Dr. Norbert Cheung's Series in Electrical Engineering

Level 4 Topic no: 10

Linear Difference Equations and the z-transform

Contents

Reference:

"Modern Digital Control Systems, 2nd edition" Raymond G. Jacquot, Longman.

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1. Using difference equation to describe a system

- Analogue description: y(t) = F(x(t)). For example: y(t)= sin(x(t))
- For discrete control, we no longer have a continuous time domain function. Instead we have a series of numbers: 0, 1, 2, 3, 5, 10,……….
- Let us consider a set of real numbers with index value k, where k = (0, 1, 2, 3…).

For a transfer function with input u(k), and output y(k):

$$u(k) = f[y(k), y(k-1), \ldots, y(k-m), u(k-1), u(k-2), \ldots, u(k-n)] \quad (2.2.1)$$

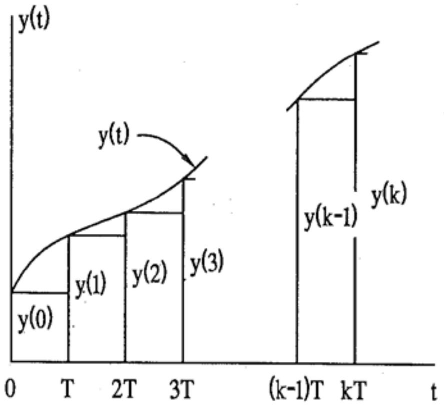
Of course there are an infinite number of ways the $n + m + 1$ values on the right side can be combined to form $u(k)$, but for the majority of this book we shall be interested only in the case where the right side involves a linear combination of the measurements and past controls, or

$$u(k) = b_{n-1}u(k-1) + \cdots + b_0 u(k-n) + a_m y(k) + a_{m-1}y(k-1) + \cdots + a_0 y(k-m) \quad (2.2.2)$$

k: present
k-1: previous sample
a_i, b_j: weighing factors

For example, if we want to integrate the function of curve below by rectangular approximation:

$$x(k) = x(k-1) + y(k-1)T$$



2. z- transform of simple sequence

Assumptions:

- Only interesting in the positive time values (i.e. one-sided z-transform)
- There is some region of the complex z-plane where the series of F(z) will converge to a limit value.
- z^{-1} is the previous value, z^{-2} is the previous previous value.
- Under the above assumptions, the z-transform is denoted by:

$$F(z) = \mathfrak{Z}[f(k)] = f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots$$

or

$$F(z) = \mathfrak{Z}[f(k)] = \sum_{k=0}^{\infty} f(k)z^{-k}$$

Example 1: unit step sequence

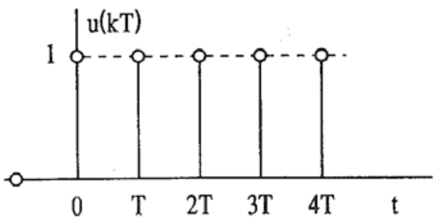
Consider the unit-step sequence of Fig. 2.2a. The function is defined as

$$u(kT) = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases} \qquad (2.3.3)$$

By application of the definition of the z-transform (2.3.2) and that of the function (2.3.3) which defines the samples, we get

$$U(z) = \mathfrak{Z}[u(kT)] = \sum_{k=0}^{\infty} z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \qquad (2.3.4)$$

It is clear that this series converges for $|z| > 1$, and a glance at a set of



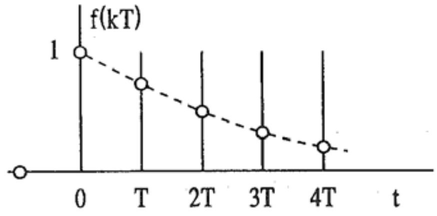
(a)

ordinary math tables will give the limiting form of such a convergent geometric series as

$$\mathfrak{Z}[u(kT)] = \frac{z}{z-1} \qquad \text{for } |z| > 1 \qquad (2.3.5)$$

which can easily be verified by long division. The requirement that $|z| > 1$ defines what is known as the region of convergence, which in this case is the area of the complex z-plane exterior to the unit circle.

Example 2: exponential function



(b)

Consider now the sampled exponential function illustrated in Fig. 2.2b. The sequence is defined by

$$f(k) = f(kT) = \begin{cases} 0 & k < 0 \\ e^{-akT} & k \geq 0 \end{cases} \tag{2.3.6}$$

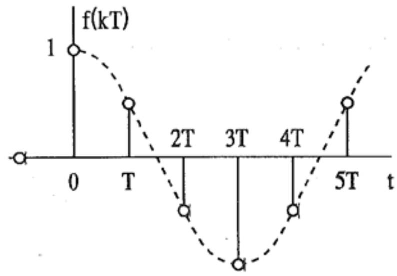
Substitution of (2.3.6) into (2.3.3) yields

$$\mathcal{Z}[e^{-akT}] = \sum_{k=0}^{\infty} (e^{-aT}z^{-1})^k \tag{2.3.7}$$

and from the previous transform the limit of the series is

$$\mathcal{Z}[e^{-akT}] = \frac{z}{z - e^{-aT}} \qquad \text{for } |z| > e^{-aT} \tag{2.3.8}$$

Example 3: cosine function



(c)

Consider the sampled cosine function of radian frequency Ω which is shown in Fig. 2.2c. The sequence is defined by

$$f(k) = f(kT) = \begin{cases} 0 & k < 0 \\ \cos k\Omega T & k \geq 0 \end{cases} \tag{2.3.13}$$

The cosine function can be rewritten using the Euler identity as

$$\cos k\Omega T = \frac{1}{2}(e^{jk\Omega T} + e^{-jk\Omega T}) \tag{2.3.14}$$

Since the z-transform of a sum is the sum of individual z-transforms, the result of (2.3.8) can be used to give

$$\mathcal{Z}[\cos k\Omega T] = \frac{1}{2}\left(\frac{z}{z - e^{j\Omega T}} + \frac{z}{z - e^{-j\Omega T}}\right) \tag{2.3.15}$$

and finding a common denominator yields

$$\mathcal{Z}[\cos k\Omega T] = \frac{z^2 - z\cos \Omega T}{z^2 - z \cdot 2\cos \Omega T + 1} \tag{2.3.16}$$

The region of convergence is the region of the z-plane exterior to the unit circle. The sampled sine function will be left as an exercise for the reader but is given in Table 2.1.

Example 4: impulse function

Consider a sequence $\delta(k)$ that is defined by

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \qquad (2.3.17)$$

Using the definition of (2.3.2) the result is

$$\mathcal{Z}[\delta(k)] = 1 \qquad (2.3.18)$$

In a similar fashion we can show that a delayed impulse function defined by

$$\delta(k - n) = \begin{cases} 0 & k \neq n \\ 1 & k = n > 0 \end{cases} \qquad (2.3.19)$$

has a z-transform

$$\mathcal{Z}[\delta(k - n)] = z^{-n} \qquad (2.3.20)$$

Example 5: ramp function

The sampled ramp function is defined by

$$f(k) = kT \qquad k = 0, 1, 2, \ldots \qquad (2.3.21)$$

and application of definition (2.3.2) yields

$$\mathcal{Z}[kT] = T \sum_{k=0}^{\infty} kz^{-k} \qquad (2.3.22)$$

Again consulting a table of mathematical functions, the limit of the series is

$$\mathcal{Z}[kT] = \frac{Tz}{(z - 1)^2} \qquad \text{for } |z| > 1 \qquad (2.3.23)$$

Table 2.1. Short Table of z-Transforms of Sampled Continuous Functions

$f(t),\ t \geq 0$	$L[f(t)]$	$F(z) = \mathcal{Z}[f(kT)] = \mathcal{Z}[f(k)]$
$u(t)$	$\frac{1}{s}$	$\frac{z}{z - 1}$
t	$\frac{1}{s^2}$	$\frac{Tz}{(z - 1)^2}$
e^{-at}	$\frac{1}{s + a}$	$\frac{z}{z - e^{-aT}}$
$\cos \Omega t$	$\frac{s}{s^2 + \Omega^2}$	$\frac{z^2 - (\cos \Omega T)z}{z^2 - (2\cos \Omega T)z + 1}$
$\sin \Omega t$	$\frac{\Omega}{s^2 + \Omega^2}$	$\frac{(\sin \Omega T)z}{z^2 - (2\cos \Omega T)z + 1}$

3. Useful theorems associated with the z- transform

It has been our experience that certain theorems were quite useful in the theory of the Laplace transform, and hence a few necessary and very useful theorems associated with the z-transforms will now be developed.

Linearity

Theorem 2.1. Linearity. We shall show that the z-transform is a linear transformation which implies that

$$\mathcal{Z}[\alpha f(k)] = \alpha\mathcal{Z}[f(k)] = \alpha F(z) \tag{2.4.1}$$

and

$$\mathcal{Z}[\alpha f(k) + \beta g(k)] = \alpha F(z) + \beta G(z) \tag{2.4.2}$$

Delay Theorem (shift theorem)

Theorem 2.2. Delay Theorem. The z-transform of a delayed sequence shifted one step to the right is given by

$$\mathcal{Z}[f(k - 1)] = z^{-1}F(z) \tag{2.4.5}$$

Advance Theorem (shift theorem)

Theorem 2.3. Advance Theorem. The z-transform of a sequence that has been shifted one step to the left is

$$\mathcal{Z}[f(k + 1)] = zF(z) - zf(0) \tag{2.4.10}$$

General Exponential Function

$$\mathcal{Z}[r^k] = \frac{z}{z - r} \qquad \text{for } |z| < |r| \tag{2.3.11}$$

4. Inversion of the z- transform

Method 1: By long division

This is best to illustrate by an example

Example 2.3. Find the inverse sequence for the following function:

$$F(z) = \frac{z^2 + z}{z^2 - 3z + 4}$$

Multiplication by z^{-2} in numerator and denominator gives

$$F(z) = \frac{1 + z^{-1}}{1 - 3z^{-1} + 4z^{-2}}$$

Now carry out formal long division to yield

$$\begin{array}{r|l}
 & 1 + 4z^{-1} + 8z^{-2} \\
\hline
1 - 3z^{-1} + 4z^{-2} & 1 + z^{-1} \\
 & \underline{1 - 3z^{-1} + 4z^{-2}} \\
 & \quad 4z^{-1} - 4z^{-2} \\
 & \quad \underline{4z^{-1} - 12z^{-2} + 16z^{-3}} \\
 & \qquad 8z^{-2} - 16z^{-3} \\
 & \qquad \underline{8z^{-2} - 24z^{-3} + 32z^{-4}} \\
 & \qquad\qquad 8z^{-3} - 32z^{-4} \\
 & \qquad\qquad \vdots
\end{array}$$

Now upon examination of the coefficients of the infinite series answer, the sequence is

$$f(0) = 1$$
$$f(1) = 4$$
$$f(2) = 8$$

Method 2: By partial fraction expansion

This is best to illustrate by an example (for case with distinct real roots)

Example 2.4. Consider the following z-domain function:

$$F(z) = \frac{z^2 + z}{(z - 0.6)(z - 0.8)(z - 1)}$$

Find the partial fraction expansion and invert the resulting transform. The expansion will be of the form

$$F(z) = \frac{A_1 z}{z - 0.6} + \frac{A_2 z}{z - 0.8} + \frac{A_3 z}{z - 1}$$

where the coefficients are given by (2.6.5). The constant term in the expansion has been omitted bcause there is not a constant term in the numerator polynomial of the original function, or

$$A_1 = \left.\frac{z+1}{(z-0.8)(z-1)}\right|_{z=0.6} = \frac{1.6}{(-0.2)(-0.4)} = 20$$

$$A_2 = \left.\frac{z+1}{(z-0.6)(z-1)}\right|_{z=0.8} = \frac{1.8}{(0.2)(-0.2)} = -45$$

and

$$A_3 = \left.\frac{z+1}{(z-0.6)(z-0.8)}\right|_{z=1} = \frac{2}{(0.4)(0.2)} = 25$$

So upon inversion of the transform,

$$f(k) = 20(0.6)^k - 45(0.8)^k + 25$$

5. Solving linear difference equations with the z- transform

By employing the delay and advance theorems (Theorems 2.2 and 2.3) and the transforms of known functions, we are now prepared to solve linear constant-coefficient difference equations, but first let us write down the results of these valuable theorems:

$$\text{Theorem 2.2:} \quad \mathfrak{Z}[f(k-n)] = z^{-n}F(z) \tag{2.7.1}$$

$$\text{Theorem 2.3:} \quad \mathfrak{Z}[f(k+n)] = z^{n}F(z) - z^{n}f(0) - \cdots - zf(n-1) \tag{2.7.2}$$

The technique is similar to that of using Laplace transforms and is best illustrated by some example problems, which follow.

Example

Example 2.7. Consider the same example as before with starting condition $x(0) = 2$ and an inhomogeneous term on the right side, or

$$x(k+1) - 0.8x(k) = 1$$

Taking the z-transform yields

$$zX(z) - 2z - 0.8X(z) = \frac{z}{z-1}$$

Solving for $X(z)$ yields

$$X(z) = \frac{2z}{z-0.8} + \frac{z}{(z-1)(z-0.8)}$$

We can readily invert the first term as in Example 2.6, but we must now expand the second term as

$$\frac{z}{(z-1)(z-0.8)} = \frac{Az}{z-1} + \frac{Bz}{z-0.8}$$

Solving for A and B yields

$$A = \frac{z-1}{z} \frac{z}{(z-1)(z-0.8)}\bigg|_{z=1} = 5$$

and

$$B = \frac{z-0.8}{z} \quad (z-1)(z-0.8)$$

So the z-domain solution is

$$X(z) = \frac{2z}{z-0.8} + \frac{5z}{z-1} - \frac{5z}{z-0.8}$$

and the total solution is

$$x(k) = -3(0.8)^k + 5$$

<u>6. z- domain transfer function</u>

Consider now a general discrete-time system described by the linear constant-coefficient difference equation

$$y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k)$$
$$= d_{n-1}u(k+n) + d_{n-1}u(k+n-1) + \cdots + d_0 u(k) \quad (2.8.1)$$

If we now take the a transform employing the advance theorem (Theorem 2.3) and ignoring the starting conditions $y(0), \ldots, y(n-1)$ and $u(0), \ldots, u(n-1)$, we get

$$(z^n + a_{n-1}z^{n-1} + \cdots a_1 z + a_0)Y(z)$$
$$= (d_n z^n + \cdots + d_1 z + d_0)U(z) \qquad (2.8.2)$$

We can solve for the ratio of output $Y(z)$ to the input $U(z)$ to give

$$H(z) = \frac{Y(z)}{U(z)} = \frac{d_n z^n + \cdots + d_1 z + d_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} \qquad (2.8.3)$$

which is the z-domain transfer function.

7. The z-plane pole locations

First let us consider a z-domain function of the form

$$F(z) = \frac{Az}{z - p} \tag{2.9.1}$$

where p is a real number or a real pole of the function $F(z)$; the associated time-domain sequence is

$$f(k) = Ap^k \qquad k > 0 \tag{2.9.2}$$

Clearly, if $p < -1$, the solution will oscillate and increase in magnitude for large k. If $-1 < p < 0$, the solution will decay in an oscillatory fashion, and if $0 < p < 1$, it will decay in an exponential manner as k becomes large. Also, if $p > 1$, the sequence will grow with an exponential nature.

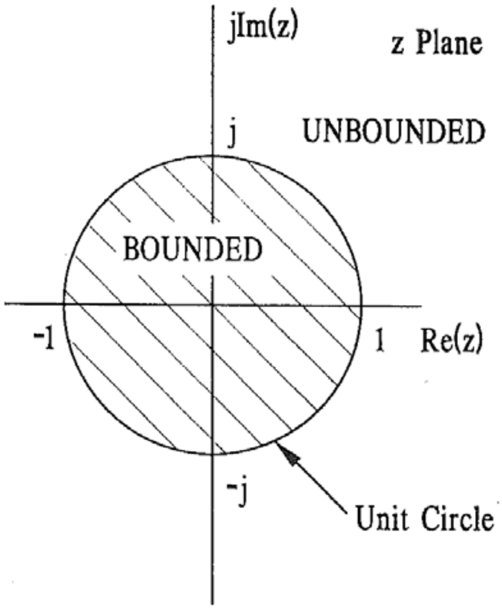


Figure 2.4. Regions of poles for bounded and unbounded sequences in the z-plane.

Now let us consider a function with a quadratic denominator of the form

$$F(z) = \frac{N(z)}{z^2 - bz + c} \tag{2.9.3}$$

The denominator has complex roots p and p^*, and hence a partial fraction expansion of the form of (2.9.4) can be made to yield

$$F(z) = \frac{Az}{z - p} + \frac{A^*}{z - p^*} \tag{2.9.4}$$

where the asterisk denotes the complex conjugate. We can write the complex pole in polar form as

$$p = Re^{j\theta} \tag{2.9.5}$$

and the conjugate pole as

$$p^* = Re^{-j\theta} \tag{2.9.6}$$

The coefficients A and A^* are also complex conjugates, and we can let

$$A = \alpha + j\beta \tag{2.9.7}$$

If we substitute (2.9.5), (2.9.6), and (2.9.7) into (2.9.4) and invert the z-transforms, we get

$$f(k) = (\alpha + j\beta)R^k e^{jk\theta} + (\alpha - j\beta)R^k e^{-jk\theta} \tag{2.9.8}$$

We may rewrite this as

$$f(k) = R^k[\alpha(e^{jk\theta} + e^{-jk\theta}) + j\beta(e^{jk\theta} - e^{-jk\theta})] \tag{2.9.9}$$

and upon employing the Euler identities we get

$$f(k) = R^k(2\alpha \cos k\theta - 2\beta \sin k\theta) \tag{2.9.10}$$

If we now examine this, we see that the sine and cosine functions are bounded by plus and minus unity, and hence the R^k factor determines the asymptotic nature of the discrete-time sequence. If R is greater than unity, the sequence will be unbounded for large k, and if R is less than unity (but still positive), the sequence will converge to zero for large k.

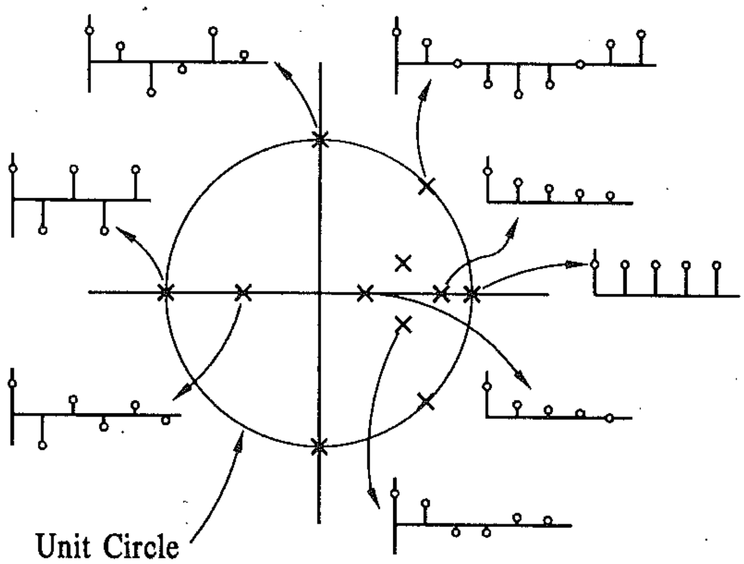


Figure 2.5. Pole locations and associated time-domain sequences.

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