applies only to linear systems.
2. The equations for SFG must be in the form of algebraic functions.
3. Variables are represented by nodes, and are usually from left to right.
4. Signal Travel along the branches only in the direction specified.
5. The branch directing from node y_k to y_j represents the dependence of the variable y_j upon y_k, but not the reverse.
6. A signal y_k traveling along a branch between nodes y_k and y_j is multiplied by the gain of the branch a_{kj}, so that a signal $a_{kj}y_k$ is delivered at node y_j.

The Signal Flow:

$$X_i = \sum_{j=1}^{n} A_{ij} X_j$$

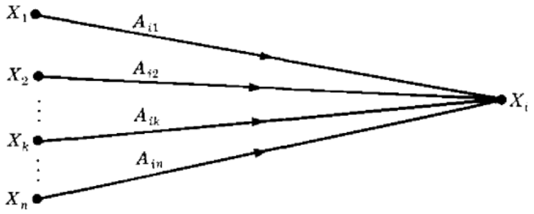


Fig. 8-3

$$X_i = A_{ik} X_k \qquad i = 1, 2, \ldots, n,\ k \text{ fixed}$$

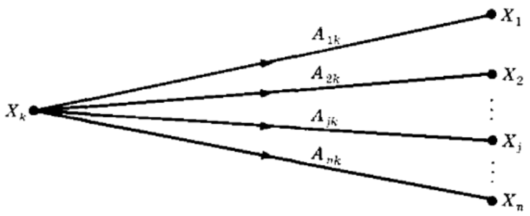


Fig. 8-5

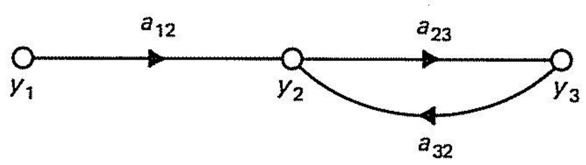
$$X_n = A_{21} \cdot A_{32} \cdot A_{43} \cdots A_{n(n-1)} \cdot X_1$$

Fig. 8-7

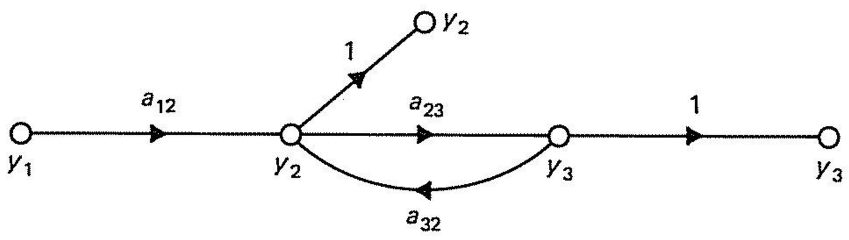
3. Definitions of Signal Flow Graphs (SFG)

Input Node (Source) – Node that has only outgoing branches.

Output Node (Sink) – Node that has only incoming branches.

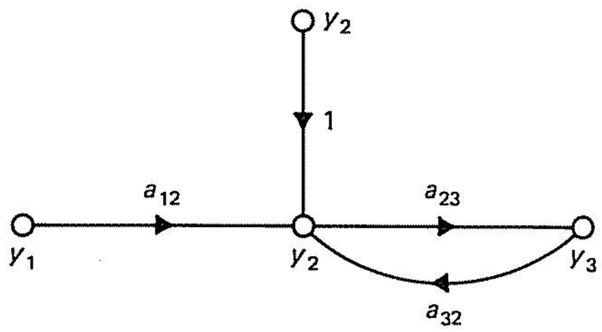


(a) Original signal-flow graph



(b) Modified signal-flow graph

Ways to make y_2 and y_3 as output nodes



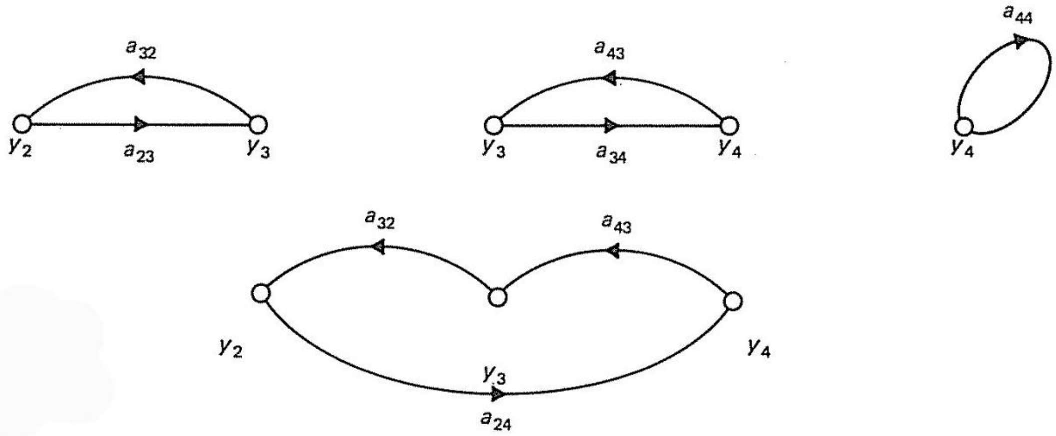
Ways to make y_2 as in put node

Terminology

Path – Collection of successive branches traveling in the same direction.

Forward Path – Path that starts at an input node and ends at an output node, and no node is traversed more than once.

<u>Loop</u> – Path that originates and terminates on the same node, along which no other node is encountered more than once.



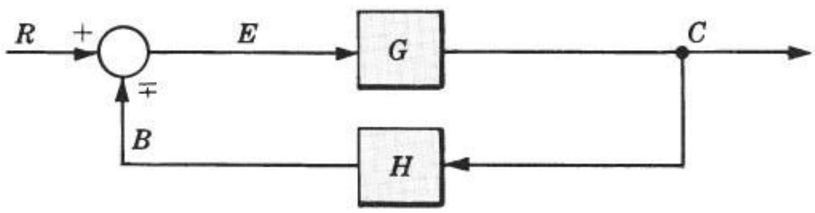
The four loops of SFG in page 2.

<u>Path Gain</u> – The product of the branch gain in traveling along a path.

<u>Forward Path Gain</u> – Path gain of a forward path

<u>Loop Gain</u> – Path gain of a loop

Example 1:



The signal flow graph is easily constructed from Fig. 8-12. Note that the − or + sign of the summing point is associated with H.

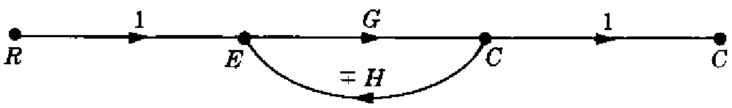
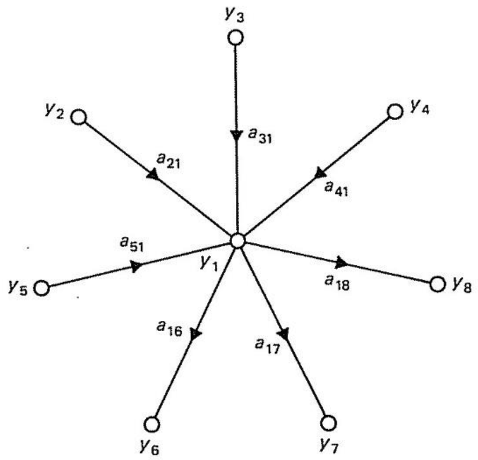


Fig. 8-12

4. SFG Algebra

Rule 1 - Value of a variable is the sum of all incoming signals.

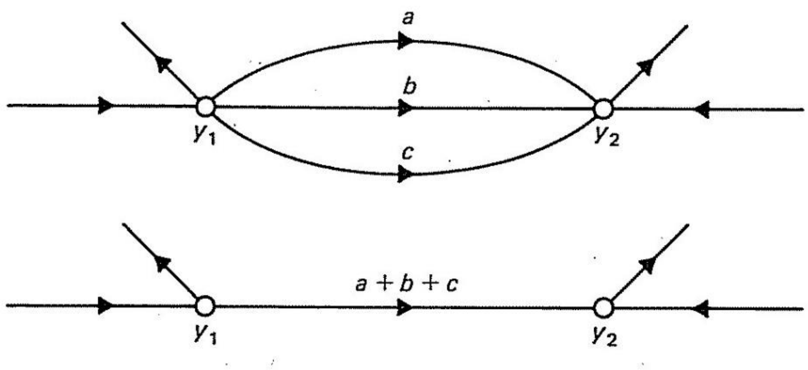


$$y_1 = a_{21}y_2 + a_{31}y_3 + a_{41}y_4 + a_{51}y_5$$

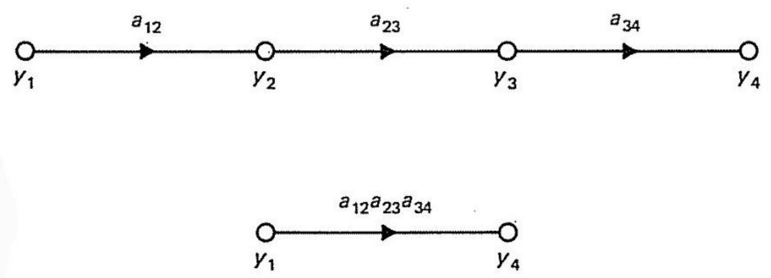
Rule 2 – Value of the variable in a node is transmitted to all branches leaving the node.

$y_6 = a_{16}y_1$ $\quad$ $y_7 = a_{17}y_1$ $\quad$ $y_8 = a_{18}y_1$

Rule 3 – Parallel branches in the same direction can be replaced by a single branch with gain equal to the sum of the parallel branches' gains.

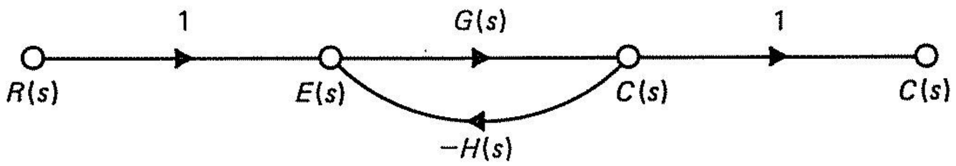


Rule 4 – A series of unidirectional branches can be replaced by a single branch with gain equal to the product of the branch gains



Rule 5 – A feedback control system with equation $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

can be represented as below.



How to construct the SFG?

The signal flow graph of a system described by a set of simultaneous equations can be constructed in the following general manner.

$$X_1 = A_{11}X_1 + A_{12}X_2 + \cdots + A_{1n}X_n$$

$$X_2 = A_{21}X_1 + A_{22}X_2 + \cdots + A_{2n}X_n$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$X_m = A_{m1}X_1 + A_{m2}X_2 + \cdots + A_{mn}X_n$$

1. Write the system equations in the form as above.
2. Arrange the m or n (whichever is larger) nodes from left to right. The nodes may be rearranged if the required loops later appear too cumbersome.
3. Connect the nodes by the appropriate branches A_{11}, A_{12}. etc.
4. If the desired output node has outgoing branches, add a dummy node and a unity gain branch.
5. Rearrange the nodes and/or loops in the graph to achieve maximum pictorial clarity.

Example of constructing SFG

Example 1

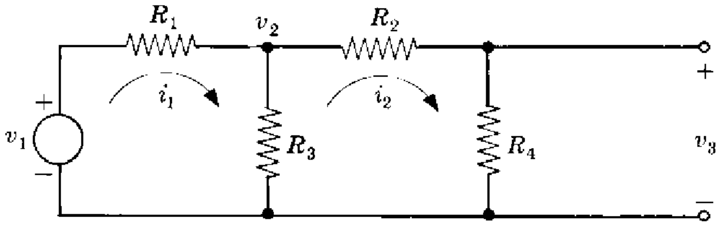


Fig. 8-13

$$i_1 = \left(\frac{1}{R_1}\right) v_1 - \left(\frac{1}{R_1}\right) v_2 \qquad v_2 = R_3 i_1 - R_3 i_2 \qquad i_2 = \left(\frac{1}{R_2}\right) v_2 - \left(\frac{1}{R_2}\right) v_3 \qquad v_3 = R_4 i_2$$

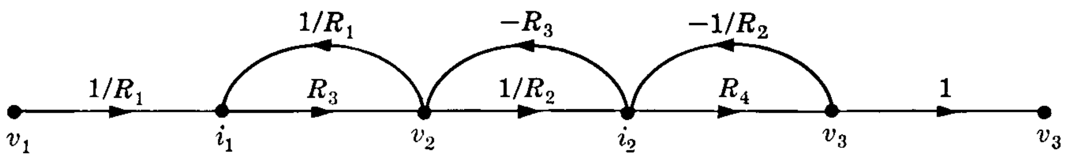


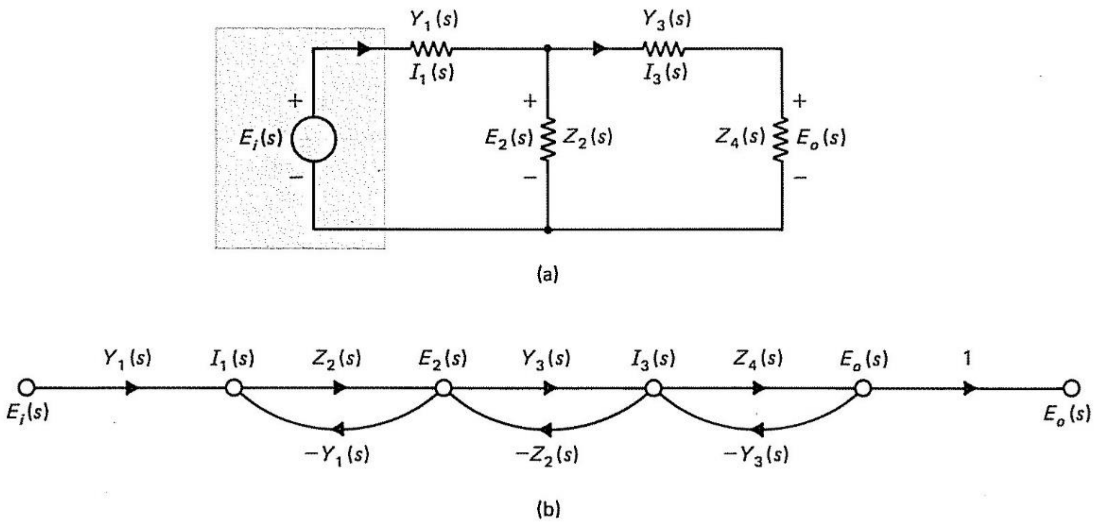
Fig. 8-15

Note that signal flow graph representations of equations are not unique. For example, the addition of a unity gain branch followed by a dummy node changes the graph, but not the equations it represents.

Example 2

The circuit below can be represented by the set of equations:

$$I_1(s) = [E_i(s) - E_2(s)]Y_1(s)$$
$$E_2(s) = [I_1(s) - I_3(s)]Z_2(s)$$
$$I_3(s) = [E_2(s) - E_0(s)]Y_3(s)$$
$$E_o(s) = Z_4(s)I_3(s)$$



Example 3

First establish the differential equation:

$$L\frac{di(t)}{dt} = e_1(t) - Ri(t) - e_c(t) \qquad C\frac{de_c(t)}{dt} = i(t)$$

Convert them to algebraic form using Laplace Transform

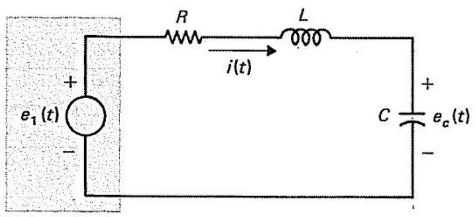
$$sI(s) = i(0) + \frac{1}{L}E_1(s) - \frac{R}{L}I(s) - \frac{1}{L}E_c(s) \qquad sE(c) = e_c(0) + \frac{1}{C}I(s)$$

Then solve for *I(s)* and $E_c(s)$

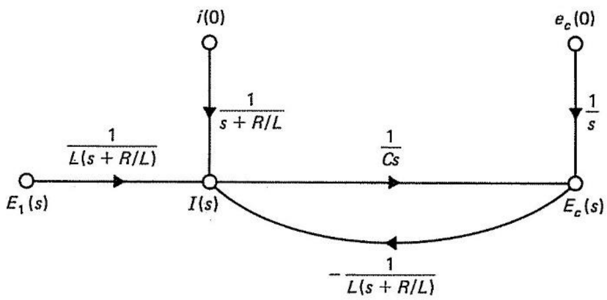
$$I(s) = \frac{1}{s+R/L}i(0) + \frac{1}{L(s+R/L)}E_1(s) - \frac{1}{L(s+R/L)}E_c(s)$$

$$E_c(s) = \frac{1}{s}e_c(0) + \frac{1}{Cs}I(s)$$

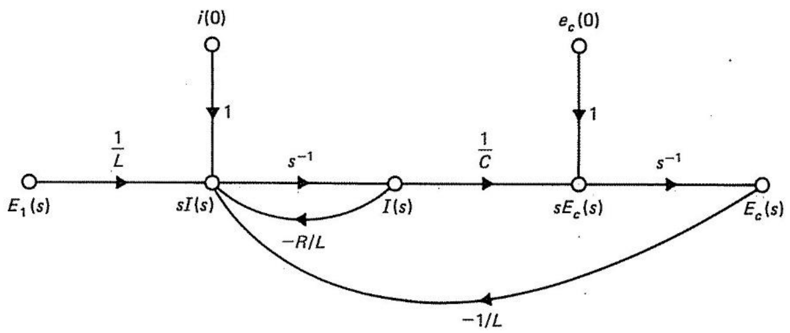
The obtained answer is (b). For your reference, (c) is obtained using different input variables. This shows that the answer is not unique.



(a)



(b)



(c)

5. General Gain Formula (Mason's Gain Formula) for SFG

A structured method to obtain the gain formula for a given SFG.

$$M = \frac{y_{out}}{y_{in}} = \sum_{k=1}^{N} \frac{M_k \Delta_k}{\Delta}$$

y_{in}	input node variable
y_{out}	output node variable
M	gain between y_{in} and y_{out}
N	total number of forward paths between y_{in} and y_{out}
M_k	gain of the k^{th} forward path between y_{in} and y_{out}

(Note that the gain formula can only be applied between the input and the output)

Δ = 1 – (sum of gains of all individual loops) + (sum of products-of-gains of all possible combinations of two non-touching loops) – (sum of products-of-gains of all possible combinations of three non-touching loops) + …..

Δ_k = the Δ of the part of signal flow graph that is non-touching with the forward path.

Example of General Gain Formula

1 There is only one forward path between E_i and E_o $N = 1$ $M_1 = Y_1Z_2Y_3Z_4$

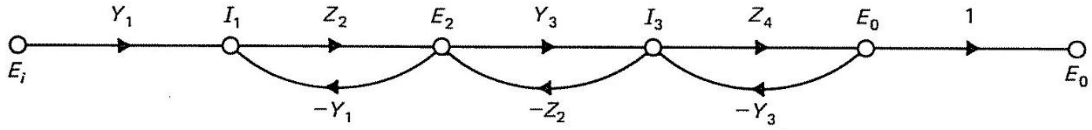
2 There are 3 loops, the loop gains are: $-Z_2Y_1$ $-Z_2Y_3$ $-Z_4Y_3$

3 There is one pair of non-touching loop, and their product gain is: $Z_2Z_4Y_1Y_3$

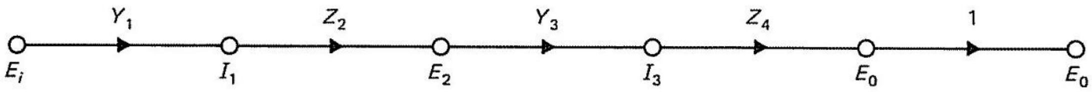
4 $\Delta = 1 - (-Z_2Y_1 - Z_2Y_3 - Z_4Y_3) + Z_2Z_4Y_1Y_3$

5 Since all 3 loops are in touch with the forward path: $\Delta_1 = 1$

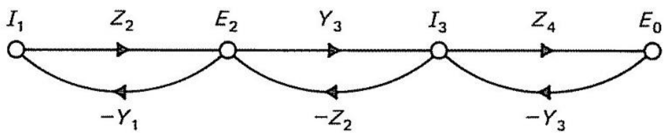
6 $$M = \frac{M_1\Delta_1}{\Delta} = \frac{Y_1Y_3Z_2Z_4}{1 + Z_2Y_1 + Z_2Y_3 + Z_4Y_3 + Z_2Z_4Y_1Y_3}$$



(a)



(b)



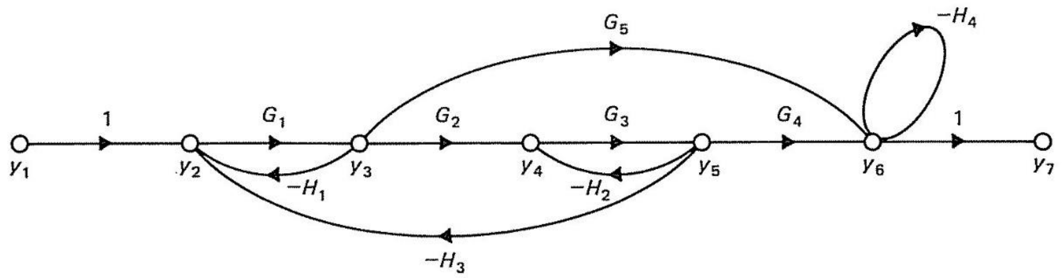
(c)

(d)

A more complicated example

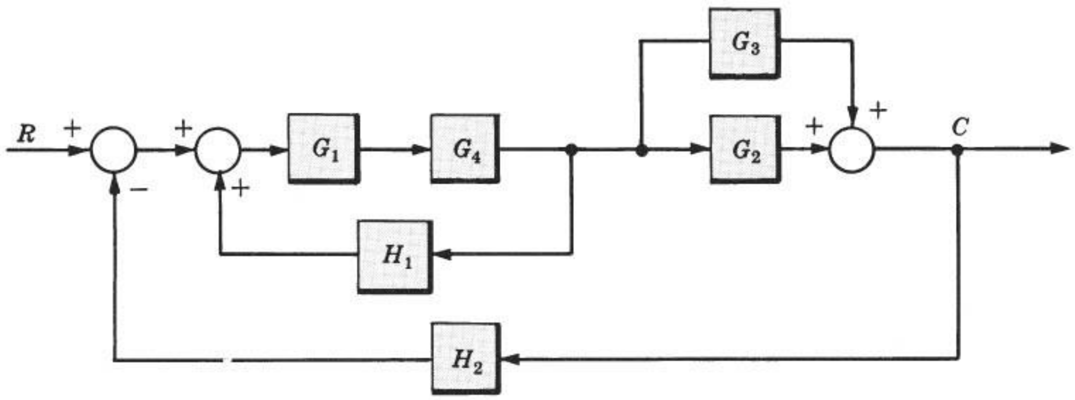
$$\frac{y_7}{y_1} = \frac{G_1G_2G_3G_4 + G_1G_5(1+G_3H_2)}{\Delta}$$

$$\Delta = 1 + \left(G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + H_4\right) \quad \text{.....individual}$$
$$+\left(G_1H_1G_3H_2 + G_1H_1H_4 + G_3H_2H_4 + G_1G_2G_3H_3H_4\right) \quad \text{.....2 non-touching}$$
$$+ G_1H_1G_3H_2H_4 \quad \text{.....3 non-touching}$$



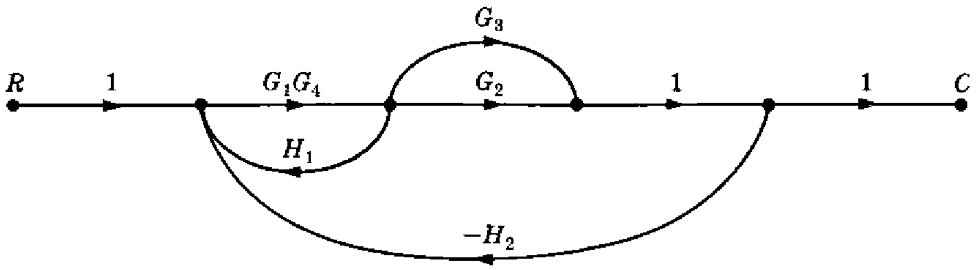
An example with 2 forward paths

Original block diagram



The SFG, with 2 paths:

$P_1 = G_1G_2G_4 \qquad P_2 = G_1G_3G_4$



There are three feedback loops:

$$P_{11} = G_1 G_4 H_1 \qquad P_{21} = -G_1 G_2 G_4 H_2 \qquad P_{31} = -G_1 G_3 G_4 H_2$$

There are no nontouching loops, and all loops touch both forward paths; then

$$\Delta_1 = 1 \qquad \Delta_2 = 1$$

Therefore the control ratio is

$$T = \frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_4 + G_1G_3G_4}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

$$= \frac{G_1G_4(G_2 + G_3)}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

--- END ---

Glossary – English/Chinese Translation

English	Chinese
signal flow graph	信号流图
nodes and branches	节点和分支
algebraic function	代数函数
source and sink	源和接收器
forward path	前进路径
loop gain	环路增益
Mason's Gain Formula	梅森增益公式
General Gain Formula	一般增益公式
two non-touching loops	两个非接触式循环