Dr. Norbert Cheung's Series in Electrical Engineering

Level 4 Topic no: 11

Digital Control Implementation and Characterization

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1. Continuous time plant with digital control
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Reference:

"Modern Digital Control Systems, 2nd edition" Raymond G. Jacquot, Longman.

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1. Continuous time plant with digital control

In Chapter 2 we developed the mathematics of discrete-time systems, but the systems examined there were essentially open loop in nature. In this chapter we explore techniques for digital control system design that are similar to those conventionally employed in continuous-time control system design. The cornerstone of this chapter is the z-transform and the z-domain representation of the discrete-time dynamics of the plant and proportional controllers which are inserted into the control loop to improve the system dynamics and error character.

Zero order hold (ZOH), A/D, and D/A

We shall model the D/A converter as a zero-order hold because the device output has no slope information. In short, a zero-order hold (ZOH) is a

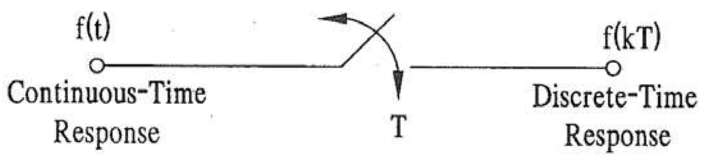

Figure 3.1. Symbol for analog-to-digital converter.

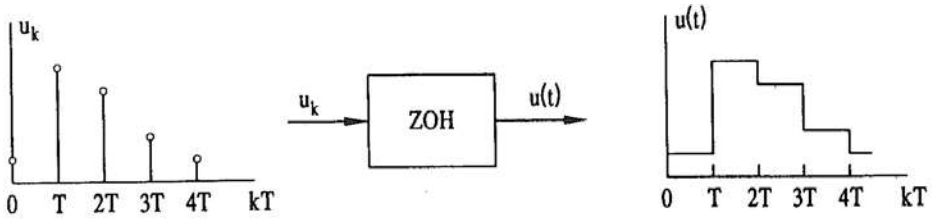

Figure 3.2. Zero-order-hold operation.

device such that given a uniformly spaced sequence of numbers u_0, u_1, u_2, . . . , a stairstep output $u(t)$ is produced such that

$$u(t) = u_k \qquad kT \leq t < (k + 1)\, T$$

A digitally controlled continuous time plant

The continuous-time single-input/single-output plant represented by transfer function $G(s)$ is shown in Fig. 3.3 driven by a zero-order hold (a model of the D/A converter) and followed by an output sampler (A/D converter). These devices function as outlined in Section 3.2. The operation of the zero-order hold is shown in Fig. 3.2, whereby a discrete-time signal is converted to a stairstep function according to expression (3.2.1). The signal at the output of the zero-order hold could be decomposed into a series of gate functions as shown in Fig. 3.4. The u_0 gate function of Fig. 3.4a could be further decomposed into step functions as shown in Fig. 3.5.

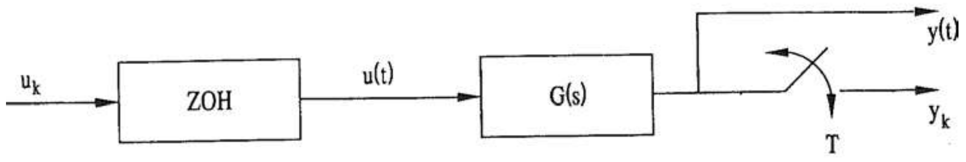

Figure 3.3. Continuous-time plant to be digitally controlled.

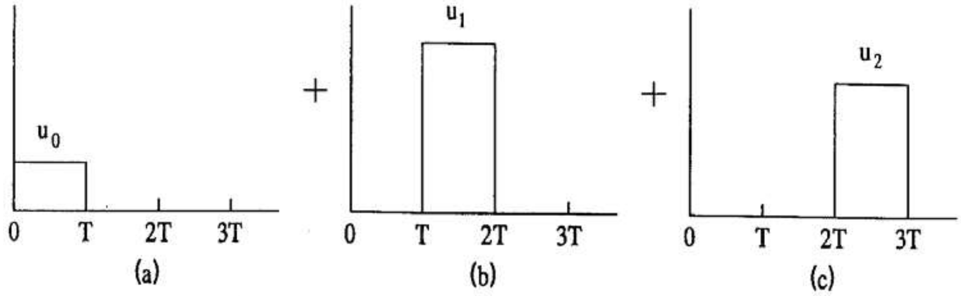

Figure 3.4. Gate function decomposition of the zero-order-hold output.

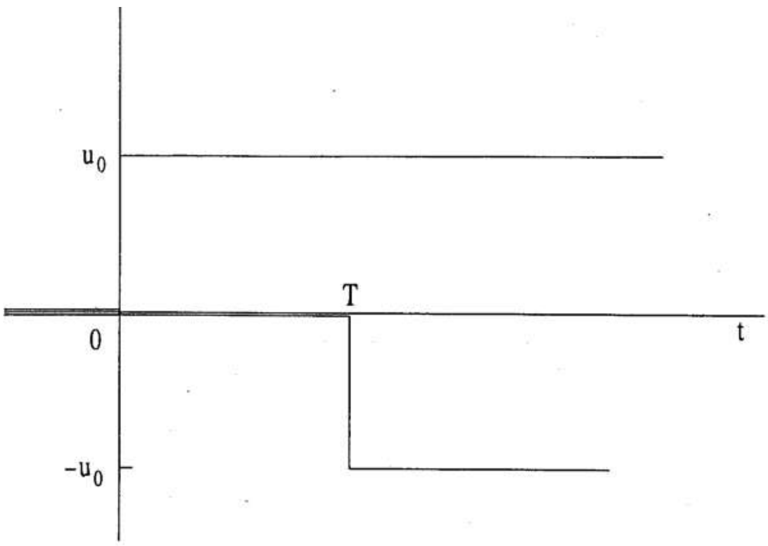

Figure 3.5. Step function decomposition of a gate function.

The continuous-time response to the first step of magnitude u_0 is

$$y(t) = u_0 \mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] \tag{3.3.1}$$

and the z-transform of the sampled sequence y_k is

$$Y(z) = u_0 \mathcal{Z}\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] \tag{3.3.2}$$

The response due to the negative-going step is exactly the same except that it is negative and delayed by one sample period so that the total z-domain response to the u_0 *gate function* may be calculated by superposition to be

$$Y(z) = u_0\left\{\mathfrak{Z}\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] - z^{-1}\mathfrak{Z}\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right]\right\} \tag{3.3.3}$$

where the z^{-1} represents the delay by one sample period as given by Theorem 2.2. Extending this idea to the second gate function, we get the response to the first two gate functions to be

$$\begin{aligned} Y(z) &= u_0(1 - z^{-1})\mathfrak{Z}\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] \\ &\quad + u_1(z^{-1} - z^{-2})\mathfrak{Z}\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] \end{aligned} \tag{3.3.4}$$

which can be simplified to be

$$Y(z) = (u_0 + u_1 z^{-1})(1 - z^{-1})\mathfrak{Z}\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] \tag{3.3.5}$$

Now for the whole series of gate functions representing $u(t)$, the transform of the sampled output is

$$Y(z) = \left(\sum_{k=0}^{\infty} u_k z^{-k}\right)(1 - z^{-1})\mathfrak{Z}\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] \tag{3.3.6}$$

We may now recognize the first term as the z-transform of the u_k sequence that is driving the zero-order hold. We will call this z-transform $U(z)$, so we see that the discrete-time transfer function between the input and output sequences is

$$\begin{aligned} G(z) &= \frac{Y(z)}{U(z)} = (1 - z^{-1})\mathfrak{Z}\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] \\ &= \frac{z-1}{z}\mathfrak{Z}\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] \end{aligned} \tag{3.3.7}$$

- (3.3.7) is referred to the pulse transfer function of Fig. 3.3
- It is used to convert the continuous time plant to digital z domain.
- By converting everything to z domain, calculation of the system characteristics is made easier.
- The two other approaches are: (i) converting everything to s-domain equivalent, (ii) direct computing of a mixed analogue-digital system.

Example 1

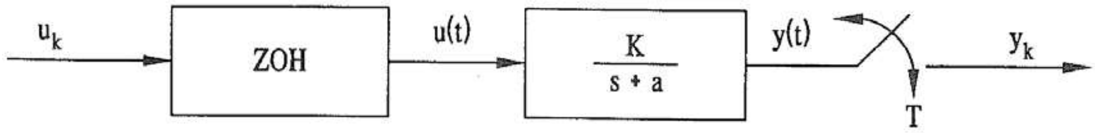

Figure 3.6. First-order plant to be digitally controlled.

Example 3.1. A large number of systems, such as thermal and fluid reservoir control problems as encountered in chemical engineering applications, may be described by a first-order differential equation or a transfer function of the form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{s + a}$$

This system will be driven by a zero-order hold and followed by a sampler as illustrated in Fig. 3.6. Relation (3.3.7) may be used to find the z-domain transfer function of

$$G(z) = (1 - z^{-1})\mathcal{Z}\mathcal{L}^{-1}\left[\frac{K}{s(s + a)}\right]$$

and upon making a partial fraction expansion, we get

$$G(z) = (1 - z^{-1})\mathcal{Z}\mathcal{L}^{-1}\left[\frac{K}{a}\left(\frac{1}{s} - \frac{1}{s + a}\right)\right]$$

Application of a table of z-transforms yields

$$G(z) = \frac{z - 1}{z}\left(\frac{z}{z - 1} - \frac{z}{z - e^{-aT}}\right)\frac{K}{a}$$

and upon simplification,

$$G(z) = \frac{K}{a}\frac{(1 - e^{-aT})}{z - e^{-aT}}$$

Now let us simplify this expression by choosing $K = a = 1$ and choose a

sampling interval to be $T = 0.2$ to yield

$$G(z) = \frac{1 - e^{-0.2}}{z - e^{-0.2}}$$

and upon evaluating the exponentials,

$$G(z) = \frac{0.1813}{z - 0.8187}$$

Example 3.2. Consider another simple example, which is the pure inertial plant driven by zero-order hold as shown in Fig. 3.7. This plant could represent the rotational dynamics of a simple satellite. Relation (3.3.7) gives us the ability to calculate the overall transfer function by

$$G(z) = \frac{z-1}{z}\mathscr{Z}\mathscr{L}^{-1}\left[\frac{G(s)}{s}\right]$$

or

$$G(z) = \frac{z-1}{z}\mathscr{Z}\mathscr{L}^{-1}\left[\frac{1}{s^3}\right]$$

Consulting a table of z-transforms yields

$$G(z) = \frac{z-1}{z}\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3}$$

which after the simplification yields

$$G(z) = \frac{T^2}{2}\frac{(z+1)}{(z-1)^2}$$

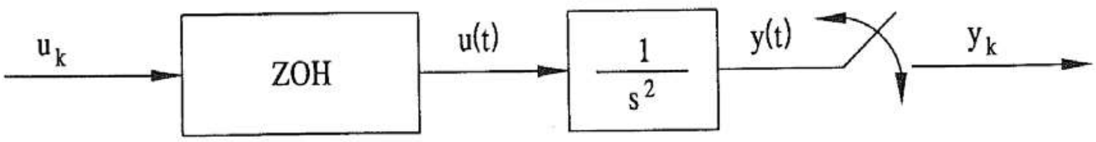

Figure 3.7. Inertial plant.

2. Digital control implementation

In this section, we see how a digital controller is implemented using a general computing hardware platform. The general control law to be implemented is:

$$u(k) = a_n e(k) + a_{n-1}e(k-1) + \cdots + a_0 e(k-n) + b_{n-1}u(k-1) + \cdots + b_0 u(k-n) \qquad (3.4.1)$$

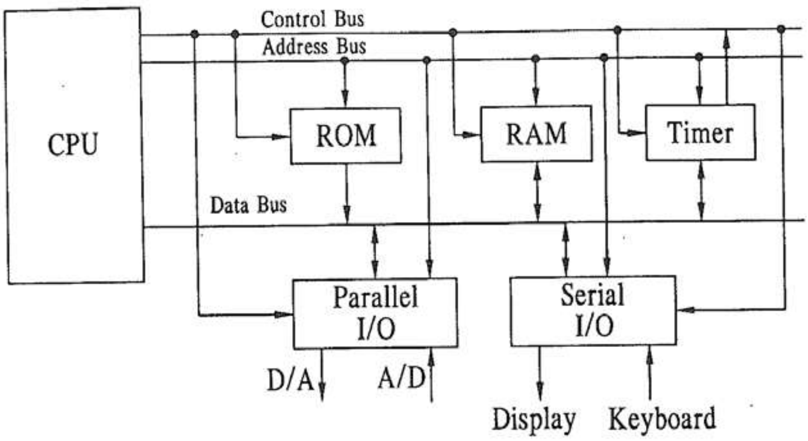

Figure 3.11. Architecture of a typical control computer.

This difference equation has an equivalent controller transfer function

$$D(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0}{z^n - b_{n-1} z^{n-1} - \cdots - b_1 z - b_0} \tag{3.4.2}$$

In the implementation of (3.4.1) all the past values of $e(k)$ and $u(k)$ are initially set to zero and the parameters a_i and b_i and the sampling interval desired are input. The sampling interval is converted to an integer count desired based on the clock rate to be counted. The timer is started counting down and a start-conversion signal is issued to the A/D converter and the system checks the A/D until conversion is complete. When conversion is complete, the value of $e(k)$ is read from the A/D, the first term of (3.4.1) is evaluated, and the resulting $u(k)$ is output to the D/A converter. Then the data $e(k)$, $e(k-1)$ . . . and $u(k)$, $u(k-1)$ . . . is all shifted down one address in the memory and all terms of (3.4.1) but the first are precalculated for the next cycle. At this point the timer is interrogated continuously until it has counted down, at which time control is transferred to the point in the software where the timer was started and the process is started again. The flowchart for such a program is illustrated in Fig. 3.12, which most certainly will clarify the explanation above.

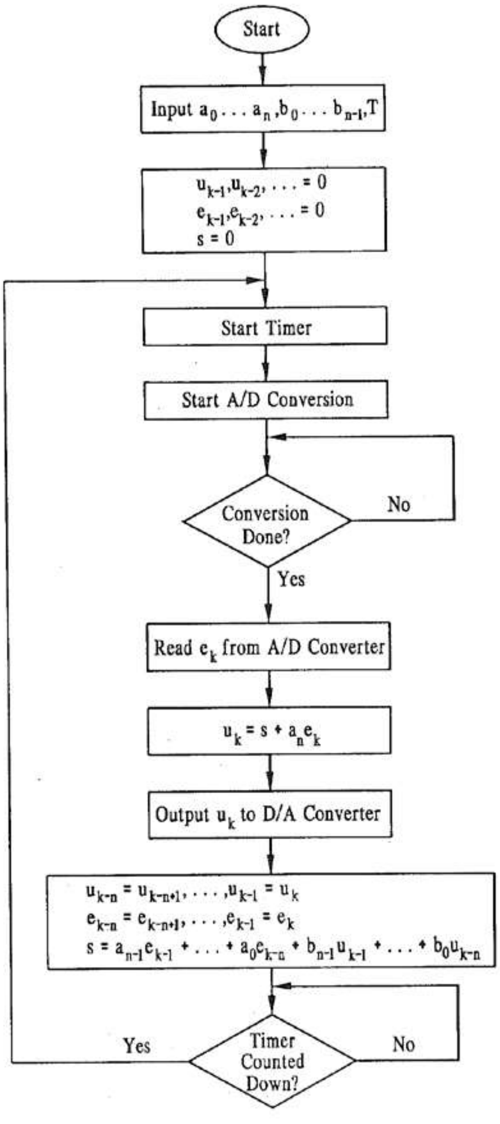

Figure 3.12. Flowchart for digital control algorithm implementation.

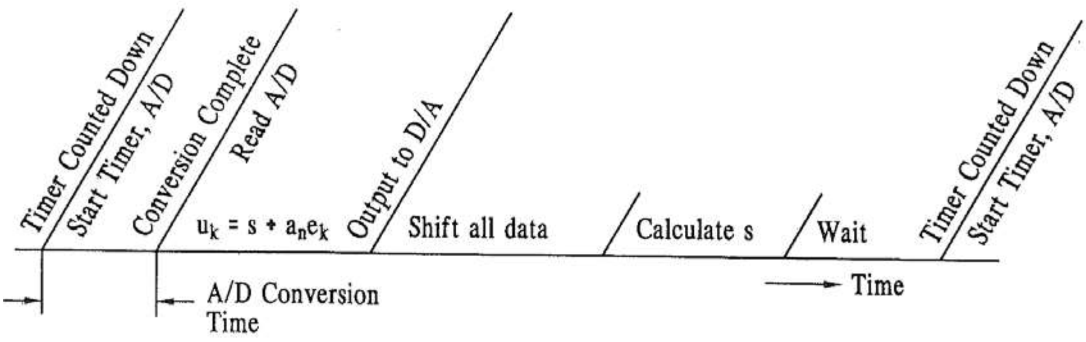

Figure 3.13. Timing diagram for a single sample interval.

- In actual hardware implementation, timer does not go into a polling loop
- Polling loop is a waste of system resource
- The control algorithm is hosted in an interrupt service routine (ISR).
- During the start of operation, a timer is initiated
- The timer will invoke the ISR at each end count.
- The control ISR will have the highest interrupt priority in the system.

Exercise: Redraw the Fig. 3.12 flow chart using Counter and ISR execution.

3. Closed loop characteristic equation

For the remainder of this chapter we shall be concerned with implementation of proportional control strategies, which are simply the first term on the right side of relation (3.4.1) or

$$u(k) = Ke(k) \tag{3.4.3}$$

or, from (3.5.2),

$$D(z) = K \tag{3.4.4}$$

where a_n has been replaced by the controller gain K. Of course, this strategy simplifies the programming implied by Fig. 3.12 but results in a control system that often does not have as high a performance as the designer might desire.

The common technique for linear closed-loop control of a discrete-time plant represented by transfer function $G(z)$ is to calculate the stepwise effort u_k based on a finite number of error measurements $e_k, e_{k-1}, \ldots,$ e_{k-m} and perhaps past values of the control sequence $u_{k-1} \cdots$ as discussed in Section 3.4. This situation is illustrated in Fig. 3.14a. If a control algorithm in the form of a linear difference equation with constant coefficients is employed, the entire system can be represented by z-domain transfer functions as shown in Fig. 3.14b.

The relation for the summing point is

$$E(z) = R(z) - Y(z) \tag{3.5.1}$$

and for the controller

$$U(z) = D(z)E(z) \tag{3.5.2}$$

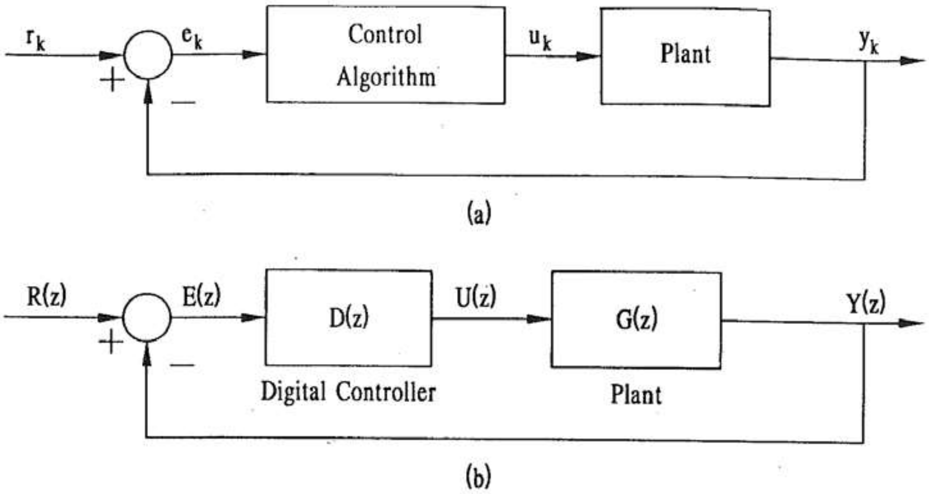

Figure 3.14. Closed-loop control system: (a) time domain; (b) z-domain.

while that for the plant is

$$Y(z) = G(z)U(z) \tag{3.5.3}$$

Substitution of (3.5.1) into (3.5.2) yields

$$U(z) = D(z)[R(z) - Y(z)] \tag{3.5.4}$$

Multiplication of (3.5.4) by $G(z$ will give $Y(z)$ according to expression (3.5.3) or

$$Y(z) = G(z)D(z)[R(z) - Y(z)] \tag{3.5.5}$$

Consolidating like terms gives

$$Y(z)[1 + G(z)D(z)] = G(z)D(z)R(z) \tag{3.5.6}$$

The closed-loop transfer function $M(z)$ is now the ratio of system output to reference input, or

$$M(z) = \frac{Y(z)}{R(z)} = \frac{G(z)D(z)}{1 + G(z)D(z)} \tag{3.5.7}$$

The quantity $1 + G(z)D(z)$, after rationalization, is the ratio of two polynomials, the numerator of which is the denominator polynomial of $M(z)$, which is the closed-loop system characteristic equation. The location of the roots of this characteristic equation in the z-plane determines the dynamic character of the closed-loop system, as discussed in Chapter 2. If one or more of these roots lie outside the unit circle, the closed-loop system is unstable, which is undesirable.

Example 3.5. Consider the closed-loop feedback control system, consisting of the first-order plant given in Example 3.1, which is to be controlled with a proportional control algorithm with a gain denoted as K_p. The complete feedback system is shown in Fig. 3.15.

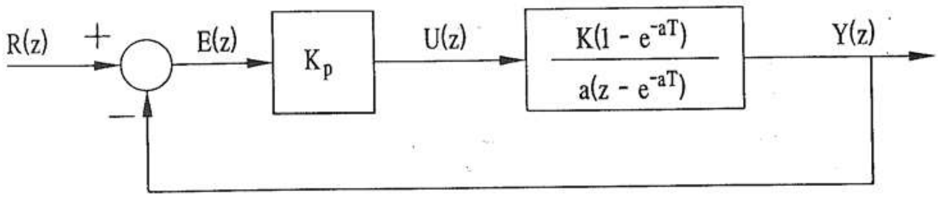

Figure 3.15. Proportional control of a first-order plant.

The closed-loop characteristic equation is given from relation (3.5.7) to be

$$1 + G(z)D(z) = 1 + \frac{K_pK}{a}\left(\frac{1 - e^{-aT}}{z - e^{-aT}}\right) = 0$$

or

$$z - e^{-aT} + \frac{K_pK}{a}(1 - e^{-aT}) = 0 \tag{a}$$

The complete root locus is shown in Fig. 3.16 as a function of the parameter K_pK/a. It is clear that the system becomes marginally stable when the closed-loop characteristic root is at $z = -1$. If we let $z = -1$ in (a) we may evaluate the value of the gain parameter for which marginal stability occurs.

$$\left(\frac{K_pK}{a}\right)_{\text{crit}} = \frac{1 + e^{-aT}}{1 - e^{-aT}} \tag{b}$$

If we multiply both numerator and denominator of relation (b) by $e^{aT/2}$, we get

$$\left(\frac{K_pK}{a}\right)_{\text{crit}} = \frac{e^{aT/2} + e^{-aT/2}}{e^{aT/2} - e^{-aT/2}} = \coth\left(\frac{aT}{2}\right)$$

The stability boundary for this system as a function of the parameter $aT/2$ is illustrated in Fig. 3.17.

If we wish to locate the closed-loop pole at an arbitrary location $z = b$ we may do so by letting $z = b$ in relation (a) and solve for the gain parameter to get

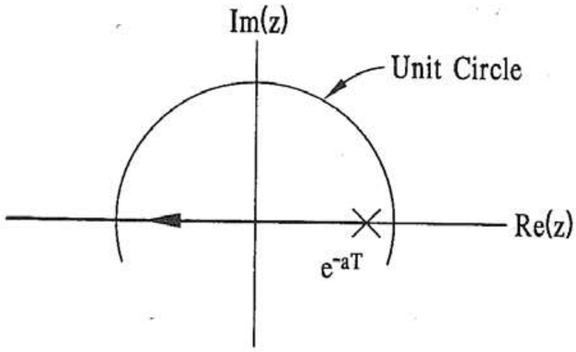

Figure 3.16. Root locus for proportionally controlled first-order plant.

$$\frac{K_pK}{a} = \frac{-b + e^{-aT}}{1 - e^{-aT}} \tag{c}$$

The loci for various values of b are also illustrated in Fig. 3.17. If we wish to evaluate the effect of the gain parameter on the steady-state error to a reference input, we can use the transfer function between the reference input and the error, which is

$$\frac{E(z)}{R(z)} = \frac{1}{1 + G(z)D(z)} = \frac{z - e^{-aT}}{z - e^{-aT} + \dfrac{K_pK}{a}(1 - e^{-aT})}$$

If the reference input $r(k)$ is a step of height r_0, then $R(z) = r_0z/(z - 1)$. The steady-state error then can be given by the final value theorem to be

$$e_{ss} = \lim_{z\to 1} \frac{z-1}{z}\,\frac{r_0 z}{z-1}\,\frac{z - e^{-aT}}{z - e^{-aT} + \dfrac{K_pK}{a}(1 - e^{-aT})}$$

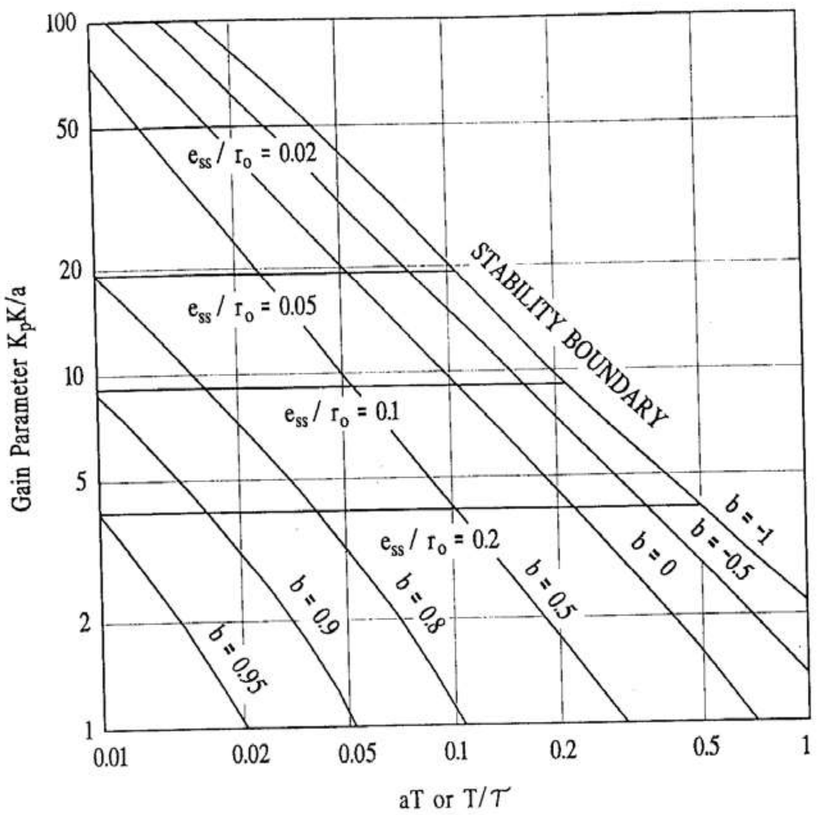

Figure 3.17. Stability boundary, closed-loop pole loci, and steady-state error loci for the proportionally controlled first-order system.

If we carry out the indicated limiting process, then

$$e_{ss} = r_0 \frac{1}{1 + \dfrac{K_p K}{a}}$$

Note that this steady-state error is independent of the sampling interval T and thus that loci of constant steady-state error are presented as horizontal lines in Fig. 3.17.

- The system can be unstable under digital control.
- The system is always stable under continuous control.
- Digital control is formulated under only a finite amount of information.
- The system gets “out of hand” during the inter-sampling period.
- Could improve stability by reducing T.

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