Dr. Norbert Cheung's Series

in

Electrical Engineering

Level 4 Topic no: 15

Overview of Digital State Space Control

Contents

Reference:

"Modern Control Engineering", Ogata, Prentice Hall

Email: norbert.cheung@polyu.edu.hk **Web Site:** www.ncheung.com

1. Converting the plant into digital state space form

Let us start with an example, in the continuous time domain:

Example 6.1. Consider the inertial plant which is described in Example 3.2 by the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2}$$

The equivalent differential equation is

$$\ddot{y} = u(t)$$

Now define the two required state variables as

$$x_1 = y$$

and

$$x_2 = \dot{y} = \dot{x}_1$$

so the differential equations governing the system are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u(t)$$

or in matrix form,

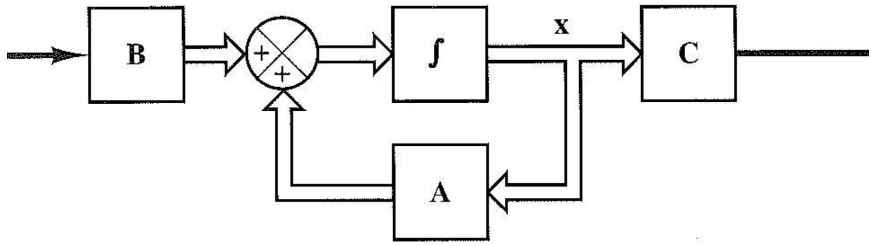
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

and the measurement equation is

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx}$$



Next, we consider another case, in which we want to convert the system into digital form:

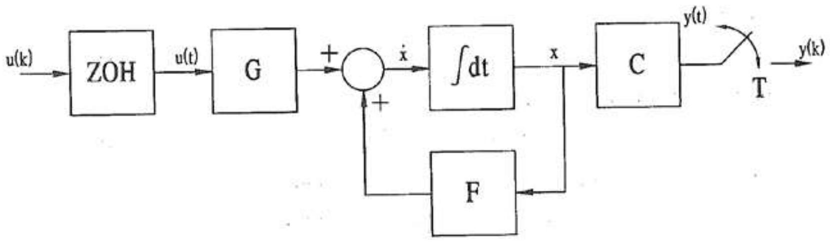
Example 6.2. Consider now the problem of the thermal system of Example 3.3. Let us define the temperatures as the state variables, so the state equations in matrix form are

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0.5 & -0.75 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} T_0(t) \\ u(t) \end{bmatrix}$$

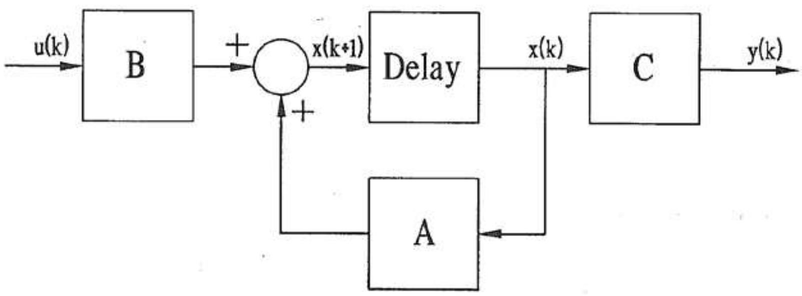
and if temperature $x_1(t)$ is the measured quantity on which to base control, the output equation is

$$y(t) = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Take the original state space equation above, add ZOH and sampler on front and back.



Through MATLAB/SIMULINK (or through solution of state equation calculation), we can transform the above block diagram into an equivalent control block diagram as below:



For T=0.25s:

The discrete-time state equations for this system are then

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.6227 & 0.3606 \\ 0.09016 & 0.8526 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.02516 \\ 0.1150 \end{bmatrix} u(k)$$

2. Jordan (diagonal) Canonical Form

We shall be concerned with systems specified as a z-domain transfer function of the form

$$H(z) = \frac{d_n z^n + d_{n-1} z^{n-1} + \cdots + d_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_0} = \frac{y(z)}{u(z)} \qquad (6.9.1)$$

$$H(z) = \frac{y(z)}{u(z)} = d_n + \frac{b_{n-1} z^{n-1} + \cdots + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_0} \qquad (6.9.5)$$

The d_n term represents a direct feedforward of the input sequence to the output sequence.

If the poles of the transfer function (6.9.5) are known, expression (6.9.5) can be written as

$$H(z) = d_n + \frac{b_{n-1} z^{n-1} + \cdots + b_0}{(z - p_1)(z - p_2) \cdots (z - p_n)} \qquad (6.9.6)$$

where we shall assume that all the p_i are distinct. Now let us make a partial fraction expansion of the second term of (6.9.6) to yield

$$H(z) = d_n + \frac{A_1}{z - p_1} + \cdots + \frac{A_n}{z - p_n} \qquad (6.9.7)$$

where the coefficients are given by the Heaviside method as

$$A_i = \lim_{z \to p_i} [(z - p_i) H(z)] \qquad i = 1, 2, \ldots, n \qquad (6.9.8)$$

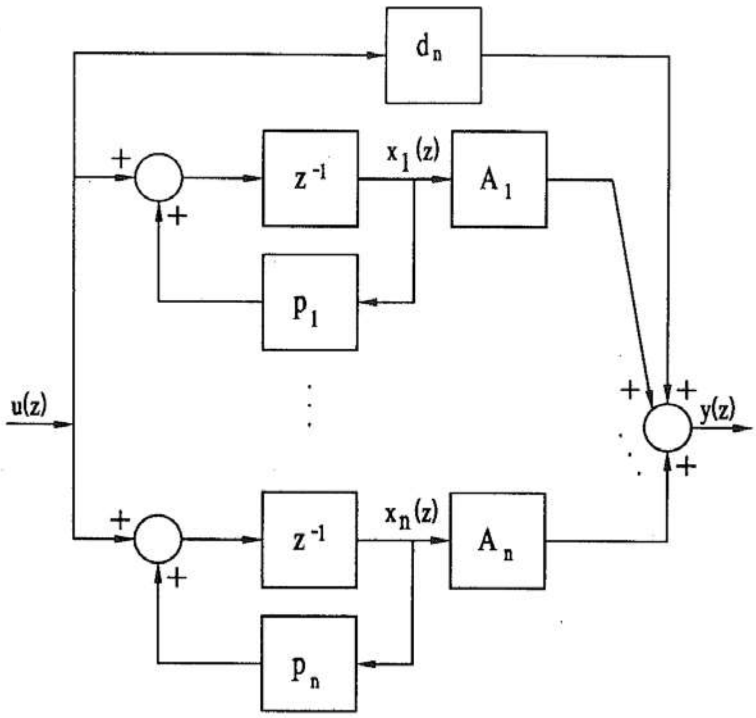
Let one of the terms of (6.9.7) be a transfer function between the input sequence and the ith-state variable

$$\frac{x_i(z)}{u(z)} = \frac{1}{z - p_i} \qquad i = 1, 2, \ldots, n \qquad (6.9.9)$$

$$x_i(k+1) = p_i x_i(k) + u(k) \qquad i = 1, 2, \ldots, n \qquad (6.9.10)$$

Then from Fig. 6.4 or expression (6.9.7), the output sequence is given by

$$y(k) = d_n u(k) + A_1 x_1(k) + \cdots + A_n x_n(k) \qquad (6.9.11)$$



The matrix forms of equations (6.9.10) and (6.9.11) are

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & \cdots & & 0 \\ 0 & p_2 & & & \vdots \\ \vdots & & \ddots & & \\ & & & p_{n-1} & 0 \\ 0 & & & 0 & p_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k) \tag{6.9.12}$$

and

$$y(k) = [A_1 \quad A_2 \quad \cdots \quad A_n] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + d_0 u(k) \tag{6.9.13}$$

Let us use an example to illustrate this:

Example 6.12. Find the parallel realization of the following transfer function:

$$H(z) = \frac{z^2 + 2z + 1}{z^2 + 5z + 6}$$

Long division of the transfer function yields

$$H(z) = 1 + \frac{-3z - 5}{z^2 + 5z + 6}$$

Let us make a partial fraction expansion of the second term to yield

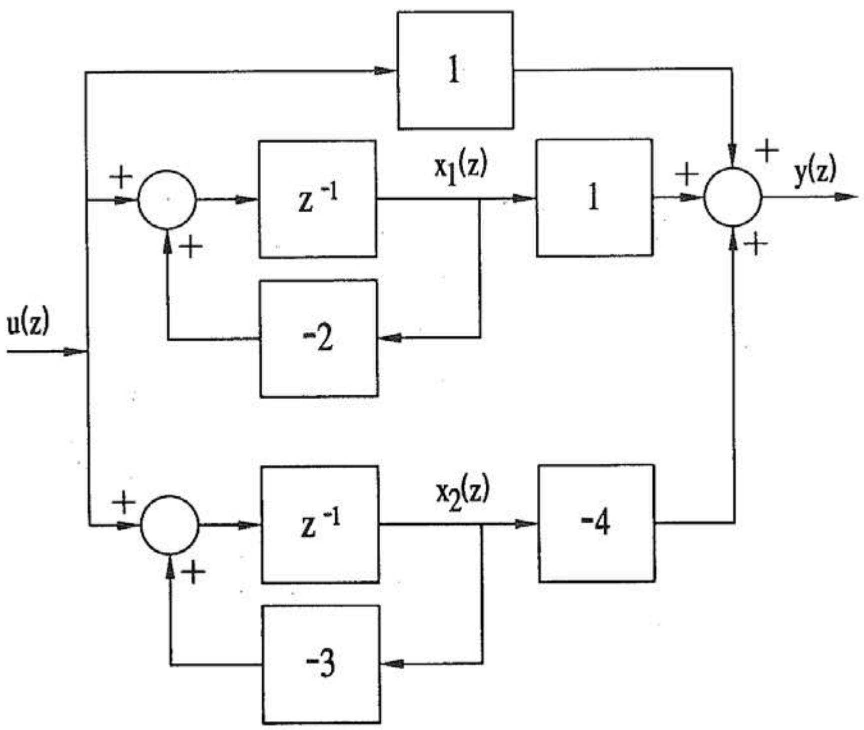
$$H(z) = 1 + \frac{1}{z + 2} + \frac{-4}{z + 3}$$

The state-variable form will be, from relations (6.9.12) and (6.9.13),

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

with an output relation

$$y(k) = [1 \quad -4] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + u(k)$$



3. Controllable Canonical Form

The controllable canonical form is sometimes also referred to as direct programming, although this should not be confused with what will be referred to as the direct form. We shall be concerned with transfer functions with numerators of lower order than denominators (otherwise, perform long division and form a feedforward of the input sequence). The transfer function of interest is given, after possible long division, to be

$$H(z) = \frac{b_{n-1}z^{n-1} + \cdots + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_0} + d_n = \frac{y(z)}{u(z)} \qquad (6.9.20)$$

Let us define an intermediate variable $w(z)$ such that

$$\frac{w(z)}{u(z)} = \frac{1}{z^n + a_{n-1}z^{n-1} + \cdots + a_0} \qquad (6.9.21)$$

and thus

$$y(z) = (b_{n-1}z^{n-1} + \cdots + b_0)w(z) + d_n u(z) \qquad (6.9.22)$$

Inverting (6.9.21) we get an nth-order difference equation

$$w(k+n) + a_{n-1}w(k+n-1) + \cdots + a_0 w(k) = u(k) \qquad (6.9.23)$$

Inversion of (6.9.22) yields an output equation

$$y(k) = b_{n-1}w(k+n-1) + \cdots + b_0 w(k) + d_n u(k)$$

Define the state variables as

$$\begin{aligned} x_1(k) &= w(k) \\ x_2(k) &= w(k+1) = x_1(k+1) \\ x_3(k) &= w(k+2) = x_2(k+1) \\ &\vdots \\ x_n(k) &= w(k+n-1) = x_{n-1}(k+1) \end{aligned} \qquad (6.9.24)$$

Substitution of relations (6.9.24) into expression (6.9.23) gives

$$x_n(k+1) = -a_{n-1}x_n(k) - a_{n-2}x_{n-1}(k) - \cdots - a_0 x_1(k) + u(k) \qquad (6.9.25)$$

with the other $(n-1)$ state equations defined by (6.9.24). The output relation becomes

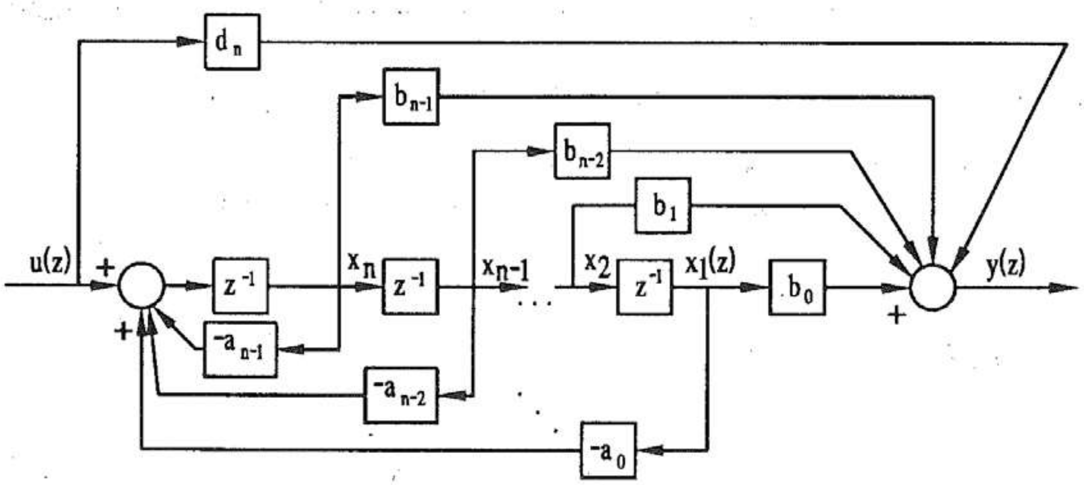
$$y(k) = b_{n-1}x_n(k) + b_{n-2}x_{n-1}(k) + \cdots + b_0 x_1(k) + d_n u(k) \qquad (6.9.26)$$

The matrix form of this representation is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & & & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ 1 \end{bmatrix} u(k) \tag{6.9.27}$$

with an output of

$$y(k) = [b_0 \quad \cdots \quad b_{n-1}] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + d_n u(k) \tag{6.9.28}$$



Example 6.14. Consider the transfer function of Example 6.12 when long division has been performed; it is of the form

$$H(z) = \frac{-3z - 5}{z^2 + 5z + 6} + 1$$

The state representation is given by inspection

$$x_1(k + 1) = x_2(k)$$

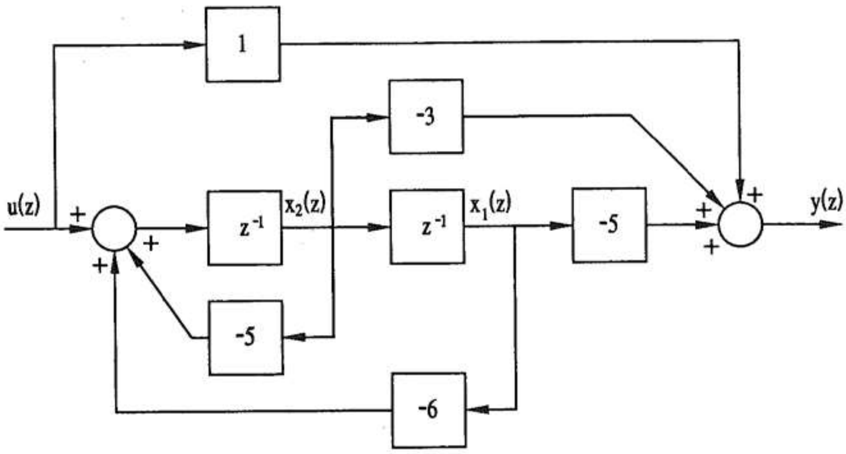
and from the denominator

$$x_2(k + 1) = -5x_2(k) - 6x_1(k) + u(k)$$

or in matrix form,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [-5 \quad -3] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + u(k)$$



4. Observable Canonical Form

The observable canonical form is sometimes referred to in the literature as the direct form. We start this discussion with the following form of the transfer function:

$$H(z) = \frac{d_n + d_{n-1}z^{-1} + \cdots + d_0 z^{-n}}{1 + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}} \tag{6.9.29}$$

Let us define the state variables as follows:

$$\begin{aligned} x_1(k + 1) &= -a_{n-1}y(k) + d_{n-1}u(k) + x_2(k) && \text{(a)} \\ x_2(k + 1) &= -a_{n-2}y(k) + d_{n-2}u(k) + x_3(k) && \text{(b)} \\ &\vdots && \vdots \\ x_n(k + 1) &= -a_0 y(k) + d_0 u(k) && (n) \end{aligned} \tag{6.9.30}$$

where

$$y(k) = x_1(k) + d_n u(k) \tag{6.9.31}$$

Expression (6.9.31) could be substituted into expressions (6.9.30) to give the following state equations:

$$
\begin{aligned}
x_1(k+1) &= -a_{n-1}x_1(k) + x_2(k) + (d_{n-1} - a_{n-1}d_n)u(k) \\
x_2(k+1) &= -a_{n-2}x_1(k) + x_3(k) + (d_{n-2} - a_{n-2}d_n)u(k) \\
&\vdots \\
x_n(k+1) &= -a_0x_1(k) + (d_0 - a_0d_n)u(k)
\end{aligned}
\tag{6.9.32}
$$

To verify the validity of this representation, let us substitute relations (6.9.30) successively into (6.9.31). Substitute the first relation of (6.9.30) into (6.9.31) to give

$$
\begin{aligned}
y(k) = &-a_{n-1}u(k-1) + d_{n-1}u(k-1) \\
&+ x_2(k-1) + d_nu(k)
\end{aligned}
\tag{6.9.33}
$$

Then substitute the second to give

$$
\begin{aligned}
y(k) = &-a_{n-1}y(k-1) + d_{n-1}u(k-1) - a_{n-2}y(k-2) \\
&+ d_{n-2}u(k-2) + x_3(k-2) + d_nu(k)
\end{aligned}
\tag{6.9.34}
$$

and after substitution of the last of the state equations, the resulting form is

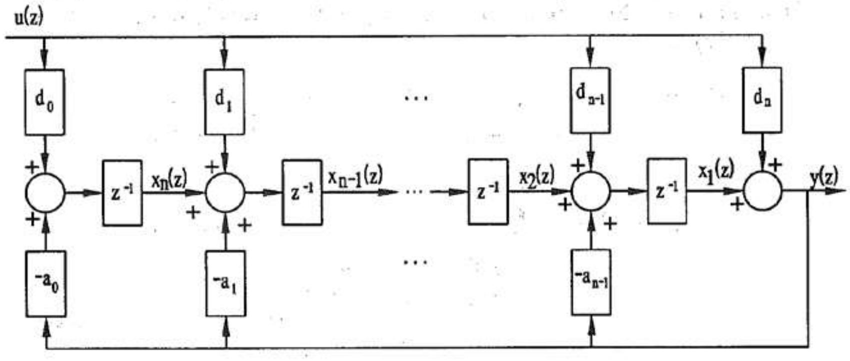
$$
y(k) = -\sum_{i=1}^{n} a_{n-i}y(k-i) + \sum_{i=0}^{n} d_{n-i}u(k-i) \tag{6.9.35}
$$

which is a difference equation which if z-transformed will give the transfer function of (6.9.29). The matrix form of the state-variable representation of (6.9.32) is

$$
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix}
=
\begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & & \\ \vdots & & & & 1 \\ -a_0 & 0 & & \cdots & 0 \end{bmatrix}
\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}
+
\begin{bmatrix} d_{n-1} - a_{n-1}d_n \\ d_{n-2} - a_{n-2}d_n \\ \vdots \\ d_0 - a_0d_n \end{bmatrix} u(k)
\tag{6.9.36}
$$

with an output expression

$$
y(k) = [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + d_nu(k) \tag{6.9.37}
$$



Example 6.15. Consider the transfer function of the previous examples:

$$H(z) = \frac{z^2 + 2z + 1}{z^2 + 5z + 6} = \frac{1 + 2z^{-1} + z^{-2}}{1 + 5z^{-1} + 6z^{-2}}$$

The state-variable form can be given by inspection to be

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} -3 \\ -5 \end{bmatrix} u(k)$$

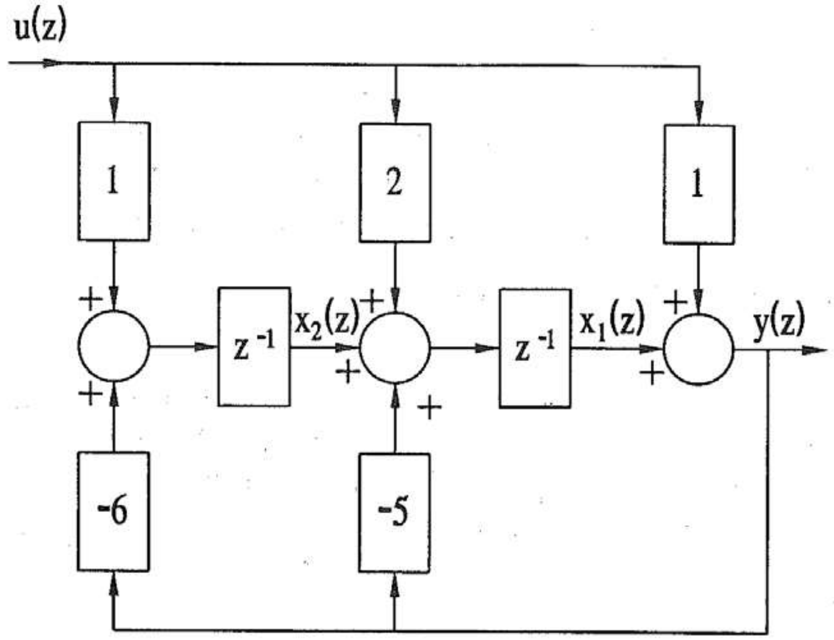


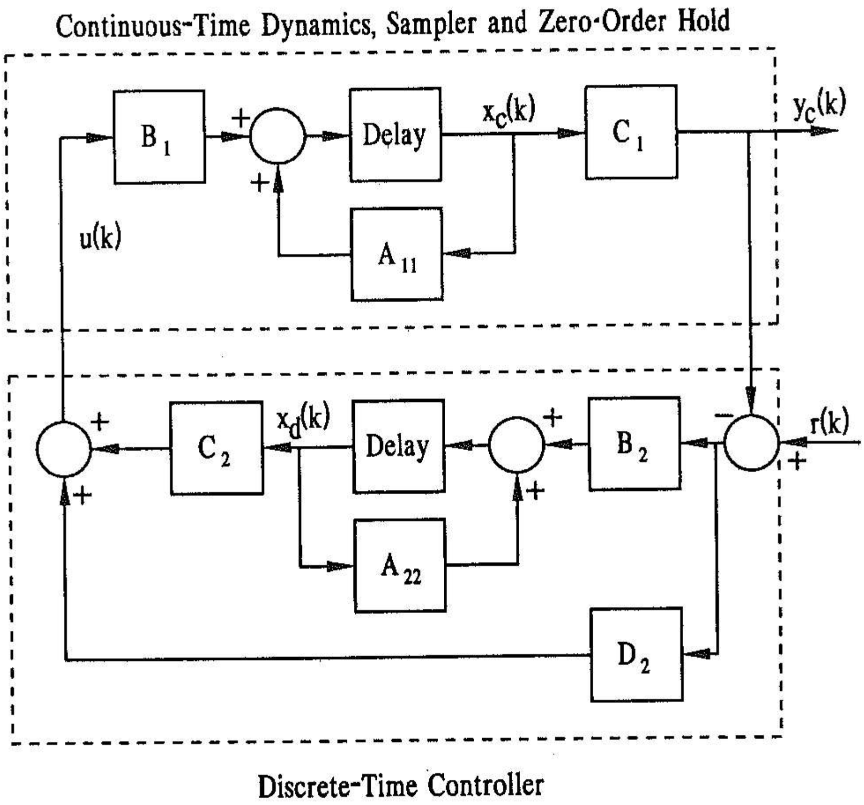
Figure 6.11. Observable canonical realization of Example 6.13.

with the output relation

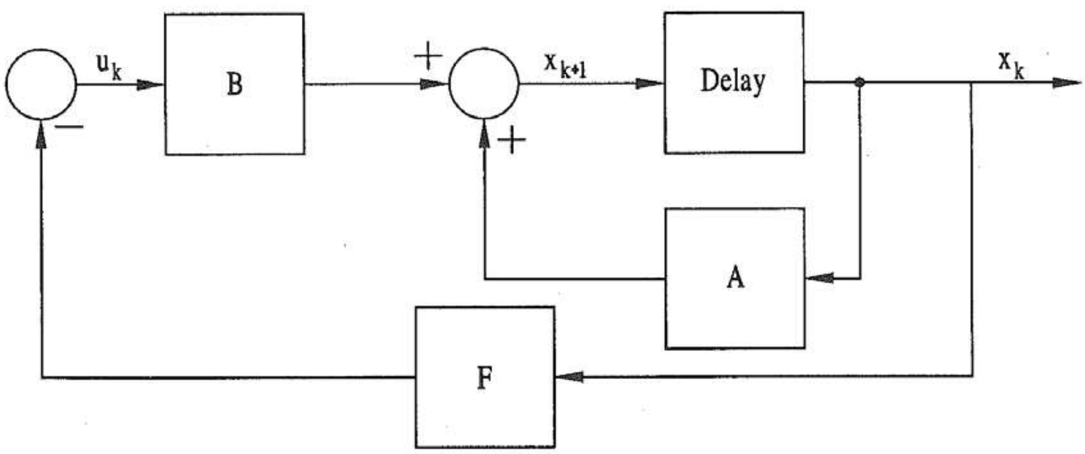
$$y(k) = x_1(k) + u(k)$$

5. Composite control system

General block diagram of a discretized continuous plant, being controlled by a digital controller.



General block diagram of using state feedback to control a linear discrete system:



Example of a complete state feedback control system:

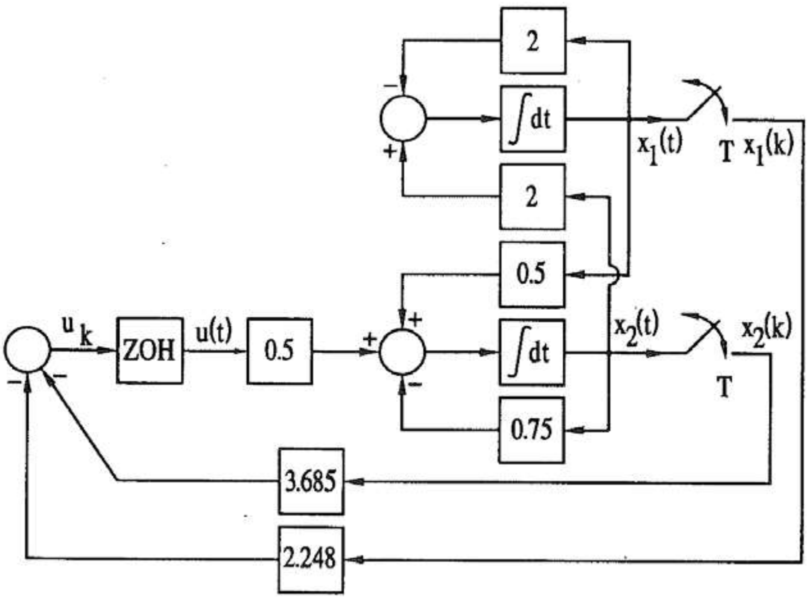


Figure 8.2. Complete state feedback control system.

But in practice, there are problems:

Often in a large system it is unfeasible to measure all the state variables, while sometimes we can only measure a few or a linear combination of the states. For feedback control we would like to be able to reconstruct the state variables from the measured variables.

An alternative approach is that of estimating the state based on measurements that are a linear combination of the states and then generating the control effort based on the estimated states. We shall assume that at best only some of the states are measured directly. We want to design a state estimator or observer (Luenberger, 1964) which when given a sequence $\mathbf{y}_k$ and the input $\mathbf{u}_k$ reconstructs an estimate of the $\mathbf{x}_k$ sequence. An observer or state estimator is another dynamic system that has inputs $\mathbf{u}_k$ and $\mathbf{y}_k$, the output of which is an estimate of $\mathbf{x}_k$ which we shall call $\hat{\mathbf{x}}_k$.

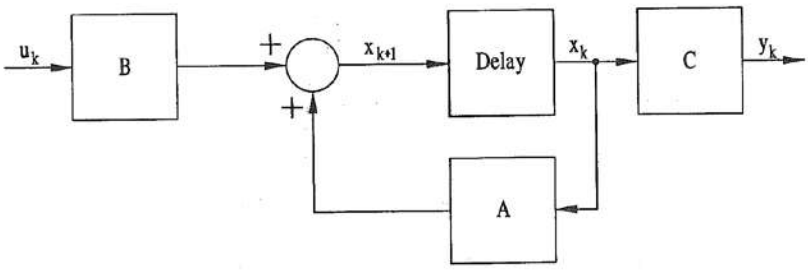
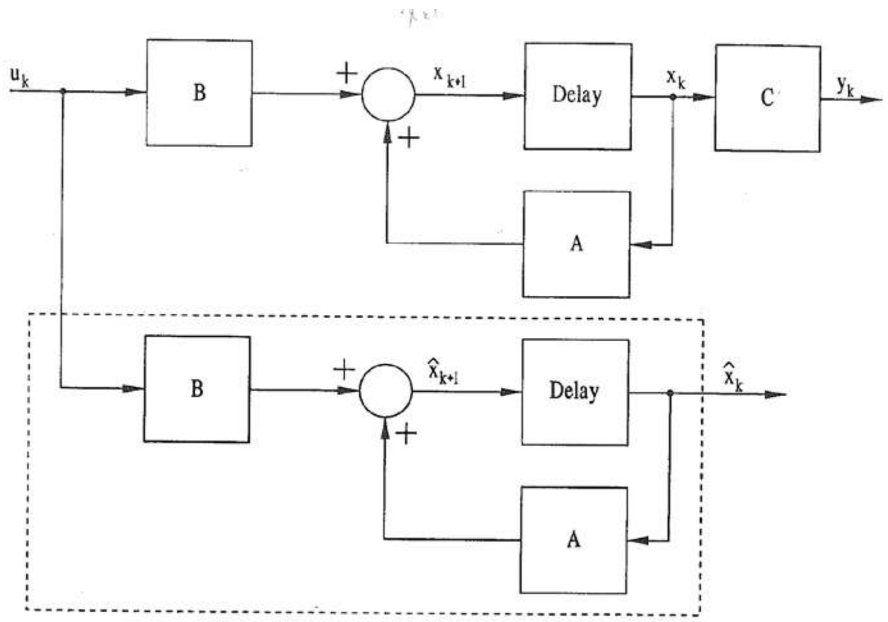


Figure 8.4. Discrete linear system.



Open-Loop Observer

In the scheme proposed above, the state was reconstructed without regard to the measurement sequence $\mathbf{y}_k$. Surely we can devise a scheme that will do a better job by employing the measurements in the estimation task.

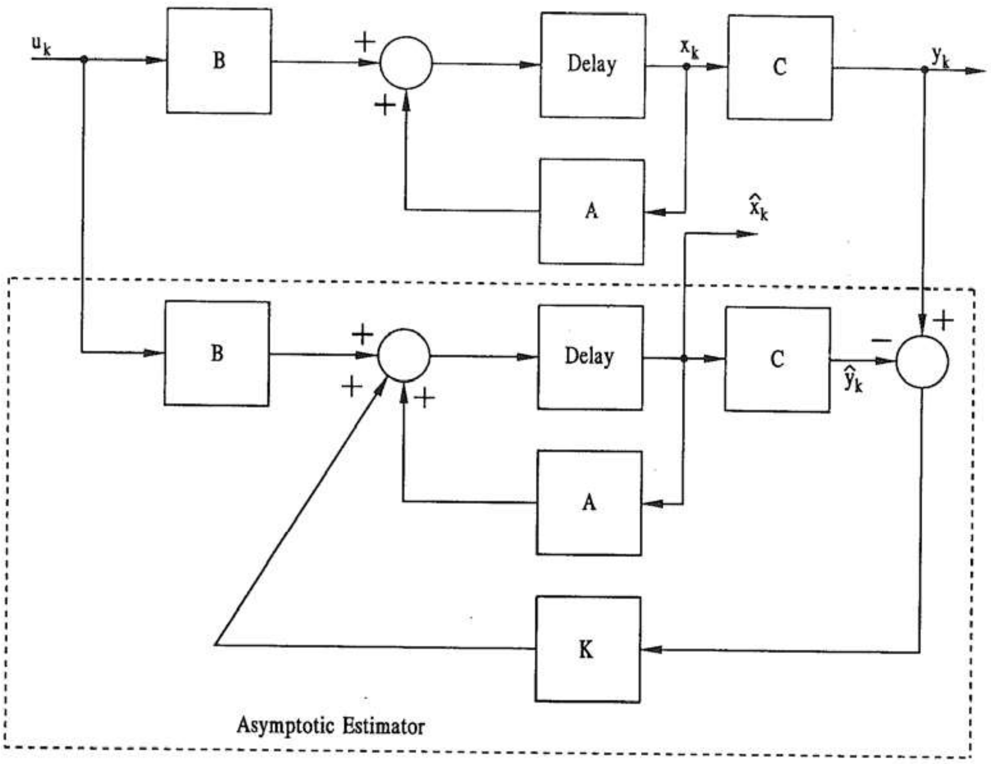


Figure 8.6. Asymptotic prediction estimator.

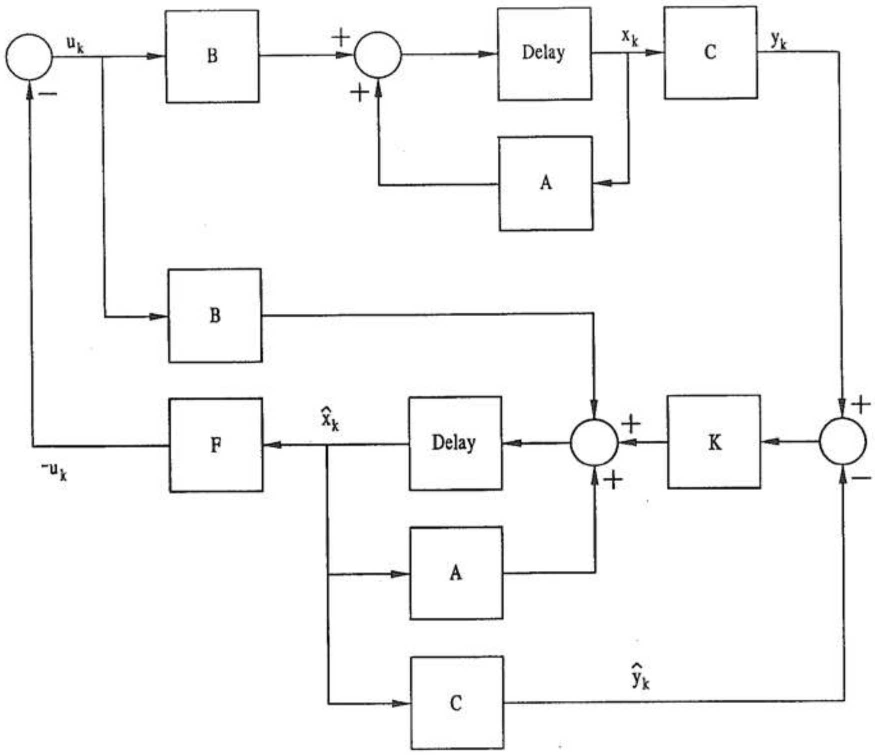


Figure 8.7. Feedback-controlled plant using the estimated state.

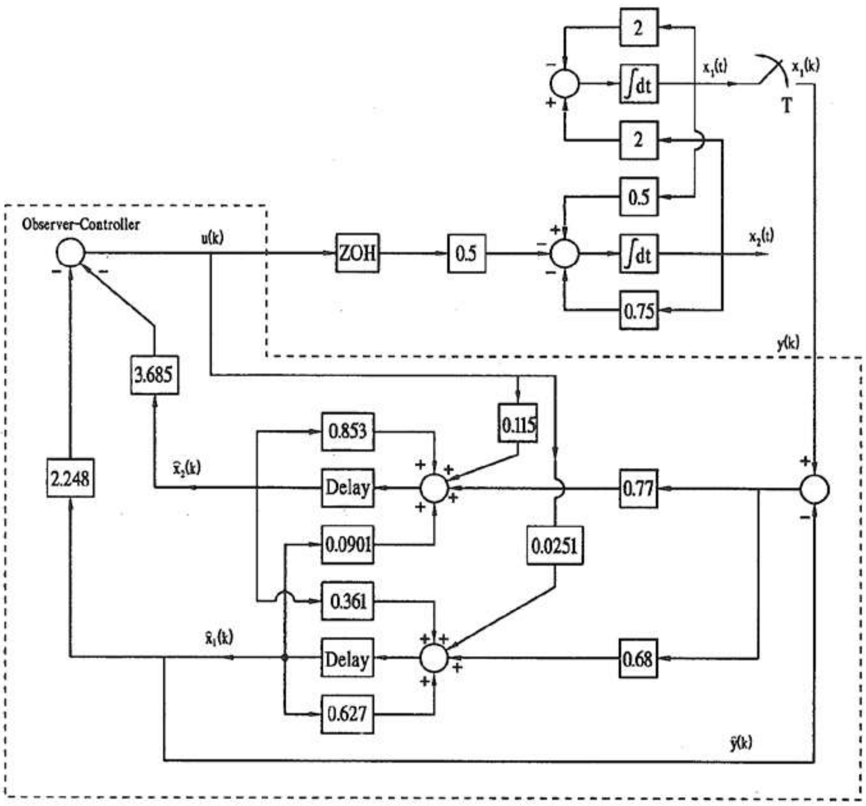


Figure 8.11. Complete estimated state feedback control system for thermal plant.

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