Dr. Norbert Cheung's Lecture Series

Level 1 Topic no: 01-b

Differential Equation and Laplace Transform

Contents

Reference:
Feedback and Control Systems – Schaum's Outline Series – 3rd edition

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1. System Equations

A differential equation is any algebraic or transcendental equality which involves either differentials or derivatives.

Mechanical - Newton's second law of motion: $f = Ma$
Rewrite as the rate of change of velocity v of the mass with respect to time t:

$$f = M(dv/dt)$$

Electrical – Ohm's Law: $v=Ri$
Rewrite as a relationship between voltage v, resistance R, and the time rate of passage of charge through the resistor:

$$v = R\{dq/dt)$$

General Form of differential equation:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = u(t)$$

Or

$$\sum_{i=0}^{n} a_i \frac{d^i y(t)}{dt^i} = u(t)$$

where $a_0, a_1, \ldots, a_n$ are constants, is an ordinary differential equation. $y(t)$ and $u(t)$ arc dependent variables, and t is the independent variable.

A time-invariant equation is an equation in which none of the terms depends *explicitly* on the independent variable time.

Consider the n^th-order linear constant-coefficient differential equation:

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = u$$

It is convenient to define a <u>differential operator:</u>

$$D \equiv \frac{d}{dt}$$

and more generally an <u>nth-order differential operator</u>

$$D^n \equiv \frac{d^n}{dt^n}$$

The differential equation can now be written as:

$$D^n y + a_{n-1} D^{n-1} y + \cdots + a_1 D y + a_0 y = u$$

$$\left(D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 \right) y = u$$

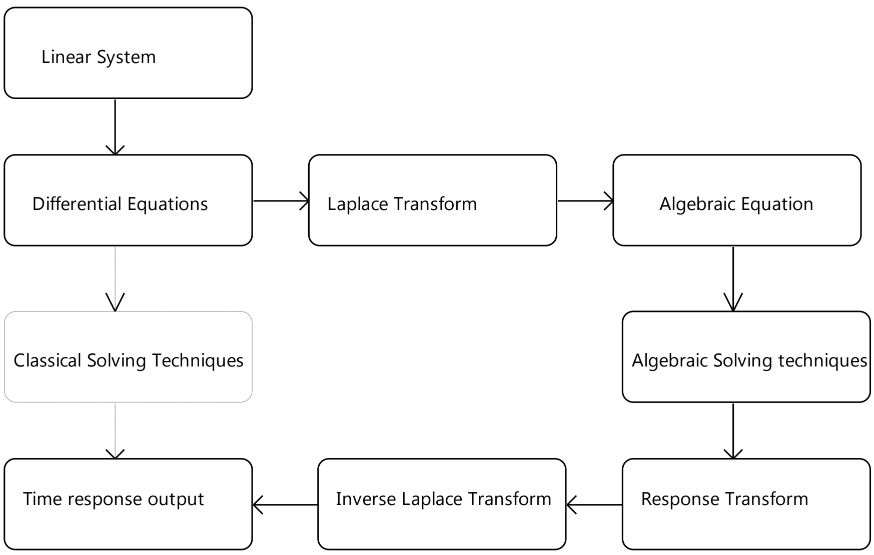
And:

$$D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 = 0$$

is called the <u>characteristic equation.</u>

<u>2. Laplace Transform</u>

We derive a differential equation describing the system response and then transform this equation into the frequency domain, where it becomes an algebraic equation. Algebraic techniques are then used to solve the transformed equation for the circuit response. The inverse Laplace transformation then changes the frequency domain response into the system response in the time domain.



Why do we need Laplace transform in Control?

Take an example, to calculate

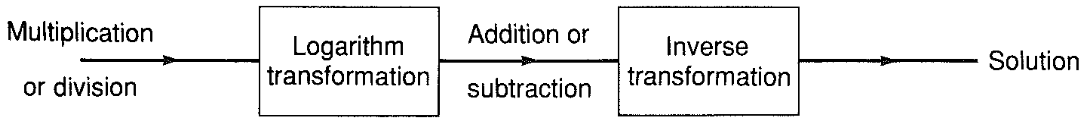
$$A = B \times C$$

We may, transform it into logarithms

$$\log A = \log BC = \log B + \log C = D$$

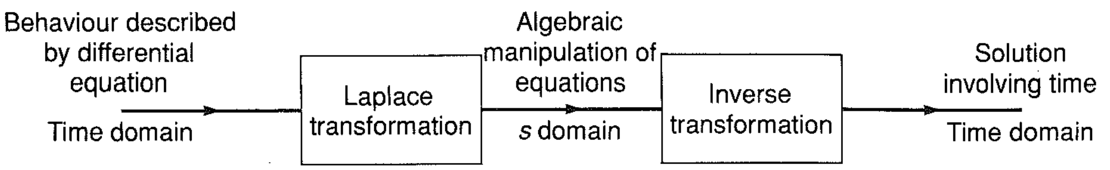
Then: $A = \text{antilog } D$

So, multiplication and division will be transformed into addition and subtraction.



Laplace transform is a similar type of transform.

Transform system behavior from time domain into a complex frequency of s domain. In this way differential equations will be much easier to manage.



Symbolically, we represent the Laplace transformation as

$L\{f(t)\} = F(s)$

Laplace transform. Thus the Laplace transform of some term which is a function of time is

$$\int_0^\infty (\text{term})\, e^{-st}\, dt$$

Because the term is a function of time it is usually written as $f(t)$ with the Laplace transform, since it is a function of s, written as $F(s)$. It is usual to use a capital letter F for the Laplace transform and a lower-case letter f for the time-varying function $f(t)$. Thus

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \qquad [1]$$

Example in Laplace Transform 1

Show that the Laplace transform of the unit step function *f(t) = u(t)* is *F(s) = 1/s.*

Solution

$$F(s) = \int_0^\infty u(t)e^{-st}dt$$

Since *u(t) = 1* throughout the range of integration this integral becomes

$$F(s) = \int_{0-}^\infty e^{-st}dt = -\left.\frac{e^{-st}}{s}\right|_{0-}^{\infty} = -\left.\frac{e^{-(\sigma+j\omega)t}}{\sigma+j\omega}\right|_{0-}^{\infty}$$

Example in Laplace Transform 2

Find the Laplace transform of the waveform

$$f(t) = 2u(t) - 5[e^{-2t}]u(t) + 3[\cos 2t]u(t) + 3[\sin 2t]u(t)$$

Solution

Using the linearity property, we write the transform of f(t) in the form

$$L\{f(t)\} = 2L\{u(t)\} - 5L\{e^{-2t}u(t)\} + 3L\{[\cos 2t]u(t)\} + 3L\{[\sin 2t]u(t)\}$$

$$F(s) = \frac{2}{s} - \frac{5}{s+2} + \frac{3s}{s^2+4} + \frac{6}{s^2+4}$$

Rationalizing the preceding sum yields

$$F(s) = \frac{16(s^2+1)}{s(s+2)(s^2+4)}$$

3. Inverse Laplace Transform

$$L^{-1}\{F(s)\} = f(t)$$

Use the following equation:

$$f(t) = \frac{1}{2\pi j}\int_{a-\infty}^{a+\infty} F(s)e^{-st}ds$$

To perform the inverse transformation, we must find the waveform corresponding the rational functions of the form

$$F(s) = K\frac{(s-z_1)(s-z_2).....(s-z_m)}{(s-p_1)(s-p_2).....(s-p_n)}$$

where the K is the scale factor, z_i (i=1,2,....m) are the zeros, and p_i (i=1,2,....,n) are the poles of F(s).

If there are more finite poles than zeros (n > M), then F(s) is called a **proper rational function**. If a proper rational function has only simple poles, then it can be decomposed into a partial fraction expansion of the form:

$$F(s) = \frac{k_1}{(s-p_1)} + \frac{k_2}{(s-p_2)} + ... + \frac{k_n}{(s-p_n)}$$

In this case, F(s) can be expressed as a linear combination of terms with one term for each of its n simple poles. Each term in the partial fraction decomposition has the form of the transform of an exponential signal. This is, we recognize that $L^{-1}\{k/(s+\alpha)\} = [ke^{-\alpha t}]u(t)$. We can now write the corresponding waveform using the linearity property:

$$f(t) = [k_1e^{p_1t} + k_2e^{p_2t} + ... + k_ne^{p_nt}]u(t)$$

Given the poles of F(s), finding the inverse transform f(t) reduces to finding the residues.

To illustrate the procedure, consider a case in which F(s) has three simple poles and one finite zero.

$$F(s) = K\frac{(s-z_1)}{(s-p_1)(s-p_2)(s-p_3)} = \frac{k_1}{(s-p_1)} + \frac{k_2}{(s-p_2)} + \frac{k_3}{(s-p_3)}$$

We find the residue k_1 by first multiplying this equation through by the factor (s - p_1):

$$(s-p_1)F(s) = K\frac{(s-z_1)}{(s-p_2)(s-p_3)} = k_1 + \frac{k_2(s-p_1)}{(s-p_2)} + \frac{k_3(s-p_1)}{(s-p_3)}$$

If we now set s = p_1, the last two terms on the right vanish, leaving

$$k_1 = (s - p_1)F(s)\Big|_{s=p_1} = K\frac{(s-z_1)}{(s-p_2)(s-p_3)}\Bigg|_{s=p_1}$$

Using the same approach for k_2 yields

$$k_2 = (s - p_2)F(s)\Big|_{s=p_2} = K\frac{(s-z_1)}{(s-p_1)(s-p_3)}\Bigg|_{s=p_2}$$

The technique generalizes so that the residue at any simple pole p_i is

$$k_i = (s - p_i)F(s)\Big|_{s=p_i}$$

Example in Laplace Inverse Transform 1

Find the waveform in time domain corresponding to the transform

$$F(s) = 2\frac{(s+3)}{s(s+1)(s+2)}$$

Solution:

F(s) is a proper rational function and has simple poles at s = 0, s = -1, s = -2. Its partial fraction expansion is

$$F(s) = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

The cover-up algorithm yields the residues as

$$k_1 = sF(s)\Big|_{s=0} = \frac{2(s+3)}{(s+1)(s+2)}\Bigg|_{s=0} = 3$$

$$k_1 = (s+1)F(s)\Big|_{s=-1} = \frac{2(s+3)}{s(s+2)}\Bigg|_{s=-1} = -4$$

$$k_1 = (s+2)F(s)\Big|_{s=-2} = \frac{2(s+3)}{s(s+1)}\Bigg|_{s=-2} = 1$$

The inverse transform f(t) is

$$f(t) = [3 - 4e^{-t} + e^{-2t}]u(t)$$

Example: Use Laplace Transform to solve differential equation

Example 5

Use Laplace transforms to solve the following differential equation:

$$3\frac{dx}{dt} + 2x = 4$$

with $x = 0$ at $t = 0$.

Answer

The Laplace transform of $3\,dx/dt$ is 3 times the Laplace transform of dx/dt. The Laplace transform of $2x$ is 2 times the Laplace transform of x. The Laplace transform of 4 is, since this can be considered to be a step function of height 4, $4/s$. Thus

$$3[sX(s) - x(0)] + 2X(s) = 4/s$$

where $X(s)$ is the Laplace transform of x. Since $x(0) = 0$ then

$$3[sX(s) - 0] + 2X(s) = 4/s$$

and so

$$3s^2X(s) + 2sX(s) = 4$$

$$X(s) = \frac{4}{3s^2 + 2s} = \frac{2(2/3)}{s[s + (2/3)]}$$

We now need to find the functions which would give the Laplace transforms of this form in order to obtain the inverse transformation and obtain x. Since the inverse transformation of $a/[s(s + a)]$ is $(1 - e^{-at})$ then

$$x = 2(1 - e^{-2t/3})$$

4. Basic Properties of Laplace Transform

1. The Laplace transform is a *linear transformation* between functions defined in the t-domain and functions defined in the s-domain. That is, if $F_1(s)$ and $F_2(s)$ are the Laplace transforms of $f_1(t)$ and $f_2(t)$, respectively, then $a_1F_1(s) + a_2F_2(s)$ is the Laplace transform of $a_1f_1(t) + a_2f_2(t)$, where a_1 and a_2 are arbitrary constants.
2. The inverse Laplace transform is a *linear transformation* between functions defined in the s-domain and functions defined in the t-domain. That is, if $f_1(t)$ and $f_2(t)$ are the inverse Laplace transforms of $F_1(s)$ and $F_2(s)$, respectively, then $b_1f_1(t) + b_2f_2(t)$ is the inverse Laplace transform of $b_1F_1(s) + b_2F_2(s)$, where b_1 and b_2 are arbitrary constants.
3. The Laplace transform of the *derivative* df/dt of a function $f(t)$ whose Laplace transform is $F(s)$ is

$$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0^+)$$

where $f(0^+)$ is the initial value of $f(t)$, evaluated as the one-sided limit of $f(t)$ as t approaches zero from positive values.

4. The Laplace transform of the *integral* $\int_0^t f(\tau)\,d\tau$ of a function $f(t)$ whose Laplace transform is $F(s)$ is

$$\mathscr{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{F(s)}{s}$$

5. The initial value $f(0^+)$ of the function $f(t)$ whose Laplace transform is $F(s)$ is

$$f(0^+) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \qquad t > 0$$

This relation is called the *Initial Value Theorem*.

6. The final value $f(\infty)$ of the function $f(t)$ whose Laplace transform is $F(s)$ is

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

if $\lim_{t\to\infty} f(t)$ exists. This relation is called the *Final Value Theorem*.

7. The Laplace transform of a function $f(t/a)$ (*Time Scaling*) is

$$\mathscr{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$$

where $F(s) = \mathscr{L}[f(t)]$.

8. The inverse Laplace transform of the function $F(s/a)$ (*Frequency Scaling*) is

$$\mathscr{L}^{-1}\left[F\left(\frac{s}{a}\right)\right] = af(at)$$

where $\mathscr{L}^{-1}[F(s)] = f(t)$.

9. The Laplace transform of the function $f(t-T)$ (*Time Delay*), where $T > 0$ and $f(t-T) = 0$ for $t \leq T$, is

$$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$$

where $F(s) = \mathscr{L}[f(t)]$.

10. The Laplace transform of the function $e^{-at}f(t)$ is given by

$$\mathscr{L}\left[e^{-at}f(t)\right] = F(s+a)$$

where $F(s) = \mathscr{L}[f(t)]$ (*Complex Translation*).

11. The Laplace transform of the *product of two functions* $f_1(t)$ and $f_2(t)$ is given by the *complex convolution integral*

$$\mathscr{L}[f_1(t)f_2(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F_1(\omega)F_2(s-\omega)\,d\omega$$

where $F_1(s) = \mathscr{L}[f_1(t)]$, $F_2(s) = \mathscr{L}[f_2(t)]$.

12. The inverse Laplace transform of the *product of the two transforms* $F_1(s)$ and $F_2(s)$ is given by the *convolution integrals*

$$\mathscr{L}^{-1}[F_1(s)F_2(s)] = \int_{0^+}^{t} f_1(\tau)f_2(t-\tau)\,d\tau = \int_{0^-}^{t} f_2(\tau)f_1(t-\tau)\,d\tau$$

where $\mathscr{L}^{-1}[F_1(s)] = f_1(t)$, $\mathscr{L}^{-1}[F_2(s)] = f_2(t)$.

5. Initial and Final Value Properties

If a Laplace transform is multiplied by s, the value of the product as s tends to infinity is the value of the inverse transform as the time t tends to zero.

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t) \qquad [4]$$

This is known as the *initial value theorem*.

If a Laplace transform is multiplied by s, the value of the product as s tends to zero is the value of the inverse transform as t tends to zero.

If a Laplace transform is multiplied by s, the value of the product as s tends to infinity is the value of the inverse transform as the time t tends to zero.

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t) \qquad [4]$$

This is known as the *initial value theorem*.

If a Laplace transform is multiplied by s, the value of the product as s tends to zero is the value of the inverse transform as t tends to zero.

Examples in Initial and Final Value Theorems

Example 11

Without evaluating the Laplace transforms, what are the initial and final values of the functions giving the following transforms?

(a) $F(s) = \dfrac{s + a}{s^2}$

(b) $V_C(s) = \dfrac{V(1/RC)}{[s + (1/RC)]s}$

Answer

(a) If the expression is multiplied by s it becomes

$$sF(s) = \frac{s + a}{s} = 1 + \frac{a}{s}$$

Using the initial value theorem, when $s \to \infty$ then the expression tends to the value 1. So the initial value of the function is 1. Using the final value theorem, when $s \to 0$ the expression tends to the value ∞. So the final value of the function is ∞.

(b) If the expression is multiplied by s it becomes

$$sV_C(s) = \frac{V(1/RC)}{[s + (1/RC)]}$$

Using the initial value theorem, when $s \to \infty$ then the expression tends to the value 0. So the initial value of v_C is 0. Using the final value theorem, when $s \to 0$ then the expression tends to the value $V(1/RC)/(1/RC)$ or V.

----- END -----

Appendix 1 - Short Table of Laplace Transform

Time Function		Laplace Transform
Unit Impulse	$\delta(t)$	1
Unit Step	$\boxed{u(t)}$	$\frac{1}{s}$
Unit Ramp	t	$\frac{1}{s^2}$
Polynomial	t^n	$\frac{n!}{s^{n+1}}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine Wave	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
Cosine Wave	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
Damped Sine Wave	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
Damped Cosine Wave	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Appendix 2 - Three Types of Partial Fraction

There are basically three types of *partial fractions*. The form of the partial fractions for each of these types is as follows:

1 Linear factors in the denominator

Expression $\dfrac{f(s)}{(s+a)(s+b)(s+c)}$

Partial fraction $\dfrac{A}{s+a}+\dfrac{B}{s+b}+\dfrac{C}{s+c}$

2 Repeated linear factors in the denominator

Expression $\dfrac{f(s)}{(s+a)^n}$

Partial fraction $\dfrac{A}{s+a}+\dfrac{B}{(s+a)^2}+\dfrac{C}{(s+a)^3}$

$$+\dots\frac{N}{(s+a)^n}$$

3 Quadratic factors in the denominator, when the quadratic does not factorize without imaginary terms

Expression $\dfrac{f(s)}{as^2+bs+c}$

Partial fraction $\dfrac{As+B}{as^2+bs+c}$

Appendix 3- Glossary – English/Chinese Translation

English	Chinese
Differential Equation	微分方程
Newton's Second Law of Motion	牛顿第二运动定律
Ohm's Law	欧姆定律
Time-Invariant Equation	时间不变方程
Differential Operator	差分运算符
Characteristic Equation	特性方程式
Laplace Transform	拉普拉斯变换
Frequency Domain	频域
Time Domain	时域
Algebraic Technique	代数技术
Logarithm	对数
Inverse Laplace Transform	反拉普拉斯变换
Unit Step Function	单位步进函数
Linearity	线性
Poles and Zeros	极点和零点
Partial Fraction	部分分数
Linear Transformation	线性变换
Derivative and Integral	导数和积分
Initial Value Theorem	初值定理
Final Value Theorem	最终值定理
Time and Frequency Scaling	时间和频率缩放
Convolution Integral	卷积积分
Complex Convolution Integral	复卷积积分