Dr. Norbert Cheung's Series in Electrical Engineering

Level 4 Topic no: 12

Digital Controller Design

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Reference:

"Modern Digital Control Systems, 2nd edition" Raymond G. Jacquot, Longman.

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1. Control system design specifications

Time domain specification

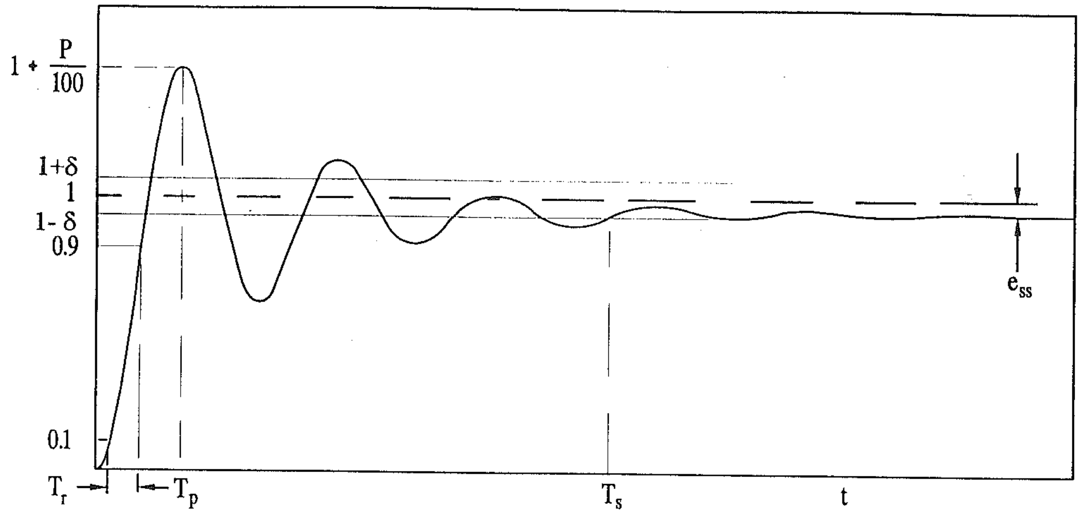


Figure 3.18. Step response of a high-performance control system.

P: Percentage overshoot

T_p: Peak time

T_s: Settling time

T_r: Rise time

e_{ss}: steady state error

Frequency domain specification

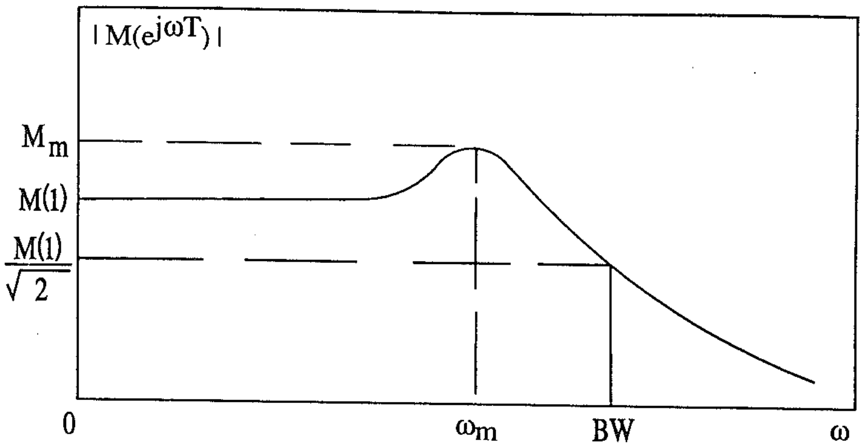


Figure 3.20. Typical high-performance control system frequency response.

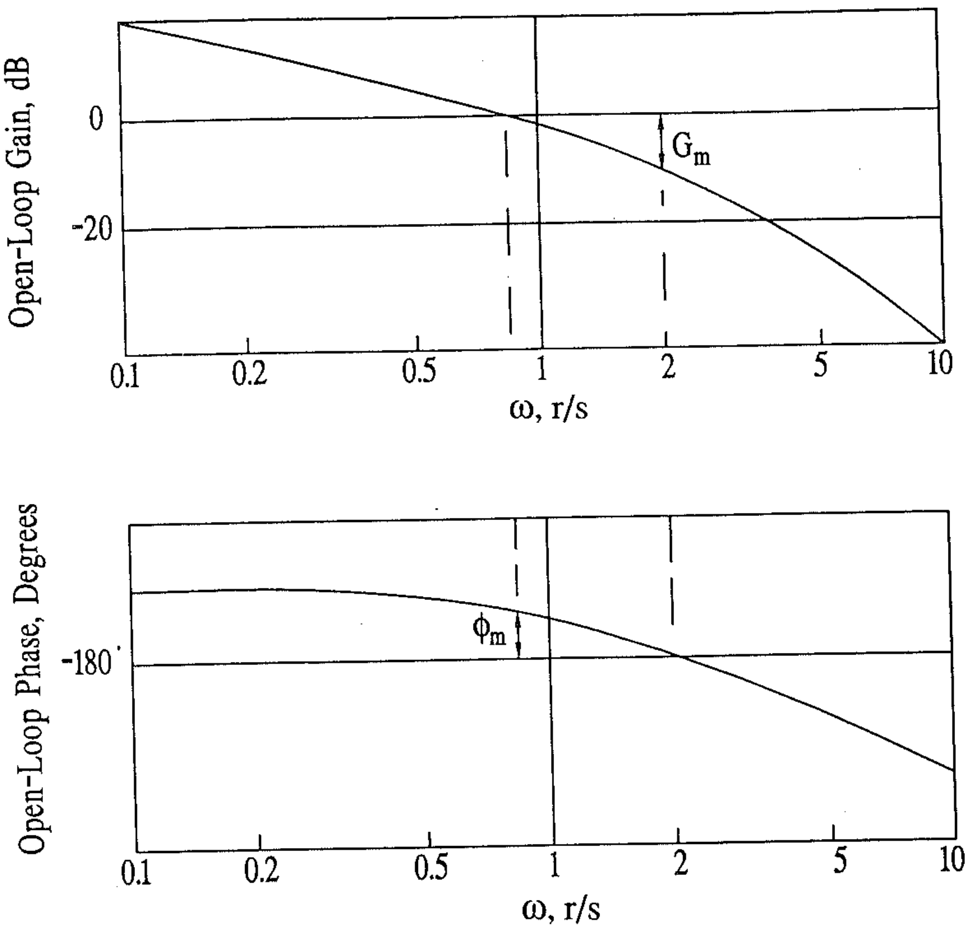
$$M(e^{j\omega T}) = M(z)\big|_{z=e^{j\omega T}} \qquad (3.7.1)$$

A plot of $|M(e^{j\omega T})|$ for a typical high-performance control system is given in Fig. 3.20. It is common to specify quantitatively the desired properties of this magnitude function.

Commonly specified quantities are indicated in Fig. 3.20 and are:

1. System bandwidth BW
2. Resonant peak gain M_m
3. Peak frequency ω_m

Gain margin and phase margin of a system



The general control law to be implemented by the digital controller is:

$$\begin{aligned} u(k) = {} & a_n e(k) + a_{n-1} e(k-1) + \cdots + a_0 e(k-n) \\ & + b_{n-1} u(k-1) + \cdots + b_0 u(k-n) \end{aligned} \qquad (3.4.1)$$

The corresponding transfer function can be written in z-domain as:

$$D(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0}{z^n - b_{n-1} z^{n-1} - \cdots - b_1 z - b_0} \qquad (3.4.2)$$

2. Elementary z-domain design considerations

We have shown that a continuous-time plant driven by a zero-order hold and followed by an output sampler can be represented by a discrete transfer function $G(z)$. The process of conventional digital control system design amounts to the synthesis of the control algorithm, reflected in the compensator transfer function $D(z)$, to yield acceptable closed-loop dynamics or error character.

In Section 3.3 we have shown that the plant pole locations in the z-plane are related to those of the continuous-time plant in the s-plane by

$$z_i = e^{s_i T} \tag{3.8.1}$$

In the design process we are often interested in systems with complex-conjugate poles whose s-plane locations are given by the roots of the quadratic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{3.8.2}$$

where ω_n is the undamped natural frequency and ζ is the damping ratio. For $\zeta < 1$ these roots are given by the quadratic formula to be

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} \tag{3.8.3}$$

which are illustrated in Fig. 3.23. The z-plane images of such s-plane poles are given by the mapping (3.8.1), or

$$z_{1,2} = e^{-\zeta\omega_n T \pm j\omega_n T\sqrt{1-\zeta^2}} \tag{3.8.4}$$

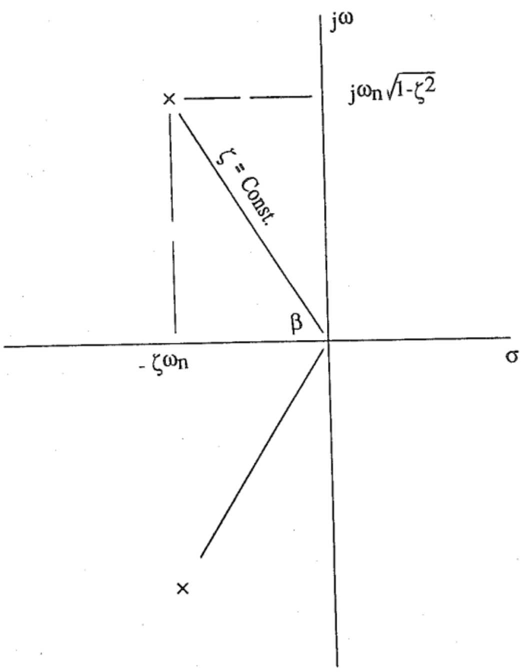


Figure 3.23. Complex-conjugate poles in the s-plane with a line of constant damping ratio shown.

In the s-plane we are commonly interested in fixing the parameter ζ so as to control overshoot and settling time of the closed-loop control system. Lines of constant damping ratio ζ in the s-plane are radial lines in the left half of the s-plane, as illustrated in Fig. 3.23. The angle β is related to the damping ratio by

$$\zeta = \cos \beta \tag{3.8.5}$$

The radial location of the poles in the z-plane is given by (3.8.4) to be

$$R = e^{-\zeta \omega_n T} \tag{3.8.6}$$

and the angular location is given to be

$$\theta = \omega_n T \sqrt{1 - \zeta^2} \tag{3.8.7}$$

Now if we solve (3.8.7) for $\omega_n T$ and substitute into (3.8.6), we get the radial pole location as a function of the angular location and the parameter ζ, so

$$R = \exp\left(\frac{-\zeta\theta}{\sqrt{1 - \zeta^2}}\right) \tag{3.8.8}$$

This is the equation of a logarithmic spiral, the "tightness" of which is

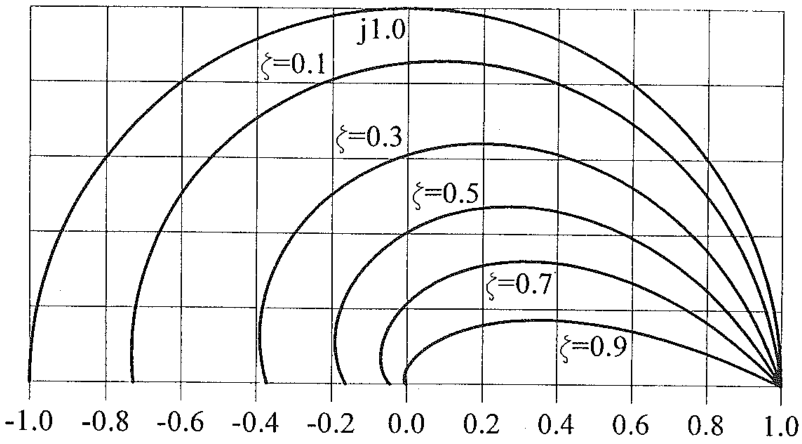


Figure 3.24. Constant damping ratio (ζ) loci in the z-plane.

controlled by the parameter ζ. These loci for various values of ζ are shown in Fig. 3.24.

Example 3.6. Consider the thermal control system considered in Example 3.3, in which the transfer function from the input of the zero-order hold to the sampler output was found to be

$$G(z) = \frac{0.025(z + 0.816)}{(z - 0.952)(z - 0.528)} = \frac{T_1(z)}{U(z)}$$

For a proportional controller $D(z) = K$, the closed-loop characteristic equation is

$$(z - 0.952)(z - 0.528) + K(0.025)(z + 0.816) = 0$$

On expanding and combining like terms in z, the result is

$$z^2 - z(1.48 - 0.025K) + (0.5026 + 0.0204K) = 0$$

The closed-loop pole locations are then

$$z_{1,2} = 0.74 - 0.0125K \pm \sqrt{(0.74 - 0.0125K)^2 - (0.5026 + 0.0204K)}$$

and if we consider only the complex roots,

$$z_{1,2} = 0.74 - 0.0125K \pm j\sqrt{0.5026 + 0.204K - (0.74 - 0.0125K)^2}$$

For a pole location on the unit circle $|z_i|^2 = 1$, or

$$1 = (0.74 - 0.0125K)^2 + (0.5026 + 0.0204K) - (0.74 - 0.0125K)^2$$

or

$$1 = 0.5026 + 0.020K$$

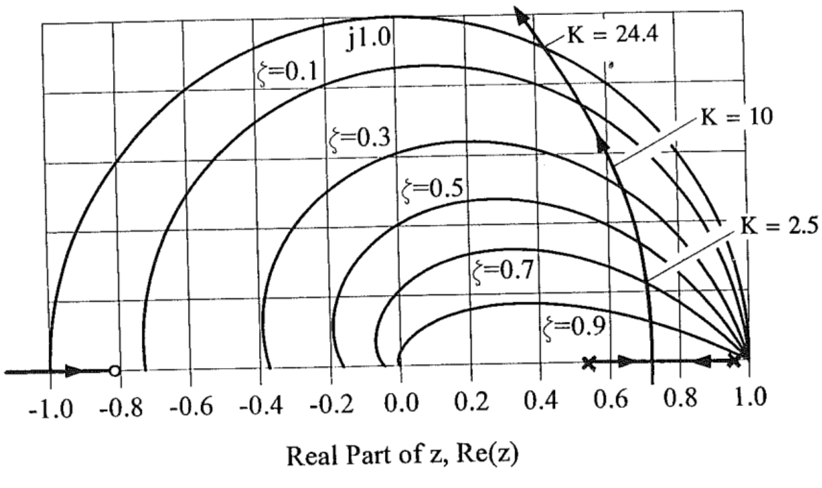
Then the critical value of the gain parameter is

$$K_{crit} = 24.38$$

The root locus for this system is shown in Fig. 3.25, with several values of the parameter K being given. For a damping ratio of about 0.7, it appears that the value of K is 2.5. The proportional control algorithm corresponding to this set of closed-loop locations is then given by

$$u(k) = 2.5[r(k) - T_1(k)] = 2.5e(k)$$

We have, in fact, adjusted the damping ratio such that there would not be excessive overshoot according to Fig. 3.21.



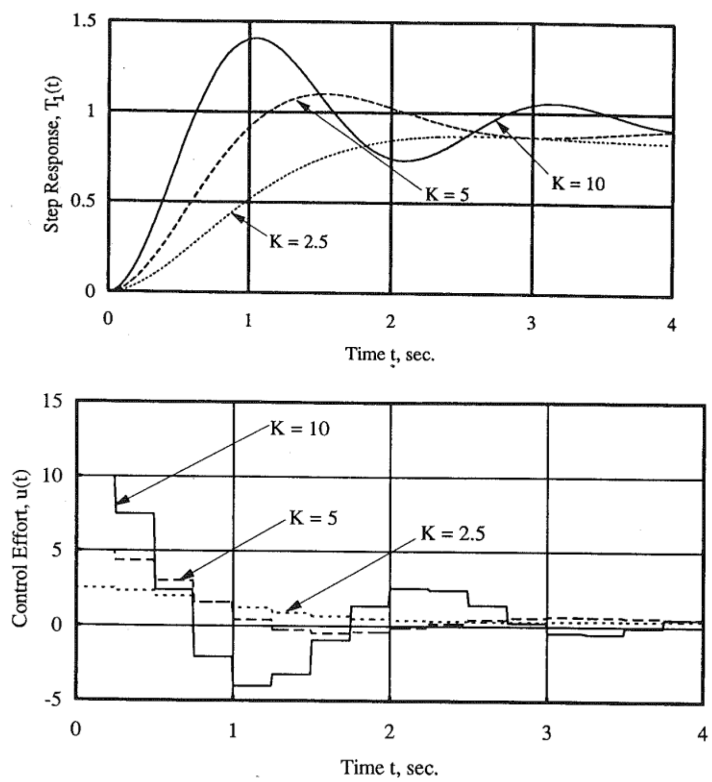


Figure 3.26. Control effort and step response of proportionally controlled thermal system.

3. Effect of disturbance on the closed loop system

Let us consider now the system of Fig. 3.27, in which the reference input is zero, which makes this system a regulator. The system is, however, subject to a disturbance $W(s)$ which may enter the control loop directly or through some dynamics represented by transfer function $G_2(s)$. A discrete-time control algorithm is represented by transfer function $D(z)$, while the continuous-time plant dynamics are represented by $G_p(s)$. The discrete-time transfer function between the discrete controller and the sampler output will be represented by $G(z)$ and can be found by the methods of Section 3.2. The output of this system $Y(z)$ can be thought of as being composed of two parts, one due to $U(z)$ and the other due to $W(s)$. Since the system is linear, the principle of superposition is applicable and the output $Y(z)$ is

$$Y(z) = G(z)U(z) + \mathfrak{Z}\mathcal{L}^{-1}\left[G_2(s)G_p(s)W(s)\right] \tag{3.9.1}$$

But the control effort $U(z)$ is given for zero reference input as

$$U(z) = -D(z)Y(z) \tag{3.9.2}$$

and substitution of this into (3.9.1) gives

$$Y(z) = -G(z)D(z)Y(z) + \mathfrak{Z}\mathcal{L}^{-1}\left[G_1(s)G_p(s)W(s)\right] \tag{3.9.3}$$

and solution for the z-domain output yields

$$Y(z) = \frac{\mathfrak{Z}\mathcal{L}^{-1}[G_2(s)G_p(s)W(s)]}{1 + G(z)D(z)} \tag{3.9.4}$$

If the time-domain response sequence is desired, this must be inverted to yield the $y(k)$ sequence. If, however, only the final value of the output

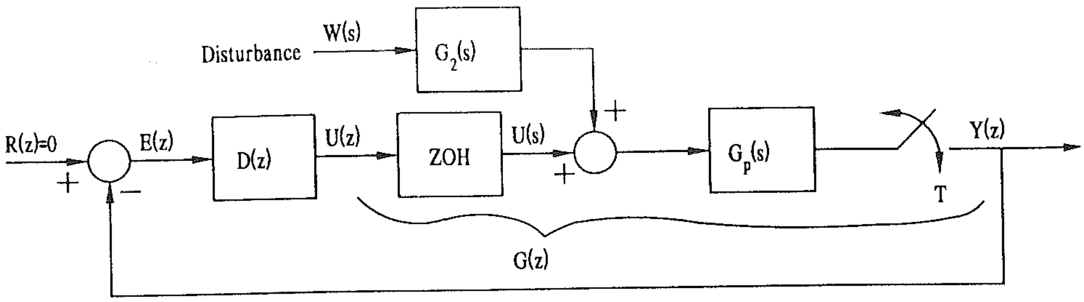


Figure 3.27. Closed-loop digital control system with continuous disturbance.

sequence is desired to a step disturbance, the final value theorem is applicable. These concepts are best illustrated by means of an example.

Example 3.7. Consider now the step disturbance of the first-order plant with transfer function

$$G_p(s) = \frac{1}{s + 1}$$

with a sampling interval of $T = 0.2$ s. The transfer function $G(z)$ was given in Examples 3.1 and 3.5 to be

$$G(z) = \frac{0.1813}{z - 0.8187}$$

The step disturbance of magnitude A is

$$W(s) = \frac{A}{s}$$

The disturbance input transfer function is unity or

$$G_2(s) = 1$$

and for a proportional controller the transfer function is

$$D(z) = K$$

The z-domain response is given by expression (3.9.4) to be

$$Y(z) = \frac{\mathfrak{Z}\mathcal{L}^{-1}[A/s(s + 1)]}{1 + K(0.1813)/(z - 0.8187)}$$

and carrying out the operations indicated in the numerator gives

$$Y(z) = \frac{A[z/(z - 1)] - z/(z - 0.8187)}{1 + K(0.1813)/(z - 0.8187)}$$

Rationalizing this fraction yields

$$Y(z) = \frac{A(0.1813)z}{(z - 1)[(z - 0.8187) + K(0.1813)]}$$

We know from Example 3.5 that the closed-loop pole position is controlled by selection of the gain parameter K. The steady-state error due to the step disturbance can be calculated by the application of the final value theorem, or

$$y(\infty) = \lim_{z\to 1} \frac{A(0.1813)}{(z - 0.8187) + K(0.1813)}$$

and carrying out the indicated operations we get

$$y(\infty) = \frac{A}{1 + K}$$

and hence the larger the value of controller gain K, the smaller the steady-state error. There is, however, an upper limit on gain K due to the instability

4. PID direct digital control algorithm

In the study of continuous-time control systems it was found that if proportional control is employed, a steady-state error was necessary in order to have a steady-state output. It was also found that if an integrator replaces the proportional controller, the steady-state error can be made zero for a steady output. Often, the introduction of the integrator into the loop will create instability or, at best, poor dynamic character, which manifests itself as overshoot and excessive ringing in the output. Several compromises are possible, one of which uses an actuating signal that has one component proportional to the error signal and the other proportional to the integral of the error. This combination reduces the steady-state error to zero and often yields acceptable dynamics. If further improvement in dynamics is required, a differentiator that is sensitive to error rate can be included in parallel with the other two devices. This continuous-time control scheme is shown in Fig. 4.1 with a continuous-time plant. The time-domain relation for the controller is

$$u(t) = K_p e(t) + K_i \int_0^t e\,dt + K_d \frac{de}{dt} \tag{4.2.1}$$

The associated s-domain transfer function for the controller is given by Laplace transformation of (4.2.1) to yield

$$D(s) = \frac{U(s)}{E(s)} = \frac{K_d s^2 + K_p s + K_i}{s} \tag{4.2.2}$$

where the choice of the constants will determine the system dynamics.

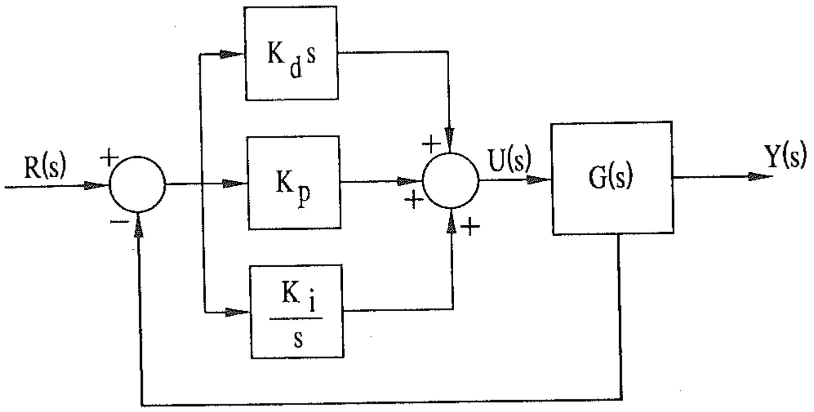


Figure 4.1. Proportional plus integral plus derivative (PID) control.

Since this technique has proven so useful for continuous-time control systems, it is desirable to develop a digital control algorithm that will be of similar character to the continuous-time scheme given above. We shall approximate the integral with trapezoidal integration and the derivative with a backward difference equation, or

$$u_k = K_p e_k + K_i T\left[\frac{1}{2}(e_0 + e_1) + \frac{1}{2}(e_1 + e_2) + \cdots + \frac{1}{2}(e_{k-1} + e_k)\right] + \frac{K_d}{T}(e_k - e_{k-1}) \quad (4.2.3)$$

and the algorithm for the previous step in time is written with the appropriate shift in subscripts, or

$$u_{k-1} = K_p e_{k-1} + K_i T\left[\frac{1}{2}(e_0 + e_1) + \cdots + \frac{1}{2}(e_{k-2} + e_{k-1})\right] + \frac{K_d}{T}(e_{k-1} - e_{k-2}) \quad (4.2.4)$$

Subtraction of (4.2.4) from (4.2.3) yields

$$u_k - u_{k-1} = K_p(e_k - e_{k-1}) + \frac{K_i T}{2}(e_{k-1} + e_k) + \frac{K_d}{T}(e_k - 2e_{k-1} + e_{k-2}) \quad (4.2.5)$$

or combining like terms yields

$$u_k = u_{k-1} + \left(K_p + \frac{K_i T}{2} + \frac{K_d}{T}\right)e_k + \left(\frac{K_i T}{2} - K_p - \frac{2K_d}{T}\right)e_{k-1} + \frac{K_d}{T}e_{k-2} \quad (4.2.6)$$

which is the direct digital control algorithm. By taking the z-transform of the difference equation [Eq. (4.2.6)] we can determine the compensator transfer function which will perform the proportional plus integral plus derivative (PID) control function:

$$D(z) = \frac{U(z)}{E(z)} = \frac{\alpha + \beta z^{-1} + \gamma z^{-2}}{1 - z^{-1}} = \frac{\alpha z^2 + \beta z + \gamma}{z(z - 1)} \tag{4.2.7}$$

where

$$\alpha = K_p + \frac{K_i T}{2} + \frac{K_d}{T} \tag{4.2.8}$$

$$\beta = \frac{K_i T}{2} - K_p - \frac{2K_d}{T} \tag{4.2.9}$$

and

$$\gamma = \frac{K_d}{T} \tag{4.2.10}$$

It is interesting to note that the transfer function of (4.2.7) has a quadratic numerator that may be chosen such as to cancel two slow, troublesome poles of a plant. As long as those poles are interior to the unit circle, the cancellation need not be exact and the resultant root-locus branches will contribute little to the closed-loop response because of the zeros of the closed-loop transfer function being coincident with the zeros of the controller.

If only proportional plus integral action is required, it is a simple matter to let K_d be zero, which yields a simplified transfer function for the compensator:

$$D(z) = \frac{U(z)}{E(z)} = \frac{\alpha z + \beta}{z - 1} \tag{4.2.11}$$

where

$$\alpha = K_p + \frac{K_i T}{2} \tag{4.2.12}$$

and

$$\beta = \frac{K_i T}{2} - K_p \tag{4.2.13}$$

Example 4.1

Example 4.1. Consider the thermal system of Examples 3.3 and 3.6, for which we would like to design a proportional plus integral (PI) controller so we will have zero steady-state error to a step input. The plant transfer function is

$$G(z) = \frac{0.025(z + 0.816)}{(z - 0.952)(z - 0.528)}$$

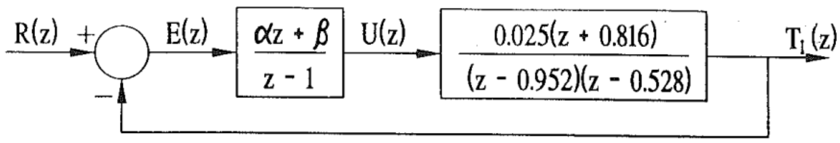


Figure 4.2. Proportional plus integral (PI) control of thermal plant.

We shall choose to locate the zero of the PI controller so as to cancel the slow pole of the plant, so the compensator transfer function is

$$D(z) = \frac{\alpha z + \beta}{z - 1} = \frac{\alpha(z + \beta/\alpha)}{z - 1}$$

where $\beta/\alpha = -0.952$. The whole feedback system structure is shown in Fig. 4.2, and the root-locus diagram for this example for variable controller gain α is shown in Fig. 4.3.

A trial-and-error design gives a closed-loop damping ratio of $\zeta = 0.7$ for a controller gain of $\alpha = 2.5$. The resulting control algorithm is

$$u_k = u_{k-1} + 2.5(e_k - 0.952e_{k-1})$$

The step response and associated control effort for this system are given in Fig. 4.4(a) and (b), respectively. It should be noted that due to the

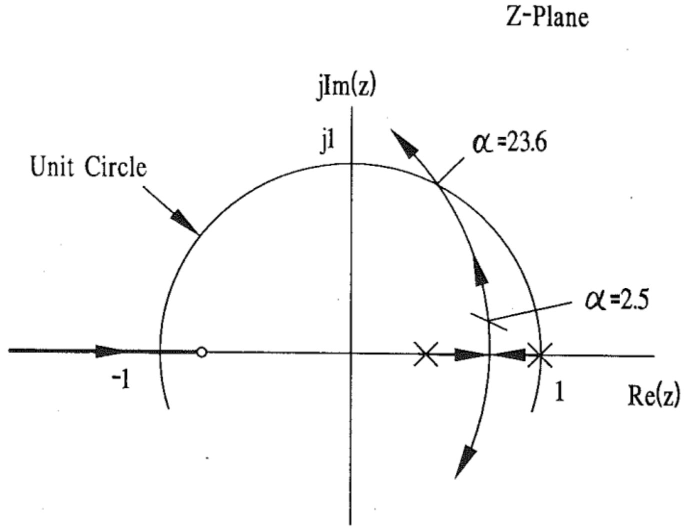


Figure 4.3. Root locus for PI control of the thermal system.

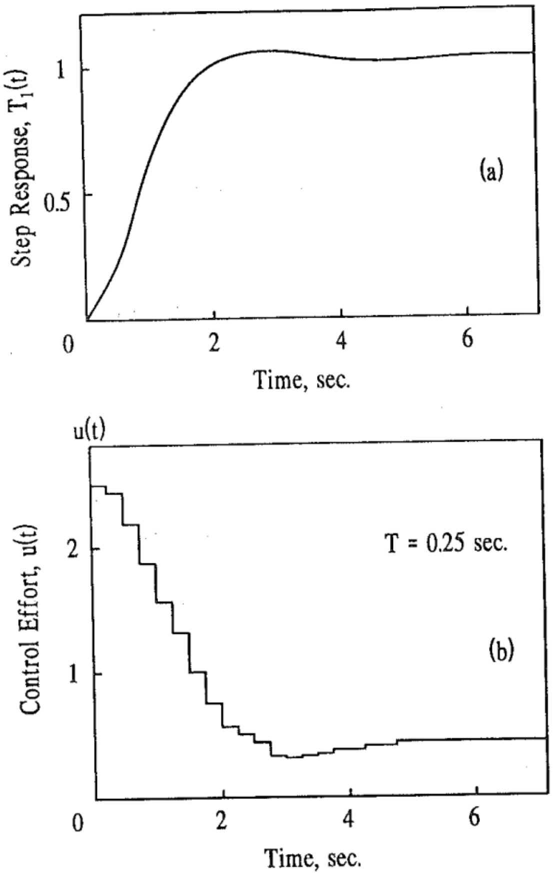


Figure 4.4. Step response (a) and control effort (b) for the PI control of the thermal plant.

integral action the steady-state error is zero and that the step response is reasonable with an overshoot of 4%; however, because the root locus was shifted to the right, the response is slightly slower than in proportionally controlled cases. To have both fast response and zero steady-state error a higher-order compensator will be necessary.

5. Ziegler Nichols tuning

The technique is based on experimental evaluation of several parameters associated with the step response of the plant to be controlled. The first task is to evaluate the plant response to a sudden constant change in the variable to be used as the control effort. Such an experimental response is illustrated in Fig. 4.5.

From this step response it is necessary to find two quantities. The first is R, the slope of the response curve at the inflection point, which is an indication of the speed of response. The second is the time L, which is a measure of the lag of the plant. For a PI controller the tuning strategy relates K_p and K_i to the values of R and L as

$$K_p = \frac{0.9}{RL} \tag{4.3.1}$$

and

$$K_i = \frac{1}{3.3L} K_p = \frac{0.272}{RL^2} \tag{4.3.2}$$

If the PID control strategy is chosen, the tuning equations are

$$K_p = \frac{1.2}{RL} \tag{4.3.3}$$

$$K_i = \frac{1}{2L} K_p = \frac{0.6}{RL^2} \tag{4.3.4}$$

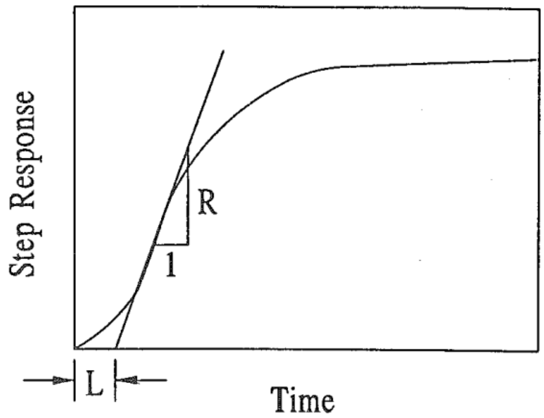


Figure 4.5. Step response of a typical plant.

and

$$K_d = 0.5LK_p = \frac{0.6}{R} \tag{4.3.5}$$

From these expressions it is possible to evaluate the parameters α, β, and γ of the controller.

Example

Example 4.2. Use the Ziegler–Nichols tuning strategy to design a PI controller for the thermal plant of Example 4.1 for which the continuous-time transfer function is

$$G(s) = \frac{1}{s^2 + 2.75s + 0.5} = \frac{T_1(s)}{U(s)}$$

The step response of this system is illustrated in Fig. 4.6.

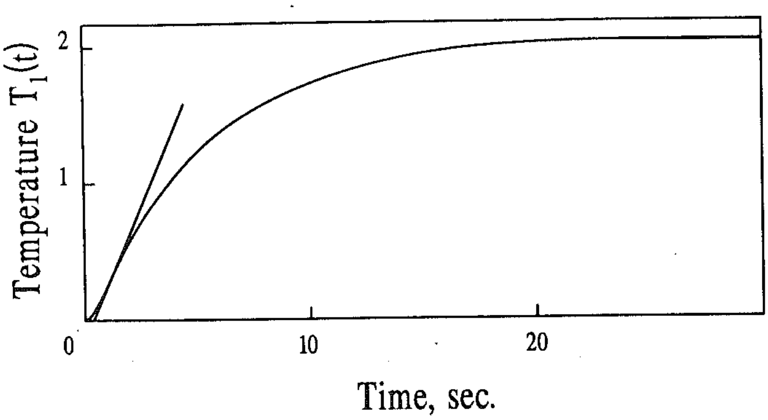


Figure 4.6. Step response of thermal plant.

Evaluation of the slope at the inflection point yields a slope of $R = 0.37$ and the lag parameter indicated is $L = 0.47$ s. Calculation of K_p and K_i yield $K_p = 5.175$ and $K_i = 3.336$. From these values and a sampling interval of $T = 0.25$ s the two controller coefficients are $\alpha = 5.592$ and $\beta = -4.758$. And thus the resultant controller is

$$D(z) = \frac{5.592(z - 0.85)}{z - 1}$$

The associated control algorithm is

$$u(k) = u(k - 1) + 5.592e(k) - 4.758e(k - 1)$$

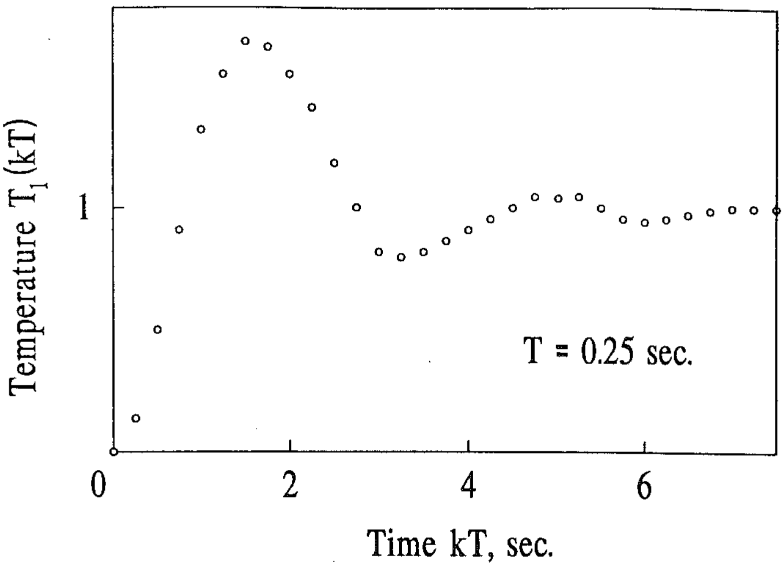


Figure 4.7. Closed-loop step response of PI controlled thermal system with Ziegler–Nichols tuning.

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